10 Review of nonlinear dynamics concepts covered in the course

1. One-parameter bifurcations:
   (a) 1d: saddle node,
   $$\dot{x} = \mu - x^2$$
   transcritical,
   $$\dot{x} = \mu x - x^2$$
   pitchfork: super-critical and sub-critical.
   $$\dot{x} = \mu x - x^3$$
   $$\dot{x} = \mu x + x^3 - x^5$$
   (b) Hysteresis in the case of two back-to-back saddle node bifurcations, and in the case of a subcritical pitchfork.
   (c) 1d bifurcations in higher dimensional systems
   (d) 2d: Hopf, super-critical
   $$\dot{r} = \mu r - r^3$$
   $$\dot{\theta} = \omega + b r^2$$
   and sub critical.
   $$\dot{r} = \mu r + r^3 - r^5$$
   $$\dot{\theta} = \omega + b r^2$$
   (e) Hysteresis in subcritical Hopf.
2. Nonlinear phase locking via analysis of flows on a circle, firefly and flashlight.
3. Two-parameter bifurcation: quasi-periodicity route to chaos in circle map.
   $$\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_n)$$
4. Relaxation oscillation, analysis via nullclines of the Van der Pol oscillator,
   $$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$
5. Saddle node bifurcation of cycles.

(F)

bifurcations in 1d systems

what is a bifurcation: a qualitative change in the nature of the solution of a dynamical system as a parameter(s) is varied.

* example: $\dot{x} = \mu - x^2$. steady state: $\dot{x} = \mu - x^2 = 0$
for $\mu < 0$, there are no steady states
$\mu = 0$ 1 steady state, $x = 0$
$\mu > 0$ 2 " " $x = \pm\sqrt{\mu}$.

* when can we expect a bifurcation

Implicite function theorem:

consider eqn $\vec{f}(\vec{x}, \mu) = 0$, where: $\vec{f}, \vec{x}$ are n-dim.
let $f(\bar{x}, \bar{\mu}) = 0$, & view this as an implicit eqn for $\vec{x}(\mu)$ satisfying $f(\vec{x}(\mu), \mu) = 0$. if $J = \left[\frac{\partial f_i}{\partial x_j}\right]$ (Jacobian) is non singular [$\det(J) \neq 0$], & f is smooth etc, then $\vec{x}(\mu)$ is unique & smooth around $\bar{x}(\bar{\mu})$.

$\Rightarrow$ given $\dot{\vec{x}} = \vec{f}(\vec{x}, \mu)$, if $J = \left\{\frac{\partial f_i}{\partial x_j}\right\}$ is not singular [in one dim: $\partial f / \partial x \neq 0$], we cannot have any bifurcation, as the solutions to $\dot{\vec{x}} = 0$ are unique at $\vec{x}, \mu$.

* in the above example:
$\dot{x} = \mu - x^2$. $J = \frac{\partial f}{\partial x} = -2x = 0$ at $x = 0$, where indeed that is a bifurcation point, due

So, near bif. pt.:

(#) $$f(x,\mu) = \underbrace{f_\mu \cdot \mu}_{1} + \underbrace{f_{x\mu} \cdot \mu x}_{2} + \underbrace{\tfrac{1}{2} f_{xx}\, x^2}_{3} + \underbrace{\tfrac{1}{2} f_{\mu\mu}\, \mu^2}_{4} + \underbrace{\tfrac{1}{6} f_{xxx}\, x^3}_{5} + \dots$$

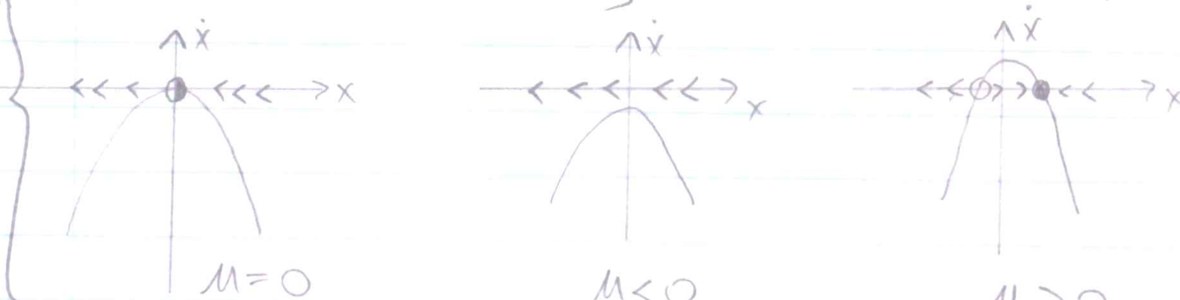
Depending on which coeffs in above expansion vanish, we can get 3 types of bifurcations.

Let us consider 3 specific examples of (#) which demonstrate these 3 types of 1d bifurcations:

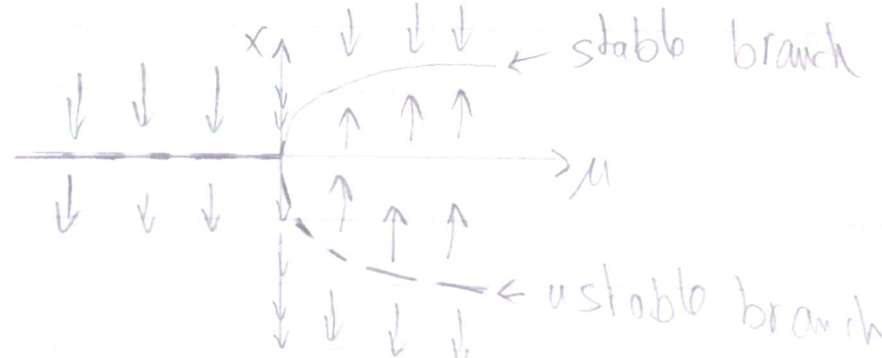
Saddle - node bifurcation

$\dot{x} = f(x,\mu) = \mu - x^2$ ⇒ fixed pts only for $\mu > 0$,
$\dot{x} = 0 \Rightarrow x = \pm\sqrt{\mu}$ — there are two way to plot this:

for a given value of μ (1)



combining all values of μ into one plot (2)



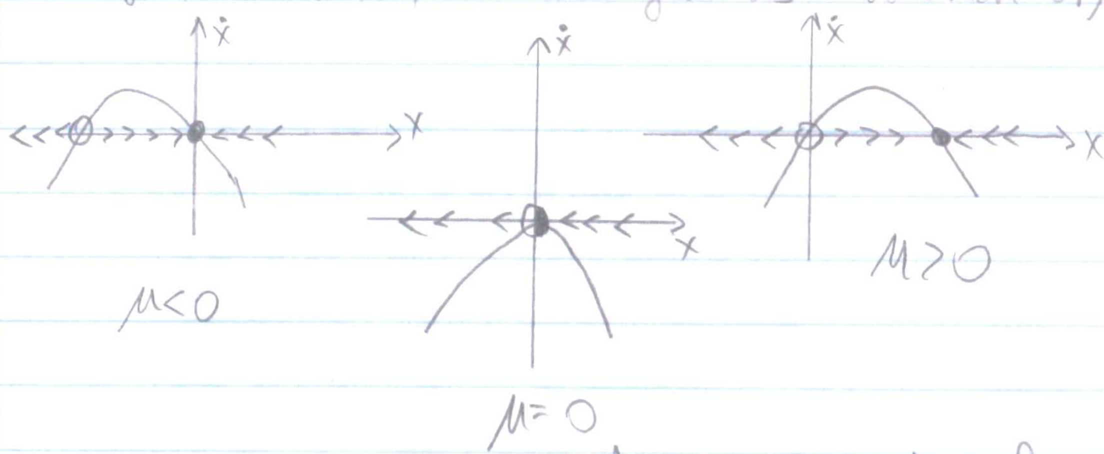
The next example of a 1d bifurcation is:

Transcritical bifurcation:

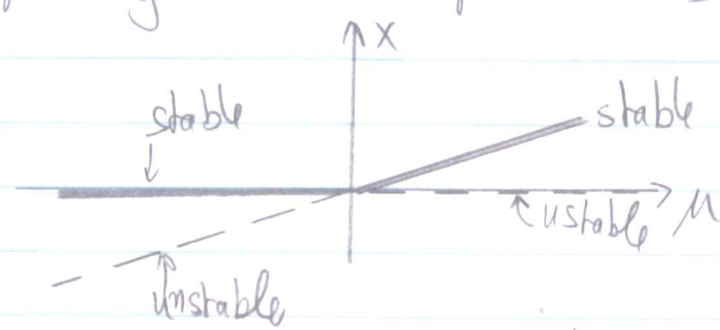
The normal form is

$$\dot{x} = f(x,\mu) = \mu x - x^2 \quad \text{[terms (2), (3) in (7), p. 47]}$$

- bif. point is at $(\mu, x) = (0,0)$.
- steady states are, for all μ: $x = 0, \mu$, but their stability is changed as function of μ:



plotting only fixed points as funct of μ:



$\Rightarrow$ exchange of stability.

zero is always a solution, so this bifurcation may be expected in systems for which a solution is known to always exist, & that it's stability may change as function of some parameter.

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example: logistic map.

$$X_{n+1} = \mu X_n (1 - X_n)$$

* conditions (symmetry requirements) for transcritical bif:

$$\left.\begin{array}{l} f(x^*) = 0 \\ \left.\frac{\partial f}{\partial x}\right|_{x^*} = 0 \end{array}\right\} \text{non hyperboic f.p.}$$

$$\left.\begin{array}{l} \left.\frac{\partial f}{\partial \mu}\right|_{\mu_0} = 0 \\ \frac{\partial^2 f}{\partial \mu \partial x} \neq 0, \quad \frac{\partial^2 f}{\partial x^2} \neq 0 \end{array}\right\}$$

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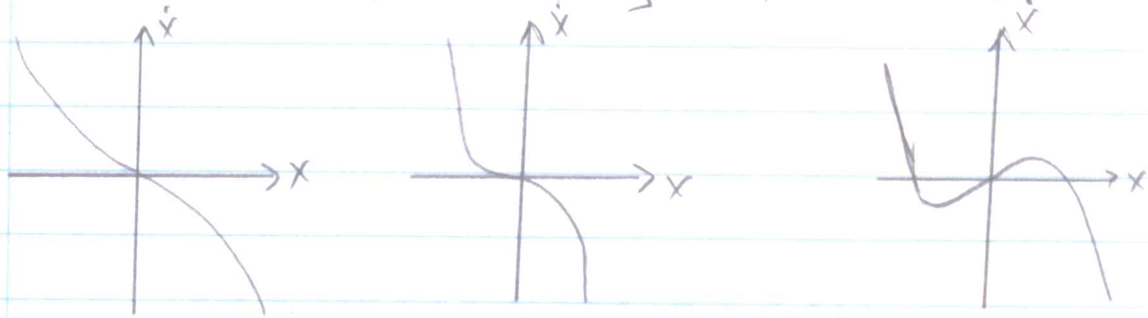
Pitchfork bifurcation.

the normal form is

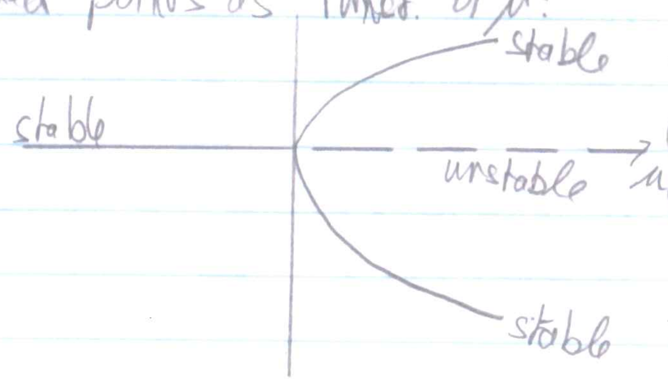
$$\dot{x} = \mu \cdot x - x^3 = f(x) \quad [\text{terms } ②, ⑤ \text{ in } (\#), \text{p. } 47]$$

$f(x^*) = 0 \Rightarrow x^* = \{0, \pm\sqrt{\mu}\}\ \mu > 0; \quad x^* = \{0\}\ \mu < 0.$



$\mu < 0$ $\mu = 0$ $\mu > 0$

& the fixed points as funct. of μ:



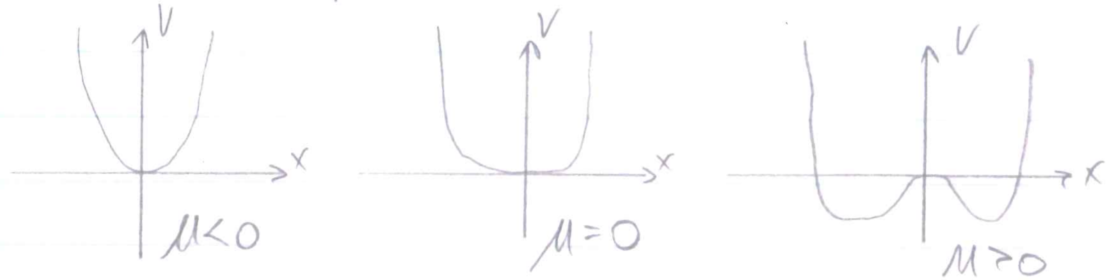
looks like pitchfork (ρεδι)

* 1 stable f.p. bifurcates into 2 stable + 1 unstable.
* occures in system with symmetry in which the existence of one solution implies that of another.

→ e.g. looking at the oceanic circulation on both sides of the equator as function of some forcing parameter (strength of wind

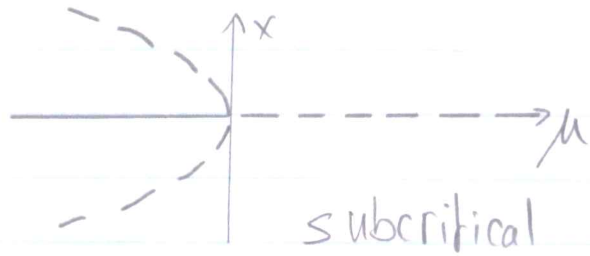
→ e.g. spatial symmetry, where eqns are invariant to $x \to -x$.

in-terms of a potential: $-\frac{dV}{dx} = f(x) = \dot{x}$



this was superferitical pitchfork bif., there is also subcritical pitch-fork bifurcation

$\dot{x} = f(x) = \mu x + x^3$. fixed points are now:



subcritical pitchfork bifurcation

* conditions for pitchfork bif:

$f(x^*) = 0$ fixed pt.

$\frac{df}{dx} = 0$ non hyperbolic

f is odd

$\frac{\partial^3 f}{\partial x^3} \neq 0 \Leftrightarrow -f(x) = f(-x)$ [because then $\dot{x} = f(x)$ is invariant under $x \to -x$]

$\frac{\partial^2 f}{\partial x^2} = 0$ ← {otherwise f is not odd}.

$\frac{\partial^2 f}{\partial x \partial \mu} \neq 0$

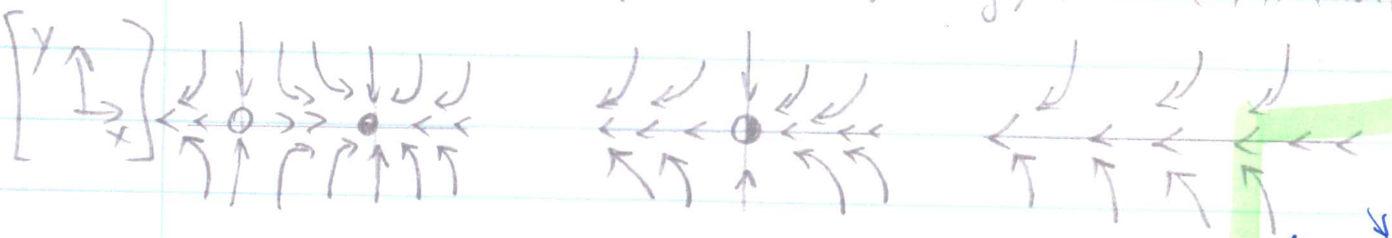
$\frac{\partial f}{\partial \mu} = 0$ ←

Center manifold theory [time permitting...]

It's important to understand that the saddle-node, transcritical & pitchfork bif's can occure in larger dimensional dynamical systems $\dot{x} = f(x,\mu)$, $x \in \mathbb{R}^n$. When this occurs, there is a systematic theory for how to reduce the n-dim system into an equivalen 1-d system via a nonlinear change of coordinates. To get a feeling for this: consider a 2-d system:

$$\dot{x} = \mu - x^2 \quad (\star)$$
$$\dot{y} = -y$$

behavior in y is very simple: $y(t) \to 0$ as $t \to \infty$.
behavior in x depends on μ; e.g. $\mu > 0 \Rightarrow (x^*, y^*) = (\pm\sqrt{\mu}, 0)$



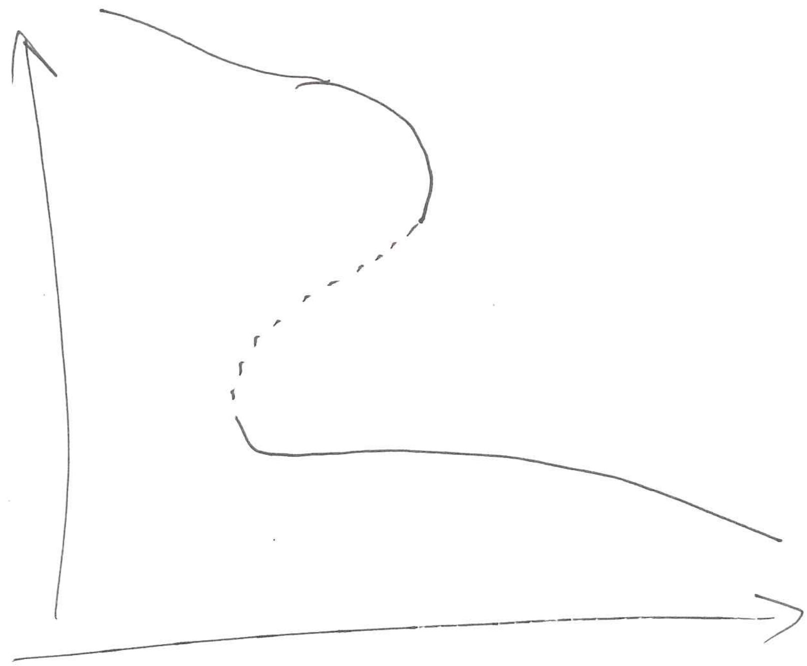
$\mu > 0$ $\quad$ $\mu = 0$ $\quad$ $\mu < 0$

this may also be rotated:

* the x-axis is the "center manifold" where all the interesting stuff happens. Note that it is "invariant", a solution with $y(t_0) = 0$ will remain on the x-axis for all t. y is a "stable manifold". There can also be "unstable manifold".
* Center manifold theory is a systematic method for finding such an invariant center manifold on which the interesting things occur...

8

hysteresis with 2 back-to-back saddle node bifs

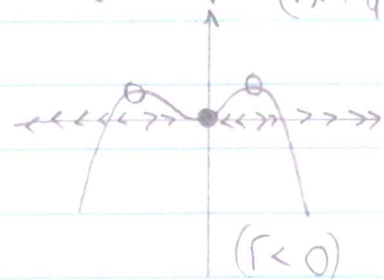


global energy balance
Stommel THC box model
⋮

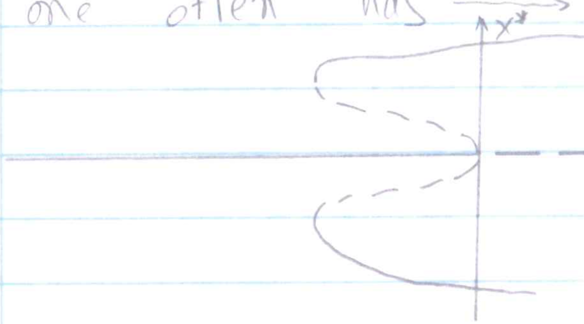
hysteresis in subcritical pitchfork bifurcation:

the equation $\dot{x} = rx + x^3$ has diverging solutions in terms of the potential function.

$V = -(rx^2 + \frac{1}{4}x^4)$



in physical systems where diverging solutions are not acceptable, yet where the symmetry $x \to -x$ is perserved, one often has $\longrightarrow$ $\dot{x} = rx + x^3 - x^5$:



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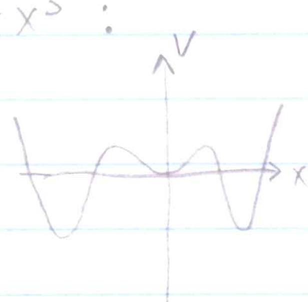
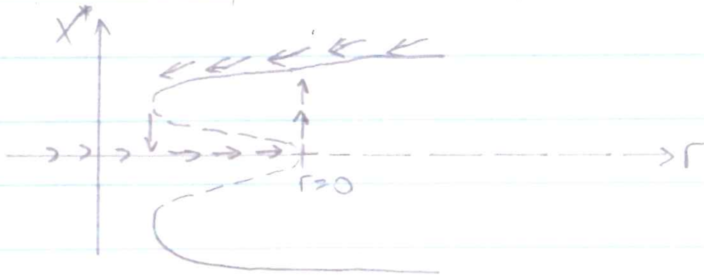


now, as r varies slowly, so that at any given moment the system is always at equilibrium (f.p.):



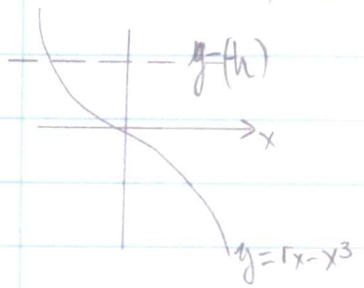
$\Rightarrow$ existence of multiple equilibria allows a hysteresis as function of r.

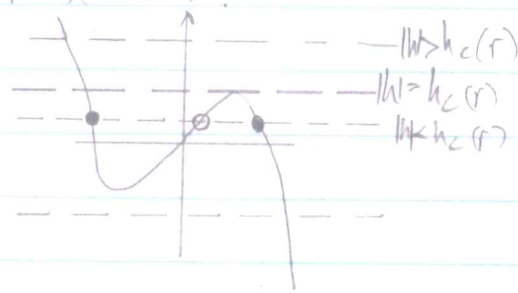
Imperfections (symmetry breaking) & Catastrophes

consider a perturbation h to a pitchfork bif:

$$\dot{x} = h + rx - x^3$$

no more symmetric under $x \to -x$.

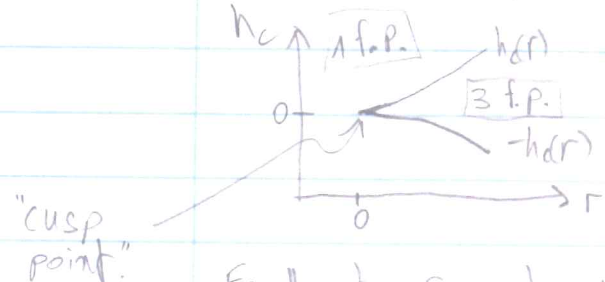




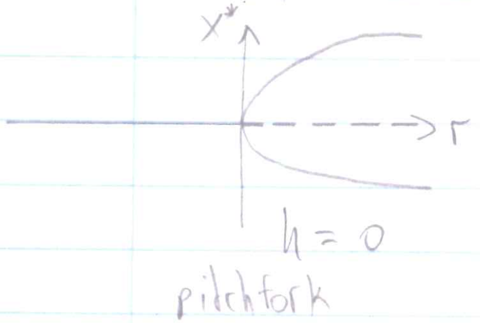
fixed pts are crossings of $y = -h$ & $y = rx - rx^3$

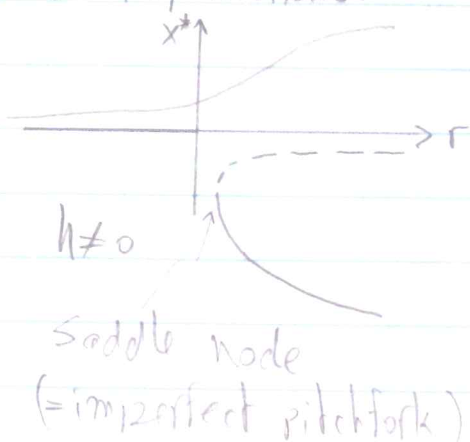
$\Rightarrow$ can have 1, 2, 3 f.p. as h varies. Find $h_c(r)$ explicitly: at $h = h_c$, $\frac{d}{dx}(rx - x^3) = 0 \Rightarrow x_{max} = \sqrt{\frac{r}{3}}$

$\Rightarrow h_c(r) = r x_{max} - x_{max}^3 = \frac{2}{3} r \sqrt{\frac{r}{3}}$, similarly for x_{min}...

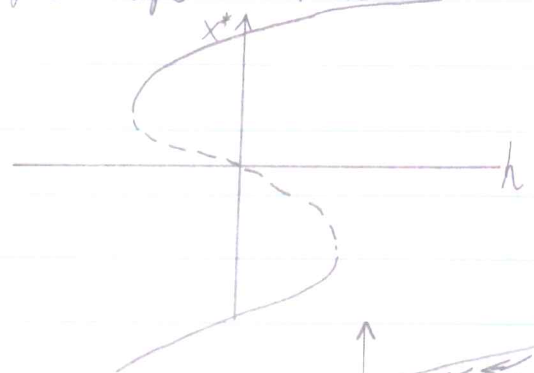


Finally, to see why this is called imperfections:

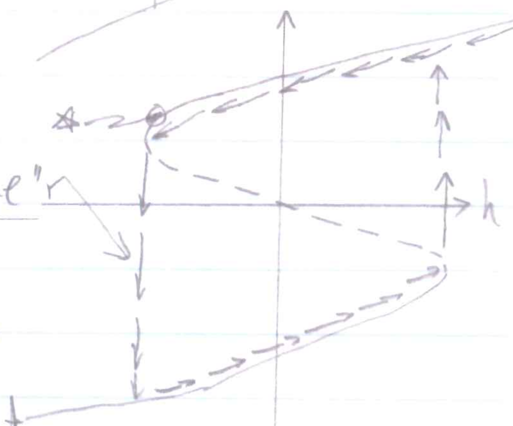




what about catastrophes: plot f.p. as function of h, for a given r:



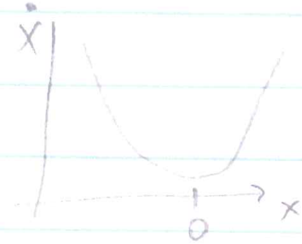
$\Rightarrow$ like in subcritical pitchfork, can have hysteresis as h varies.
also, the jump is "catastrophe"



✲ example: climate system,
$X \equiv$ global-averaged temperature,
$h =$ incoming solar ratiation
✲ = location of present-day climate

$\Rightarrow$ a small decrease in solar radiation (due to atmc polution blocking radiation) $\Rightarrow$ a catastrophe, in this case a climate state of over 100°C colder than present climate, snow-covered earth....

consider $\dot{x} = r + x^2$



the time to pass via the bottle neck is $\int_{-\infty}^{\infty} \frac{dx}{r+x^2} = \frac{\pi}{\sqrt{r}}$ again square root law.

synchronization, nonlinear phase locking:

example: Fireflies (גחליליות)

fireflies tend to flash on & of together. why? how? Also, given an external periodic light, fireflies try to follow it, & may not be able to do it if it's too fast & then try again...

a specific model: let Θ be the phase of an external stimulation (flash light) of a constant rate:

$\dot{\Theta} = \Omega$. (flash occurs at $\Theta = 0$).

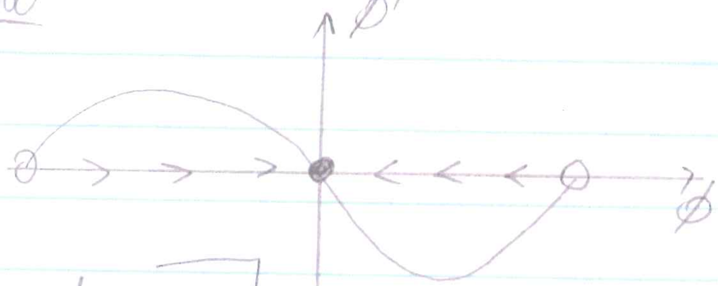
The fire fly increases/decreases it's rate of flashing depending on phase difference from external source:

$$\dot{\theta} = \underbrace{\omega}_{\text{natural rate for fireflies}} + \underbrace{A\sin(\Theta - \theta)}_{\text{adjustment to stimulus.}}$$

The phase diference between the stimulus & the firefly satisfies $\dot{\phi} = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A\sin\phi$

let $\tau = At$, $\mu = \frac{\Omega - \omega}{A}$

$\Rightarrow \phi' \equiv \frac{d\phi}{d\tau} = \mu - \sin\phi$

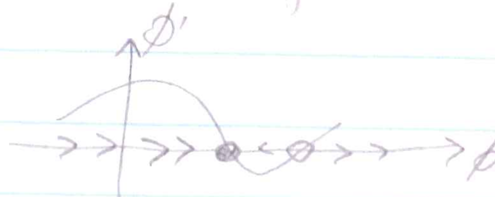


$\boxed{\mu = 0}$

* $\mu = 0 \Rightarrow$ fixed pt is $\phi = 0$

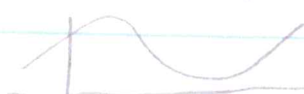
$\Rightarrow$ no phase difference between flashlight & firefly. ($\mu = 0$ means natural firefly freq = external freq $\Omega = \omega$).

* $0 < \mu < 1$



$\Rightarrow$ a constant phase lag, fireflies are "phase locked" to flashlight.

* $1 < \mu$



$\Rightarrow$ phase difference $\phi = \Theta - \theta$ varies.

$\Rightarrow$ "phase drift."

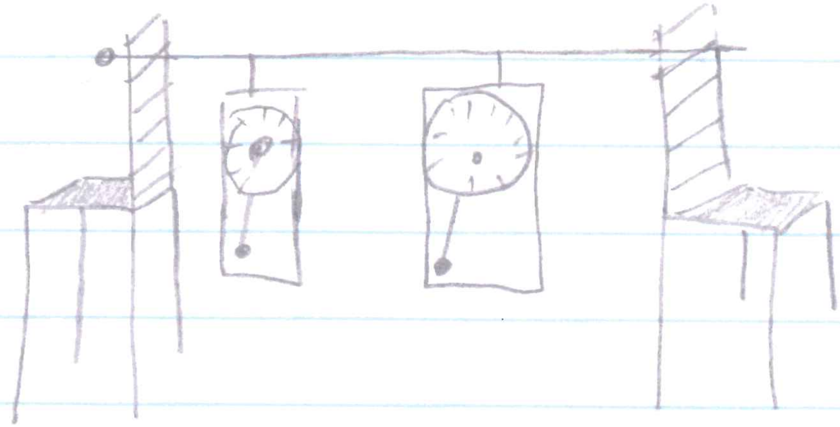
note that phase locking: (1) occurs because the nonlinear firefly oscillator can change it's period to fit the external one (Ω); (2) occurs over a specific range of parameters: $\omega - A < \Omega < \omega + A$ = "range of entrainment".

(a linear oscillator cannot change its period!!)

14

Another example of mode locking (= phase locking = nonlinear resonance)

1. 17th century Dutch physicist Huyghens: synchronisation between clocks:



2. Glacial cycles & Milankovitch forcing
... El NiNo ...

Another view at phase locking: The circle map

[Schuster, chapter 6.2]

$$\theta_{n+1} = f(\theta_n) = \theta_n + \Omega + \frac{K}{2\pi}\sin(2\pi\theta_n) \quad \boxed{\text{mod } 1}!$$

$\Rightarrow$ a simple model for a periodically forced oscillator:

Ω = forcing

$\frac{K}{2\pi}\sin(2\pi\theta_n)$ "=" gravity, nonlinear

$$\left[mr\ddot{\theta} = -b\dot{\theta} + mg\sin\theta + c\sin(\Omega\cdot t) \right]$$

when $K=0$, θ_n rotates uniformly;

* if Ω is rational, θ_n is periodic.

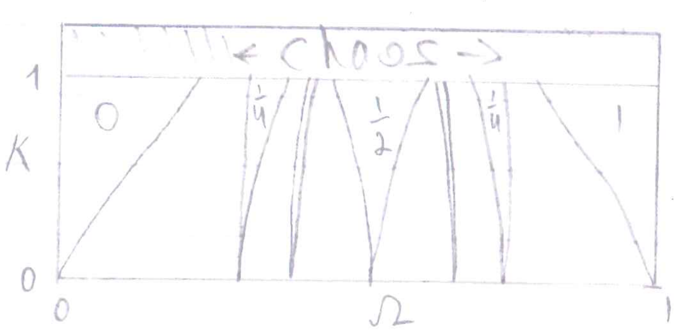
  e.g. $\Omega = \frac{1}{2} \Rightarrow$ period 2 $\theta_n = 0, \frac{1}{2}, 0, \frac{1}{2} \ldots$

  $\Omega = \frac{3}{4} \Rightarrow \theta_n = 0, \frac{3}{4}, 1\frac{1}{2}(=\frac{1}{2}), 1\frac{1}{4}(=\frac{1}{4}), 1(=0).$

  $\Rightarrow$ period 4, 3 rotations per period.

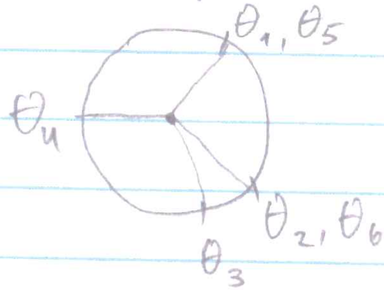
* if Ω is irrational, θ_n never repeats, & eventually covers all points on the circle.
  $\Rightarrow$ "quasi-periodic"

when $K>0$ (nonlinear regime), can have periodic solutions even for irrational Ω



Arnolds tongues

If we choose K, Ω values within the $\frac{1}{4}$ tongue, for example, then any initial conditions θ_0 will eventually converge to a specific 4-period solution, such as



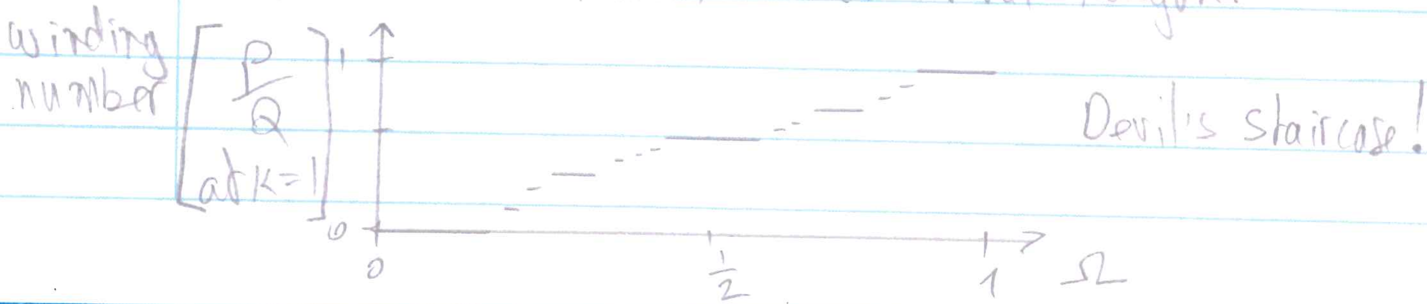
$\Rightarrow$ unequal jumps
(due to nonlinearity), but still 4-periodic.

* There are ∞ Arnold's tongues, each corresponding to a nonlinear resonance between the forcing Ω & the nonlinear oscillator, each for a different rational P/Q [$\equiv$ "Winding #" $\equiv \frac{\theta^n - \theta_0}{n}$ $(n\to\infty)$]

* [describe nonlinear resonance for an actual pendulum forced by periodic forcing, vs a linear resonance.]

Devil's staircase

At $K=1$, the tongues of periodic solutions cover the entire $\Omega=[0,1]$ interval, besides a fractal set of dimension <1 $\Rightarrow$ zero total length.



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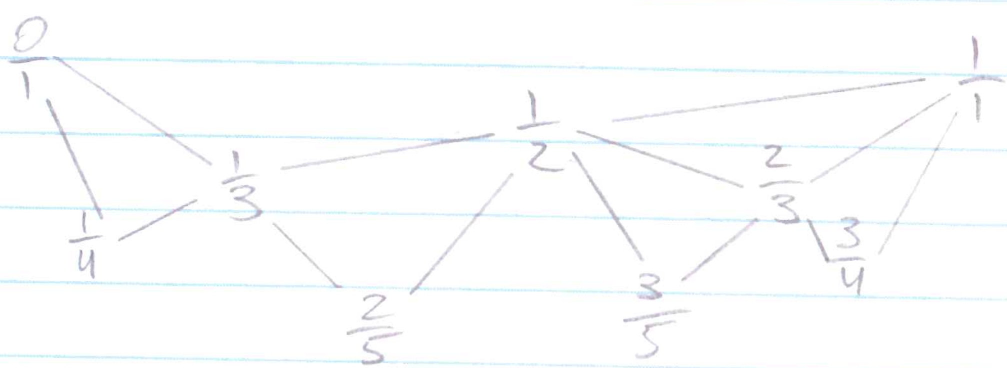
Farey tree for the Devil's staircase

* note that width of steps is smaller if the denominator in the winding # of each step is larger.
* also, given two steps $\frac{p}{q}$, $\frac{p'}{q'}$ the largest step in between is $(p+p')/(q+q')$. This is because this is the rational # with the largest denominator between p/q, p'/q'.

[examples: $0/1 < 1/2 < 1/1$
$1/2 < 2/3 < 1/1$
$1/2 < 3/5 < 2/3$]

$\Rightarrow$ can order the steps using a Farey tree, which orders rationals $\frac{p}{q}$ by denominator q:



18

(1) $f(x)$ and $g(x)$ are continuously differentiable for all x;
(2) $g(-x) = -g(x)$ for all x (i.e., $g(x)$ is an *odd* function);
(3) $g(x) > 0$ for $x > 0$;
(4) $f(-x) = f(x)$ for all x (i.e., $f(x)$ is an *even* function);
(5) The odd function $F(x) = \int_0^x f(u)\,du$ has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$.

Then the system (2) has a unique, stable limit cycle surrounding the origin in the phase plane.

This result should seem plausible. The assumptions on $g(x)$ mean that the restoring force acts like an ordinary spring, and tends to reduce any displacement, whereas the assumptions on $f(x)$ imply that the damping is negative at small $|x|$ and positive at large $|x|$. Since small oscillations are pumped up and large oscillations are damped down, it is not surprising that the system tends to settle into a self-sustained oscillation of some intermediate amplitude.

EXAMPLE 7.4.1:

Show that the van der Pol equation has a unique, stable limit cycle.

Solution: The van der Pol equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ has $f(x) = \mu(x^2 - 1)$ and $g(x) = x$, so conditions (1)–(4) of Liénard's theorem are clearly satisfied. To check condition (5), notice that

$$F(x) = \mu\left(\tfrac{1}{3}x^3 - x\right) = \tfrac{1}{3}\mu x(x^2 - 3).$$

Hence condition (5) is satisfied for $a = \sqrt{3}$. Thus the van der Pol equation has a unique, stable limit cycle. ■

There are several other classical results about the existence of periodic solutions for Liénard's equation and its relatives. See Stoker (1950), Minorsky (1962), Andronov et al. (1973), and Jordan and Smith (1987).

7.5 Relaxation Oscillations

It's time to change gears. So far in this chapter, we have focused on a qualitative question: Given a particular two-dimensional system, does it have any periodic solutions? Now we ask a quantitative question: Given that a closed orbit exists, what can we say about its shape and period? In general, such problems can't be solved exactly, but we can still obtain useful approximations if some parameter is large or small.

19

We begin by considering the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

for $\mu >> 1$. In this *strongly nonlinear* limit, we'll see that the limit cycle consists of an extremely slow buildup followed by a sudden discharge, followed by another slow buildup, and so on. Oscillations of this type are often called ***relaxation oscillations***, because the "stress" accumulated during the slow buildup is "relaxed" during the sudden discharge. Relaxation oscillations occur in many other scientific contexts, from the stick-slip oscillations of a bowed violin string to the periodic firing of nerve cells driven by a constant current (Edelstein–Keshet 1988, Murray 1989, Rinzel and Ermentrout 1989).

EXAMPLE 7.5.1:

Give a phase plane analysis of the van der Pol equation for $\mu >> 1$.

Solution: It proves convenient to introduce different phase plane variables from the usual "$\dot{x} = y$, $\dot{y} = \ldots$". To motivate the new variables, notice that

$$\ddot{x} + \mu\dot{x}(x^2 - 1) = \frac{d}{dt}\left(\dot{x} + \mu\left[\tfrac{1}{3}x^3 - x\right]\right).$$

So if we let

$$F(x) = \tfrac{1}{3}x^3 - x, \qquad w = \dot{x} + \mu F(x), \tag{1}$$

the van der Pol equation implies that

$$\dot{w} = \ddot{x} + \mu\dot{x}(x^2 - 1) = -x. \tag{2}$$

Hence the van der Pol equation is equivalent to (1), (2), which may be rewritten as

$$\begin{aligned} \dot{x} &= w - \mu F(x) \\ \dot{w} &= -x. \end{aligned} \tag{3}$$

One further change of variables is helpful. If we let

$$y = \frac{w}{\mu}$$

then (3) becomes

$$\begin{aligned} \dot{x} &= \mu\left[\, y - F(x) \,\right] \\ \dot{y} &= -\tfrac{1}{\mu}x. \end{aligned} \tag{4}$$

20

Now consider a typical trajectory in the (x, y) phase plane. The nullclines are the key to understanding the motion. We claim that all trajectories behave like that shown in Figure 7.5.1; starting from any point except the origin, the trajectory zaps horizontally onto the ***cubic nullcline*** $y = F(x)$. Then it crawls down the nullcline until it comes to the knee (point B in Figure 7.5.1), after which it zaps over to the other branch of the cubic at C. This is followed by another crawl along the cubic

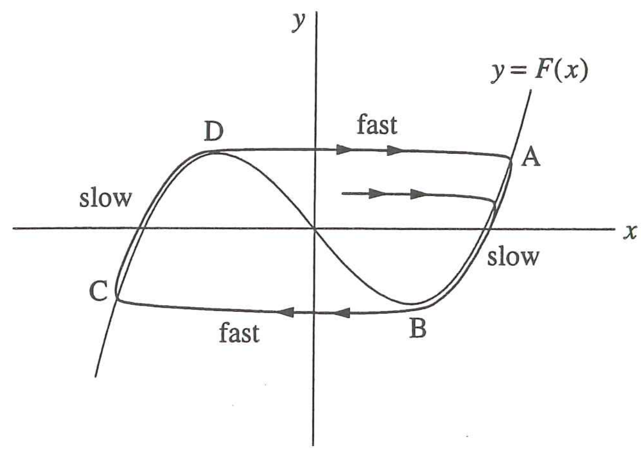


Figure 7.5.1

until the trajectory reaches the next jumping-off point at D, and the motion continues periodically after that.

To justify this picture, suppose that the initial condition is not too close to the cubic nullcline, i.e., suppose $y - F(x) \sim O(1)$. Then (4) implies $|\dot{x}| \sim O(\mu) >> 1$ whereas $|\dot{y}| \sim O(\mu^{-1}) << 1$; hence the velocity is enormous in the horizontal direction and tiny in the vertical direction, so trajectories move practically horizontally. If the initial condition is *above* the nullcline, then $y - F(x) > 0$ and therefore $\dot{x} > 0$; thus the trajectory moves sideways *toward* the nullcline. However, once the trajectory gets so close that $y - F(x) \sim O(\mu^{-2})$, then $\dot{x}$ and $\dot{y}$ become comparable, both being $O(\mu^{-1})$. What happens then? The trajectory crosses the nullcline vertically, as shown in Figure 7.5.1, and then moves slowly along the backside of the branch, with a velocity of size $O(\mu^{-1})$, until it reaches the knee and can jump sideways again. ■

This analysis shows that the limit cycle has two ***widely separated time scales***: the crawls require $\Delta t \sim O(\mu)$ and the jumps require $\Delta t \sim O(\mu^{-1})$. Both time scales are apparent in the waveform of $x(t)$ shown in Figure 7.5.2, obtained by numerical integration of the van der Pol equation for $\mu = 10$ and initial condition $(x_0, y_0) = (2, 0)$.

21

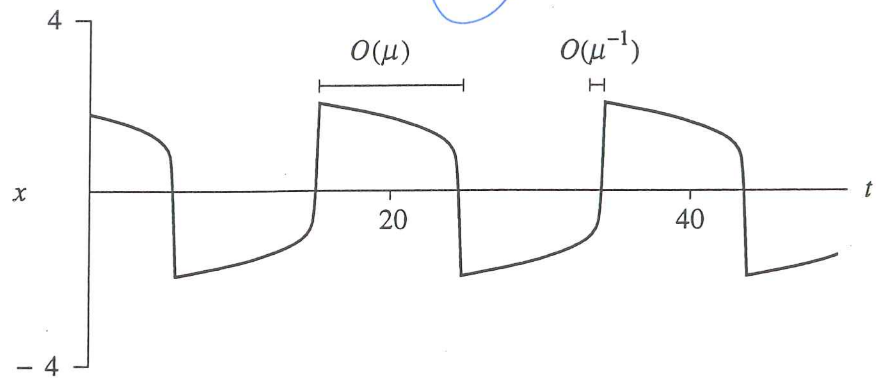


Figure 7.5.2

EXAMPLE 7.5.2:

Estimate the period of the limit cycle for the van der Pol equation for $\mu >> 1$.

Solution: The period T is essentially the time required to travel along the two ***slow branches***, since the time spent in the jumps is negligible for large μ. By symmetry, the time spent on each branch is the same. Hence $T \approx 2\int_{t_A}^{t_B} dt$. To derive an expression for dt, note that on the slow branches, $y \approx F(x)$ and thus

$$\frac{dy}{dt} \approx F'(x)\frac{dx}{dt} = (x^2-1)\frac{dx}{dt}.$$

But since $dy/dt = -x/\mu$ from (4), we find $dx/dt = -x/\mu(x^2-1)$. Therefore

$$dt \approx -\frac{\mu(x^2-1)}{x}\,dx \qquad (5)$$

on a slow branch. As you can check (Exercise 7.5.1), the positive branch begins at $x_A = 2$ and ends at $x_B = 1$. Hence

$$T \approx 2\int_2^1 \frac{-\mu}{x}(x^2-1)\,dx = 2\mu\left[\frac{x^2}{2} - \ln x\right]_1^2 = \mu\left[3 - 2\ln 2\right], \qquad (6)$$

which is $O(\mu)$ as expected. ∎

The formula (6) can be refined. With much more work, one can show that $T \approx \mu[3-2\ln 2] + 2\alpha\mu^{-1/3} + \ldots$, where $\alpha \approx 2.338$ is the smallest root of $\mathrm{Ai}(-\alpha) = 0$. Here $\mathrm{Ai}(x)$ is a special function called the Airy function. This correction term comes from an estimate of the time required to turn the corner between

22

the jumps and the crawls. See Grimshaw (1990, pp. 161–163) for a readable derivation of this wonderful formula, discovered by Mary Cartwright (1952). See also Stoker (1950) for more about relaxation oscillations.

One last remark: We have seen that a relaxation oscillation has two time scales that operate *sequentially*—a slow buildup is followed by a fast discharge. In the next section we will encounter problems where two time scales operate *concurrently*, and that makes the problems a bit more subtle.

7.6 Weakly Nonlinear Oscillators

This section deals with equations of the form

$$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0 \qquad (1)$$

where $0 \le \varepsilon << 1$ and $h(x, \dot{x})$ is an arbitrary smooth function. Such equations represent small perturbations of the linear oscillator $\ddot{x} + x = 0$ and are therefore called ***weakly nonlinear oscillators***. Two fundamental examples are the van der Pol equation

$$\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} = 0, \qquad (2)$$

(now in the limit of small nonlinearity), and the ***Duffing equation***

$$\ddot{x} + x + \varepsilon x^3 = 0. \qquad (3)$$

To illustrate the kinds of phenomena that can arise, Figure 7.6.1 shows a computer-generated solution of the van der Pol equation in the $(x, \dot{x})$ phase plane, for $\varepsilon = 0.1$ and an initial condition close to the origin. The trajectory is a slowly winding spiral; it takes many cycles for the amplitude to grow substantially. Eventually

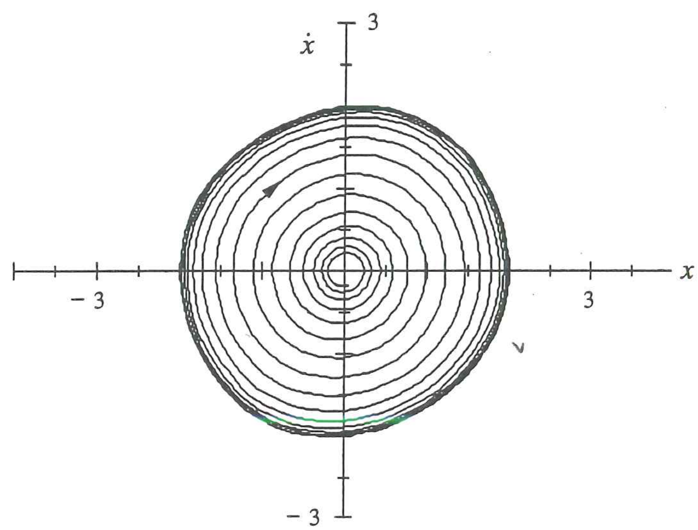


Figure 7.6.1

23

regions of the phase plane rather than just the neighborhood of a single fixed point. Hence they are called ***global bifurcations***. In this section we offer some prototypical examples of global bifurcations, and then compare them to one another and to the Hopf bifurcation. A few of their scientific applications are discussed in Sections 8.5 and 8.6 and in the exercises.

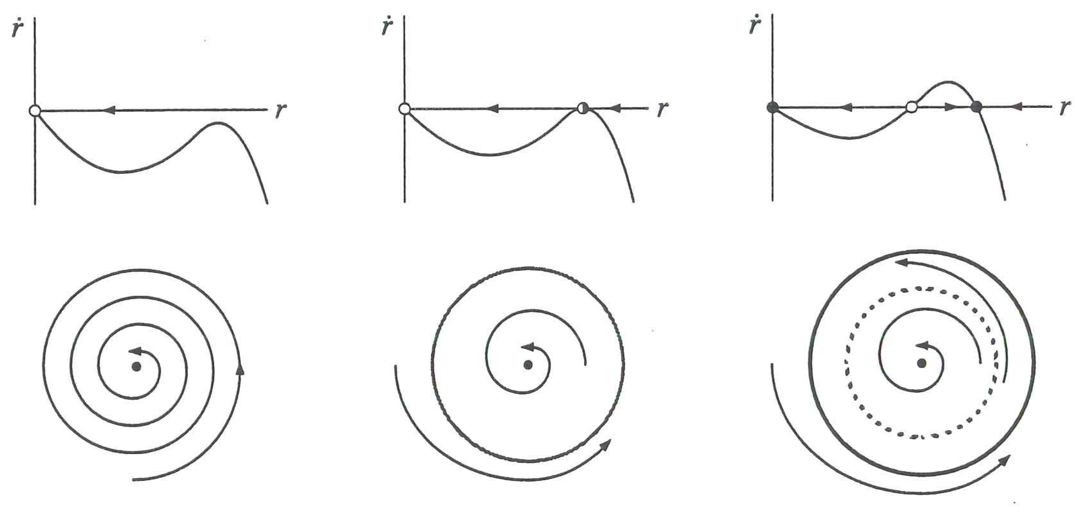
Saddle-node Bifurcation of Cycles

A bifurcation in which two limit cycles coalesce and annihilate is called a *fold* or ***saddle-node bifurcation of cycles***, by analogy with the related bifurcation of fixed points. An example occurs in the system

$$\dot{r} = \mu r + r^3 - r^5$$
$$\dot{\theta} = \omega + b r^2$$

studied in Section 8.2. There we were interested in the subcritical Hopf bifurcation at $\mu = 0$; now we concentrate on the dynamics for $\mu < 0$.

It is helpful to regard the radial equation $\dot{r} = \mu r + r^3 - r^5$ as a one-dimensional system. As you should check, this system undergoes a saddle-node bifurcation of fixed points at $\mu_c = -1/4$. Now returning to the two-dimensional system, these fixed points correspond to circular *limit cycles*. Figure 8.4.1 plots the "radial phase portraits" and the corresponding behavior in the phase plane.



$\mu < \mu_c$ $\qquad$ $\mu = \mu_c$ $\qquad$ $0 > \mu > \mu_c$

Figure 8.4.1

At μ_c a half-stable cycle is born out of the clear blue sky. As μ increases it splits into a pair of limit cycles, one stable, one unstable. Viewed in the other direction, a stable and unstable cycle collide and disappear as μ decreases through μ_c. Notice that the origin remains stable throughout; it does not participate in this bifurcation.

24

For future reference, note that at birth the cycle has $O(1)$ amplitude, in contrast to the Hopf bifurcation, where the limit cycle has small amplitude proportional to $(\mu - \mu_c)^{1/2}$.

Infinite-period Bifurcation

Consider the system

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = \mu - \sin\theta$$

where $\mu \geq 0$. This system combines two one-dimensional systems that we have studied previously in Chapters 3 and 4. In the radial direction, all trajectories (except $r^* = 0$) approach the unit circle monotonically as $t \to \infty$. In the angular direction, the motion is everywhere counterclockwise if $\mu > 1$, whereas there are two invariant rays defined by $\sin\theta = \mu$ if $\mu < 1$. Hence as μ decreases through $\mu_c = 1$, the phase portraits change as in Figure 8.4.2.

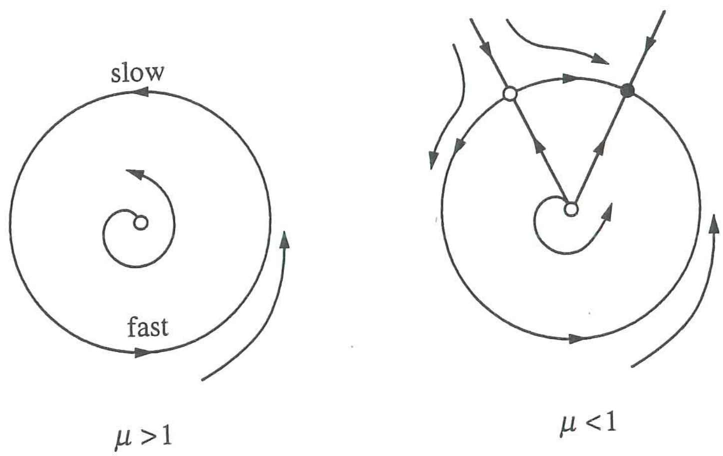


Figure 8.4.2

As μ decreases, the limit cycle $r = 1$ develops a bottleneck at $\theta = \pi/2$ that becomes increasingly severe as $\mu \to 1^+$. The oscillation period lengthens and finally becomes infinite at $\mu_c = 1$, when a fixed point appears on the circle; hence the term ***infinite-period bifurcation***. For $\mu < 1$, the fixed point splits into a saddle and a node.

As the bifurcation is approached, the amplitude of the oscillation stays $O(1)$ but the period increases like $(\mu - \mu_c)^{-1/2}$, for the reasons discussed in Section 4.3.

Homoclinic Bifurcation

In this scenario, part of a limit cycle moves closer and closer to a saddle point. At the bifurcation the cycle touches the saddle point and becomes a homoclinic or-

25

bit. This is another kind of infinite-period bifurcation; to avoid confusion, we'll call it a *saddle-loop* or ***homoclinic bifurcation***.

It is hard to find an analytically transparent example, so we resort to the computer. Consider the system

$$\dot{x} = y$$
$$\dot{y} = \mu y + x - x^2 + xy.$$

Figure 8.4.3 plots a series of phase portraits before, during, and after the bifurcation; only the important features are shown.

Numerically, the bifurcation is found to occur at $\mu_c \approx -0.8645$. For $\mu < \mu_c$, say $\mu = -0.92$, a stable limit cycle passes close to a saddle point at the origin (Figure 8.4.3a). As μ increases to μ_c, the limit cycle swells (Figure 8.4.3b) and bangs into the saddle, creating a homoclinic orbit (Figure 8.4.3c). Once $\mu > \mu_c$, the saddle connection breaks and the loop is destroyed (Figure 8.4.3d).

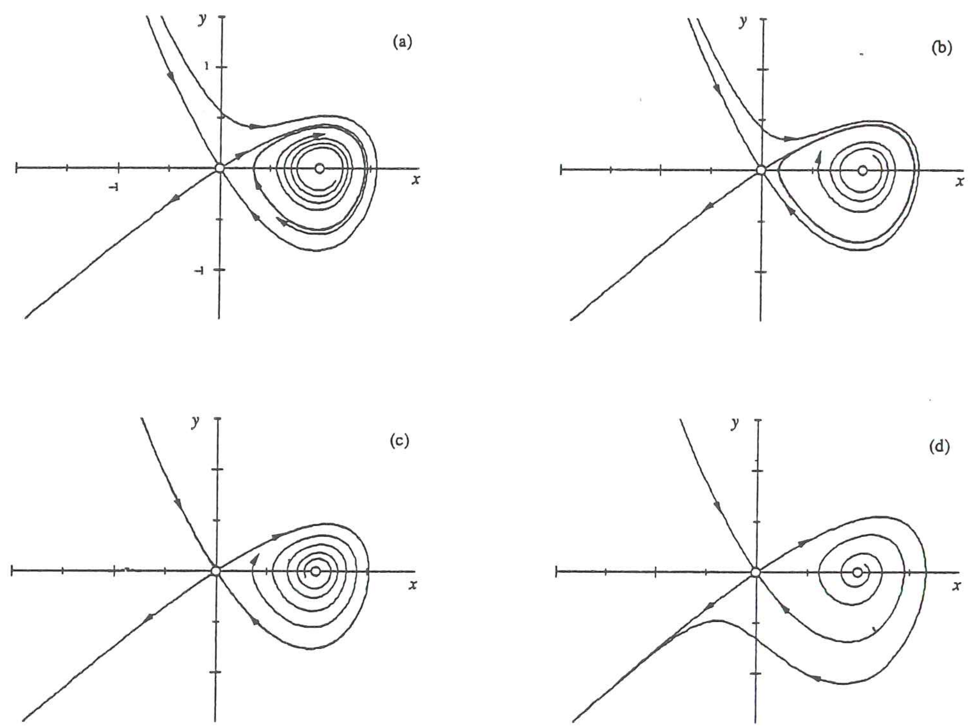


Figure 8.4.3

The key to this bifurcation is the behavior of the unstable manifold of the saddle. Look at the branch of the unstable manifold that leaves the origin to the northeast: after it loops around, it either hits the origin (Figure 8.4.3c) or veers off to one side or the other (Figures 8.4.3a, d).

26

Scaling Laws

For each of the bifurcations given here, there are characteristic *scaling laws* that govern the amplitude and period of the limit cycle as the bifurcation is approached. Let μ denote some dimensionless measure of the distance from the bifurcation, and assume that $\mu << 1$. The generic scaling laws for bifurcations of cycles in two-dimensional systems are given in Table 7.4.1.

	Amplitude of stable limit cycle	Period of cycle
Supercritical Hopf	$O(\mu^{1/2})$	$O(1)$
Saddle-node bifurcation of cycles	$O(1)$	$O(1)$
Infinite-period	$O(1)$	$O(\mu^{-1/2})$
Homoclinic	$O(1)$	$O(\ln \mu)$

Table 7.4.1

All of these laws have been explained previously, except those for the homoclinic bifurcation. The scaling of the period in that case is obtained by estimating the time required for a trajectory to pass by a saddle point (see Exercise 8.4.12 and Gaspard 1990).

Exceptions to these rules can occur, but only if there is some symmetry or other special feature that renders the problem nongeneric, as in the following example.

EXAMPLE 8.4.1:

The van der Pol oscillator $\ddot{x} + \varepsilon\dot{x}(x^2 - 1) + x = 0$ does not seem to fit anywhere in Table 7.4.1. At $\varepsilon = 0$, the eigenvalues at the origin are pure imaginary ($\lambda = \pm i$), suggesting that a Hopf bifurcation occurs at $\varepsilon = 0$. But we know from Section 7.6 that for $0 < \varepsilon << 1$, the system has a limit cycle of amplitude $r \approx 2$. Thus the cycle is born "full grown," not with size $O(\varepsilon^{1/2})$ as predicted by the scaling law. What's the explanation?

Solution: The bifurcation at $\varepsilon = 0$ is degenerate. The nonlinear term $\varepsilon\dot{x}x^2$ vanishes at precisely the same parameter value as the eigenvalues cross the imaginary axis. That's a nongeneric coincidence if there ever was one!

We can rescale x to remove this degeneracy. Write the equation as $\ddot{x} + x + \varepsilon x^2\dot{x} - \varepsilon\dot{x} = 0$. Let $u^2 = \varepsilon x^2$ to remove the ε-dependence of the nonlinear term. Then $u = \varepsilon^{1/2}x$ and the equation becomes

27

$$\ddot{u} + u + u^2\dot{u} - \varepsilon\dot{u} = 0 .$$

Now the nonlinear term is not destroyed when the eigenvalues become pure imaginary. From Section 7.6 the limit cycle solution is $x(t,\varepsilon) \approx 2\cos t$ for $0 < \varepsilon << 1$. In terms of u this becomes

$$u(t,\varepsilon) \approx \left(2\sqrt{\varepsilon}\right)\cos t .$$

Hence the amplitude grows like $\varepsilon^{1/2}$, just as expected for a Hopf bifurcation. ■

The scaling laws given here were derived by thinking about prototypical examples in *two-dimensional* systems. In higher-dimensional phase spaces, the corresponding bifurcations obey the same scaling laws, but with two caveats: (1) Many *additional* bifurcations of limit cycles become possible; thus our table is no longer exhaustive. (2) The homoclinic bifurcation becomes much more subtle to analyze. It often creates chaotic dynamics in its aftermath (Guckenheimer and Holmes 1983, Wiggins 1990).

All of this begs the question: Why should you care about these scaling laws? Suppose you're an experimental scientist and the system you're studying exhibits a stable limit cycle oscillation. Now suppose you change a control parameter and the oscillation stops. By examining the scaling of the period and amplitude near this bifurcation, you can learn something about the system's dynamics (which are usually not known precisely, if at all). In this way, possible models can be eliminated or supported. For an example in physical chemistry, see Gaspard (1990).

8.5 Hysteresis in the Driven Pendulum and Josephson Junction

This section deals with a physical problem in which both homoclinic and infinite-period bifurcations arise. The problem was introduced back in Sections 4.4 and 4.6. At that time we were studying the dynamics of a damped pendulum driven by a constant torque, or equivalently, its high-tech analog, a superconducting Josephson junction driven by a constant current. Because we weren't ready for two-dimensional systems, we reduced both problems to vector fields on the circle by looking at the heavily *overdamped limit* of negligible mass (for the pendulum) or negligible capacitance (for the Josephson junction).

Now we're ready to tackle the full two-dimensional problem. As we claimed at the end of Section 4.6, for sufficiently weak damping the pendulum and the Josephson junction can exhibit intriguing hysteresis effects, thanks to the coexistence of a stable limit cycle and a stable fixed point. In physical terms, the pendulum can settle into either a rotating solution where it whirls over the top, or a stable rest state where gravity balances the applied torque. The final state depends on the initial conditions. Our goal now is to understand how this bistability comes about.