

# Notes for climate dynamics course (EPS 231)

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# 1 Introduction

This is an evolving detailed syllabus of EPS 231, see course [web page](#), all course materials are available under the downloads [directory](#). If accessing from outside campus or via the university wireless network, you will need to connect via the [Harvard VPN](#).

Homework assignments (every 9-10 days) are 50% of final grade, and a final course project constitutes the remaining 50%. We strongly encourage you to discuss and work on homework problems and collaborate on solving them with other students (and with the teaching staff, of course), but you must write your final answers in your own words. You should ensure that any written work you submit for evaluation reflects your own work and your own understanding of the topic. There is an option to take this course as a pass/fail with approval of instructor.

## 2 Basics, energy balance, multiple climate equilibria

Downloads available [here](#).

### 2.1 Multiple equilibria, climate stability, greenhouse

1. [energy\\_balance\\_0d.pdf](#) with the graphical solutions of the steady state solution to the equation  $CT_t = (Q/4)(1 - \alpha(T)) - \epsilon\sigma T^4$  obtained using [energy\\_balance\\_0d.m](#), and then the quicktime animation of the bifurcation behavior.
2. [Some nonlinear dynamics background: saddle node bifurcation \(\(p 45, Strogatz, 1994\) or Applied Math 203 notes p 47\)](#), and then the energy balance model as two back to back saddle nodes and the resulting hysteresis as the insolation is varied;
3. Climate implications: (1) faint young sun paradox! (2) snowball ([snowball obs](#) from Ed Boyle's lecture);
4. 2-level greenhouse model ([notes](#)) and slide on actual mechanism ([pptx](#)).
5. [More 1d bifurcations](#): transcritical and pitchfork, same notes as above.

### 2.2 (Time permitting:) Small ice cap instability

1. Introduction: The Budyko and Sellers 1d models Simple diffusive energy balance models produce an abrupt disappearance of polar ice as the global climate gradually warms, and a corresponding hysteresis. The SICI eliminates polar ice caps smaller than a critical size (18 deg from pole) determined by heat diffusion and radiative damping parameters. Below this size the ice cap is incapable of determining its own climate which then becomes dominated, instead, by heat transport from surrounding regions. (Above wording from Winton (2006), teaching notes below based on North et al. (1981)).
2. Derive the 1d energy balance model for the diffusive and Budyko versions (eqns 22 and 33): Assume the ice cap extends where the temperature is less than -10C, the ice-free areas have an absorption (one minus albedo) of  $a(x) = a_f$ , and in the ice-covered areas  $a(x) = a_i$ ; approximate the latitudinal structure of the annual mean insolation as function of latitude using  $S(x) = 1 + S_2 P_2(x)$ , with  $P_2(x) = (3x^2 - 1)/3$  being the second Legendre polynomial (HW); add the transport term represented by diffusion  $(r^2 \cos \theta)^{-1} \partial / \partial \theta (D \cos \theta \partial T / \partial \theta)$ , which, using  $x = \sin \theta$ ,  $1 - x^2 = \cos^2 \theta$  and  $d/dx = (1/\cos \theta) d/d\theta$  gives the result in the final steady state 1D equation (22),

$$-\frac{d}{dx} D(1-x^2) \frac{dT(x)}{dx} + A + BT(x) = QS(x)a(x, x_0).$$

as boundary condition, use the symmetry condition that  $dT/dx = 0$  at the equator, leading to only the even Legendre polynomials.

3. (Time permitting:) Alternatively to the above, we could model the transport term a-la Budyko as  $\gamma[T(x) - T_0]$  with  $T_0$  being the temperature averaged over all latitudes, and the temperature then can be solved analytically (HW).
4. How steady state solution is calculated: eqns 22-29; then 15 and 37, the equation just before 37 and the 4 lines in the paragraph before these equations. Result is Fig. 8: plot of  $Q$  (solar intensity) as function of  $x_s$  (edge of ice cap). Analysis of results: see highlighted Fig. 8 in sources directory, unstable small ice cap (which cannot sustain its own climate against heat diffusion from mid-latitudes), unstable very-large ice cap (which is too efficient at creating its own cold global climate and grows to a snowball), and stable mid-size cap (where we are now).
5. Heuristic explanation of SICI: Compare Figs 6 and 8 in North et al. (1981), SICI appears only when diffusion is present. It is therefore due to the above mentioned mechanism: competition between diffusion and radiation. To find the scale of the small cap: it survives as long as the radiative effect dominates diffusion:  $BT \sim DT/L^2$ , implying that  $L \sim \sqrt{D/B}$ . Units:  $[D] = \text{watts}/(\text{degree K} \times \text{m}^2)$  (page 96),  $[B] = \text{m}^2/\text{sec}$  (p 93), so that  $[L]$  is non dimensional. That's fine because it is in units of sine of latitude. Size comes out around 20 degrees from pole, roughly size of present-day sea ice!
6. The important lesson(!): p 95 in North et al. (1981), left column second paragraph, apologizing for the (correct...) prediction of a snowball state.
7. This is a complex PDE (infinite number of degrees of freedom), displaying a simple bifurcation structure. In such case we are guaranteed by the central manifold and normal form theorems that it can be transformed to the normal form of a saddle node near the appropriate place in parameter space. First, transformed to center manifold and get an equation independent of stable and unstable manifolds (first page of lecture\_04\_cntr\_mnfl.d.pdf); next, transform to normal form within center manifold (lecture\_03\_bif1d2.pdf, p 89).
8. (Time permitting:) Numerically calculated hysteresis in 1D Budyko and Sellers models: figures [ebm1d-budyko.jpg](#) and [ebm1d-sellers.jpg](#) obtained using [ebm\\_1dm.m](#).
9. (Time permitting:) One noteworthy difference between Budyko and Sellers is the transient behavior, with Budyko damping all scales at the same rate, and Sellers being scale-selective. (original references are Budyko, 1969; Sellers, 1969).

## 3 ENSO

Downloads available [here](#).

### 3.1 ENSO background and equatorial waves

Sources: Woods Hole (WH) notes (Cessi et al., 2001), lectures 0, 1 [here](#).

### 3.2 Delay oscillator model

Sources: Woods Hole (WH) notes (Cessi et al., 2001), lecture 2, [here](#), plus the following: Gill's atmospheric model solution from Dijkstra (2000) technical box 7.2 p 347; recharge oscillator from Jin (1997) (section 2, possibly also section 3);

- The climatological background: easterlies, walker circulation, warm pool and cold tongue, thermocline slope ([ppt](#), and lecture 1 from WH notes).
- Dynamical basics (WH lecture 1): reduced gravity equations on an equatorial beta plane. Equatorial Rossby and Kelvin waves, thermocline slope set by balance between wind stress and pressure gradient, SST dynamics, atmospheric heating and wind response to SST from Gill's model. The coupled ocean-atmosphere feedback.
- The heuristic delayed oscillator equation from section 2.1 in WH notes. One detail to note regarding how do we transition from  $+\hat{b}h_{\text{off-eq}}(t - [\frac{1}{2}\tau_R + \tau_K])$  to  $-\bar{b}\tau_{eq}(t - [\frac{1}{2}\tau_R + \tau_K])$  and then to  $-bT(t - [\frac{1}{2}\tau_R + \tau_K])$ :  $h_{\text{off-eq}}$  depends on the Ekman pumping off the equator. In the northern hemisphere, if the wind curl is positive, the Ekman pumping is positive, upward ( $w_E = \text{curl}(\vec{\tau}/f)/\rho$ ), and the induced thermocline depth anomaly is therefore negative (a shallowing signal). The wind curl may be approximated in terms of the equatorial wind only (larger than the off-equatorial wind), consider the northern hemisphere:

$$h_{\text{off-eq}} \propto -w_{\text{off-eq}}^{\text{Ekman}} \propto -\text{curl}(\tau_{\text{off-eq}}) \approx \partial_y \tau_{\text{off-eq}}^{(x)} \approx (\tau_{\text{off-eq}}^{(x)} - \tau_{\text{eq}}^{(x)})/L \propto -\tau_{\text{eq}}^{(x)}.$$

Finally, as the east Pacific temperature is increasing, the wind anomaly in the central Pacific is westerly (positive), leading to the minus sign in front of the  $T$  term,

$$-\tau_{\text{eq}}^{(x)}(t - [\frac{1}{2}\tau_R + \tau_K])/L \propto -T(t - [\frac{1}{2}\tau_R + \tau_K]).$$

- Next, the linearized stability analysis of the Schopf-Suarez delayed oscillator from the WH notes section 2.1.1. The dispersion relation in the WH notes is  $\sigma = 1 - 3\bar{T}^2 - \alpha \exp(-\sigma\delta)$ . Its real part is  $0 = \sigma_R - (1 - 3\bar{T}^2 - \alpha \exp(-\sigma_R\delta) \cos(-\sigma_I\delta))$ , and its imaginary part is  $0 = \sigma_I - (-\alpha \exp(-\sigma_R\delta) \sin(-\sigma_I\delta))$ . The following [image](#) shows the roots for an example with two unstable roots and many stable ones. Show time series of numerical solution of this

model for values on both sides of the first bifurcation ([damped](#) and [self-sustained](#)) point, obtained using [delay\\_Schopf\\_Suarez\\_1989.m](#).

- Self-sustained vs damped: nonlinear damping term and, more importantly, the proximity of ENSO to the first bifurcation point beyond steady state, [discuss Hopf bifurcation from non linear dynamics notes pp 26-28 here](#) or Strogatz (1994).
- (Time permitting:) A more quantitative derivation of delay oscillator, starting from the shallow water equations and using Jin's two-strip approximation (WH notes, section 2.2).

### 3.3 ENSO's irregularity

#### 3.3.1 Chaos

- ENSO phase locking to seasonal cycle: flows on a circle, synchronization/ phase locking, Huygens clocks, firefly and flashlight example from Strogatz. Connection to ENSO and the seasonal cycle.
- ENSO irregularity as chaos driven by the seasonal cycle: circle map and quasi-periodicity route to chaos:

$$\theta_{n+1} = f(\theta_n) = \theta_n + \Omega - \frac{K}{2\pi} \sin 2\pi\theta_n, \quad \theta_n = \text{mod}(1)$$

$K = 0$  and quasi-periodicity,  $0 < K < 1$  and phase locking, Arnold tongues;  $K = 1$  and the devil's staircase.  $K > 1$ , overlapping of resonances and chaos. Transition from circle map to pendulum and to delayed oscillator driven by seasonal cycle. (Time permitting:) winding number:  $\lim_{n \rightarrow \infty} (f^n(\theta_0) - \theta_0)$ , not taking  $\theta_n$  as mod(1) for this calculation, Farey tree

- (Time permitting:) Some generalities on identifying quasi-periodicity route to chaos in a complex system, including delay coordinate phase space reconstruction.
- References for phase locking: Strogatz (1994), for quasi-periodicity route to chaos: Schuster (1989). For both: course notes for applied math 203, pages 71,75,77-83 in [lecture\\_03\\_bif1d2\\_eli.pdf](#). delay coordinate phase space reconstruction in [lecture\\_07\\_bif2d3\\_eli.pdf](#).
- (Time permitting:) Slides on transition to chaos in CZ model.

#### 3.3.2 Noise

- Non normal amplification (similar to WH notes, but maximizing  $\Psi(\tau)^T \Psi(\tau)$  instead of  $|d\Psi(t=0)/dt|^2$  as described in WH notes.): transient growth; geometric view using a  $2 \times 2$  example of  $d\mathbf{x}/dt = A\mathbf{x}$  where  $\mathbf{x} = (x,y)$ , with eigenvectors on unit circle, derivation of

optimal i.c that maximize  $\mathbf{x}(\tau)^T \mathbf{x}(\tau)$  as first eigenvector of  $B^T B$  where  $B$  is the propagator  $B = \exp(A\tau)$ ; corresponding eigenvalue is the amplification factor from the initial conditions to the amplified state.

- WWBs in observations: seem stochastic, seen to precede each El Nino event, affect Pacific by forcing of equatorial Kelvin waves. Show Hovmoller diagram with WWBs and SST from Yu et al. (2003), in jpg file; wind stress sequence showing WWB evolution from Vecchi and Harrison (1997) (Figs on pages 41, 42, 43 at the end of this report); effects of wind bursts on SST and thermocline depth (heat content) from Mcphaden and Yu (1999) Figs 1,2,3 (last one is model results); ocean-only model response to a strong WWB: Zhang and Rothstein (1998), Figs. 4, 5, showing the response to a wind burst after 10 days and after several months;
- Stochastic optimals: mathematical details are covered below in the context of MOC/THC variability. Here, introduce the concept without derivation, and then discuss: Do WWBs look like stochastic optimals? Show stochastic optimals in different models and discuss their model dependence, using Figs. 11, 12 and 17 from Moore and Kleeman (2001). Then: Are WWBs actually stochastic, or are their statistics a strong function of the SST (Fig. 2, Tziperman and Yu, 2007), making the stochastic element less relevant?

### 3.4 Teleconnections

1. Motivation for ENSO teleconnection: show pdf copy of impacts page from El Nino theme page.
2. Further motivation: results of barotropic model runs from Hoskins and Karoly (1981) Figures 3,4,6,8,9, showing global propagation of waves due to tropical disturbances.
3. General ray tracing theory based on notes-ray-tracing.pdf. Note that this is a non-rigorous derivation, not using multiple-scale analysis.
4. Derivation of dispersion relation Rossby wave in presence of a mean zonal flow. In Cartesian coordinates, start from potential vorticity conservation,  $(\partial_t + u\partial_x + v\partial_y)(v_x - u_y + \beta y) = 0$ ; let  $u = U(y) + u'$ ,  $v = v'$ , and linearize, to find  $(\partial_t + U\partial_x)(v'_x - u'_y) + (\beta - U_{yy})v' = 0$ ; introduce stream function  $v' = \psi_x$ ,  $u' = -\psi_y$  and effective beta  $\beta_{\text{eff}} = \beta - U_{yy}$  to find  $(\partial_t + U\partial_x)\nabla^2\psi + \beta_{\text{eff}}\psi_x = 0$ ; then mention HK1981 paper which uses the same equation, but in spherical coordinates using Mercator projection: see beginning of subsection 5b, equations 5.1-5.16. Because the mean flow and therefore the dispersion relation are  $x$  and  $t$ -independent, we have  $k = \text{constant}$  and  $\omega = \text{constant}$  along a ray, while  $d_g l/dt \equiv (\partial_t + \mathbf{c}_g \cdot \nabla)l = -\Omega_y$  and  $d\mathbf{x}/dt = \mathbf{c}_g$ .
5. Qualitative discussion based on dispersion relation, from Hoskins and Karoly (1981) after solution 5.23. Start from dispersion relation for stationary waves,  $0 = \omega = \bar{u}_M k - \frac{\beta_M k}{k^2 + l^2}$ , and define  $K_s = (\beta_M/u_M)^{1/2} = k^2 + l^2$ . Based on these only, discuss trapping by jet: if

$k > K_s$ , then  $l$  must be imaginary. Therefore, as the wave propagates northward from the low latitude, with a constant  $k$  while  $K_s$  gets smaller due to changes in mean flow and effective beta (Fig. 13a,b), this implies evanescent behavior in latitude past a critical latitude, and trapping of the ray at the critical latitude.

6. Next, see Jeff Shaman's description for why the ray is reflected rather than being trapped at the critical latitude due to prognostic  $dl/dt$  equation allowing  $dl/dt < 0$  when  $l = 0$ , leading to the ray turning back south.
7. (Time permitting:) To get some idea of the amplitude of the propagating waves, assume the meridional wavenumber varies slowly in latitude,  $l = l(\epsilon y)$ , corresponding to a medium that varies slowly, on a scale longer than the wavelength. Use the WKB solution (Bender and Orszag (1978) section 10.1): start with equation 5.18. Substituting into 5.9 this leads to  $d^2P/dy^2 + l^2(\epsilon y)P = 0$  for  $l^2(\epsilon y)$  defined in 5.20; to transform to standard WKB form, define  $Y = \epsilon y$  so that  $\epsilon^2 d^2P/dY^2 + l^2(Y)P = 0$ ; try a WKB solution corresponding to a wave-like exponential with a rapidly varying phase plus a slower correction  $P = \exp(S_0(Y)/\delta + S_1(Y))$  to find

$$\epsilon^2[(S'_0/\delta + S'_1)^2 + (S''_0/\delta + S''_1)]P + l^2P = 0;$$

let  $\delta = \epsilon$  and then  $O(1)$  equation is  $S'_0{}^2 + l^2(Y) = 0$  so that  $S_0 = i \int l(Y) dY$  (if  $l^2$  is nearly constant, this simply reduces to the usual wave solution  $e^{ily}$ ). Next, consider  $O(\epsilon)$  equation which, after using the  $O(1)$  equation, is  $2S'_0S'_1 + S''_0 = 0$  and the solution is  $S'_1 = -S''_0/(2S'_0) = -(dl/dY)/(2l) = -d/dY(\ln l^{1/2})$  so that  $S_1 = \ln l^{-1/2}$  which means that the wave amplitude is  $l^{-1/2}$ . This gives the solution in Hoskins and Karoly (1981) equation (5.21, 5.23), see further discussion there.

8. Show rays for constant angular momentum flow,  $\bar{u}_M \equiv U/\cos\phi = \bar{\omega}a$ ,  $\beta_M = (2\cos^2\phi)(\Omega + \bar{\omega})/a$  (section 5c, page 1192, Fig. 12) and then the one using realistic zonal flows (Figs. 13, 14, 15, etc);
9. Finally, mention that later works showed that stationary linear barotropic Rossby waves excite nonlinear eddy effects which may eventually dominate the teleconnection effects.
10. For baroclinic atmospheric waves,  $0 = \omega = \Omega(k, l, y) = \bar{u}_M k - \frac{\beta_M k}{k^2 + l^2 + L_R^{-2}}$ , so that  $K_s = (\beta_M/u_M)^{1/2} - L_R^{-2} = k^2 + l^2$ , and they are more easily trapped, and are going to be trapped at the equator with a scale of the Rossby radius of deformation which is some 1000km or so (see discussion on page 1195 left column).

### 3.5 ENSO diversity

See downloads folder [here](#).



## 4 Thermohaline circulation

Downloads available [here](#).

### 4.1 Phenomenology, Stommel box model

- Background, schematics of THC, animations of CFCs in ocean, sections and profiles of T, S from [here](#), meridional mass and heat transport, climate relevance; RAPID measurements (Cunningham et al., 2007); anticipated response during global warming; MOC vs THC.
- Stommel model: mixed boundary conditions, Stommel-Taylor model notes; Qualitative discussion on proximity of present day THC to a stability threshold (Tziperman, 1997; Toggweiler et al., 1996). And again the surprising ability of simple models to predict/ explain GCM results (Fig. 2 from Rahmstorf, 1995). See also (Dijkstra, 2000, section 3.1.1, 3.1.2, 3.1.3).

### 4.2 Scaling and energetics

1. (Time permitting:) Scaling for the amplitude and depth of the THC from Vallis (2006) chapter 15, section 15.1, showing that the THC amplitude is a function of the vertical mixing which in turn is due to turbulence. Mention only briefly the issue of the “no turbulence theorem” and importance of mechanical forcing (details in the next section, not to be covered explicitly).
2. (Time permitting:) Energetics, Sandstrom theorem stating that “heating must occur, on average, at a lower level than the cooling, in order that a steady circulation may be maintained against the regarding effects of friction” (eqn 15.23 Vallis, 2006). The “no turbulence theorem” in the absence of mechanical forcing by wind and tides (eqn 15.27) without mechanical mixing, and (15.30) with; hence the importance of mechanical energy/ mechanical forcing for the maintenance of turbulence and of the THC. Sections 15.2, 15.3 (much of this material is originally from Paparella and Young, 2002);
3. (Time permitting:) Tidal energy as a source for mixing energy (Munk and Wunsch, 1998, Figures 4,5).

### 4.3 Convective oscillations

- Advective feedback and convective feedback from sections 6.2.1, 6.2.2 in Dijkstra (2000), both show why we expect temperature and salinity to play independent roles.
- Flip-flop oscillations (Welander1982\_flip\_flop.m) and loop oscillations (Dijkstra, 2000, section 6.2.3);

- (Time permitting:) Multiple convective equilibria from Lenderink and Haarsma (1994), hysteresis from Fig. 8 in this paper, and “potentially convective” regions in their GCM from Fig. 11. When put in the regime with no steady states, this model shows flip-flop oscillations like Welander’s, without the need to prescribe an artificial convection threshold.
- [Analysis of relaxation oscillations following Strogatz \(1994\) example 7.5.1 pages 212-213, or nonlinear dynamics course notes: slow phase and fast phases.](#)
- (Note:) Winton’s deep decoupling oscillations are covered below as part of DO events. They are similar to the Welander’s flip-flop oscillations, except that the convection also affects the overturning circulation leading to strong MOC variability/ relaxation oscillations.

#### 4.4 Stochastically driven MOC variability

A review of classes of THC oscillations: small amplitude/ large amplitude; linear and stochastically forced/ nonlinear self-sustained; loop oscillations due to advection around the THC path, or periodic switches between convective and non convective states; relaxation oscillations; noise induced switches between steady state, stochastic resonance;

Details of noise-driven THC/MOC variability:

1. Linear Loop-oscillations due to advection around the circulation path:
  - (a) Linearized stability analysis (write equations, explain linearization, writing of linearized equations in matrix form) and bifurcations (section 6.2.4 in Dijkstra (2000) or Figs 3, 4 from Tziperman et al. (1994)); Stability regimes in a 4-box model: stable, stable oscillatory, [Hopf bifurcation], unstable oscillatory, unstable; Note changes from 2-box Stommel model: oscillatory behavior and change to the point of instability on the bifurcation diagram; Stochastic forcing can excite this damped oscillatory variability.
  - (b) The GCM study of Delworth et al. (1993) (DMS), Figs. 4, 5, 6, 8; this paper also demonstrates the link between the variability of meridional density gradients and of the THC; Note the proposed role of changes to the gyre circulation in this paper, mention related mechanisms based on ocean mid-latitude Rossby wave propagation;
  - (c) A box model fit to the DMS GCM, showing that the horizontal gyre variability may not be critical and that the variability is due to the excitation of a damped oscillatory mode (Fig. 6, Griffies and Tziperman, 1995);
  - (d) Useful and interesting analysis methods in DMS: composites (Figs. 6,7), and regression analysis between scalar indices (Figs. 8,9) and between scalar indices and fields (Figs. 10, 11, 12).
2. A complementary view of the above stochastic excitation of damped THC oscillatory mode: first, Hasselmann’s model driven by white noise and leading to a red spectrum response

(to derive the spectrum of the response, Fourier transform the first equation, and multiply transformed equation and its complex conjugate).

$$\begin{aligned} \dot{x} + \gamma x &= \xi(t) \\ P(\omega) = |\hat{x}|^2 &= \xi_0^2 / (\omega^2 + \gamma^2). \end{aligned}$$

Compare this to a damped oscillatory mode excited by noise that results in a spectral peak, using the following derivation of the spectral response,

$$\begin{aligned} \ddot{x} + \gamma \dot{x} + \Omega^2 x &= \xi(t) \\ -\omega^2 \hat{x} - i\gamma\omega \hat{x} + \Omega^2 \hat{x} &= \xi_0 \\ \hat{x} &= \xi_0 / (\Omega^2 - \omega^2 - i\gamma\omega) = \xi_0 (\Omega^2 - \omega^2 + i\gamma\omega) / ((\Omega^2 - \omega^2)^2 + \gamma^2 \omega^2) \\ \hat{x}^* &= \xi_0 (\Omega^2 - \omega^2 - i\gamma\omega) / ((\Omega^2 - \omega^2)^2 + \gamma^2 \omega^2) \\ P(\omega) = |\hat{x}|^2 = \hat{x} \hat{x}^* &= \frac{\xi_0^2}{(\Omega^2 - \omega^2)^2 + \gamma^2 \omega^2}. \end{aligned}$$

3. Stochastic variability due to noise induced transitions between steady states. First, the GCM study showing jumping between three very different MOC equilibria under sufficiently strong stochastic forcing: Figs. 2, 4 from Weaver and Hughes (1994). Then Fig. 5 from Cessi (1994) showing a careful analysis of the same type of variability in the Stommel box model.
4. (Time permitting:) details of Cessi analysis: Stommel 2 box model from section 2 with model derivation and in particular getting to eqn 2.9 with temperature fixed and salinity difference satisfying an equation of a particle on a double potential surface; section 3 with deterministic perturbation;
5. Stochastic resonance: periodic FW forcing plus noise. Matlab code `Stommel_stochastic_resonance.m`, and jpeg figures with results: `Stochastic_Resonance_a, b, c.jpg`;
6. (Time permitting:) THC oscillations due to ‘‘Thermal’’ Rossby waves analyzed by Te Raa and Dijkstra (2002), using equations 8-10 and Figure 7 of Zanna et al. (2011).
7. Stochastic forcing, non normal THC dynamics, transient amplification; first the basic derivation: Stochastic optimals: the derivation from Tziperman and Ioannou (2002): consider a stochastically forced linear system:

$$\dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{f}(t)$$

solution is

$$\mathbf{P}(\tau) = e^{\mathbf{A}\tau} \mathbf{P}(0) + \int_0^\tau ds e^{\mathbf{A}(\tau-s)} \mathbf{f}(s) = \mathbf{B}(\tau, 0) \mathbf{P}(0) + \int_0^\tau ds \mathbf{B}(\tau, s) \mathbf{f}(s)$$

where the first term is a response to the initial conditions which decays in time and can therefore be ignored, and the second is the response to the stochastic forcing. Assuming for simplicity a zero-mean state variable  $\mathbf{P}(t)$ , the variance is given by the following expression (summation convention assumed),

$$\begin{aligned} \text{var}(\|\mathbf{P}\|) &= \langle P_i(\tau)P_i(\tau) \rangle \\ &= \left\langle \int_0^\tau ds \int_0^\tau dt B_{il}(\tau, s) f_l(s) B_{in}(\tau, t) f_n(t) \right\rangle \\ &= \int_0^\tau ds \int_0^\tau dt B_{il}(\tau, s) B_{in}(\tau, t) \langle f_l(s) f_n(t) \rangle \end{aligned}$$

Specifying the noise statistics as separable in space and time, letting the  $i$ th component of the noise be, say,  $f_i v(t)$ , and with  $C_{ln} = f_l f_n$  being the noise spatial correlation matrix and  $D(t-s) = \langle v(t)v(s) \rangle$  the temporal correlation function (delta function for white noise),

$$\langle f_l(s) f_n(t) \rangle = C_{ln} D(t-s)$$

we have

$$\begin{aligned} \text{var}(\|P\|) &= \int_0^\tau ds \int_0^\tau dt B_{il}(\tau, s) B_{in}(\tau, t) C_{ln} D(t-s) \\ &= \text{Tr} \left( \int_0^\tau ds \int_0^\tau dt [B^T(\tau, s) B(\tau, t) D(t-s)] C \right) \\ &\equiv \text{Tr}(ZC) \end{aligned}$$

This implies that the most efficient way to excite the variance is to make the noise spatial structure be the first eigenvector of  $Z$ . To show this, show that eigenvectors of  $Z$  maximize  $J = \text{Tr}(CZ) = Z_{ij} C_{ji}$ ; given the above expression for  $C_{ij} = f_i f_j$  and we need to maximize  $Z_{ij} f_i f_j + \lambda(1 - f_k f_k)$ ; differentiating with respect to  $f_n$  we get that  $\mathbf{f} = \{f_n\}$  is an eigenvector of the matrix  $Z$ . The vector that maximizes the variance is, as usual, the one corresponding to the largest eigenvalue of  $Z$ .

Then more specifically to THC/MOC: The 3-box model of Tziperman and Ioannou (2002), or the spatially resolved 2d model of Zanna and Tziperman (2005)). Figures for the amplification and mechanism, taken from a talk on this subject (file nonormal\_THC.pdf).

8. (Time permitting:) A more general issue that comes up in this application of transient amplification is the treatment of singular norm kernel (appendix) and infinite amplification; show and explain the first mechanism of amplification (Figure 2); note how limited the amplification may actually be in this mechanism.
9. (Time permitting:) Use of eigenvectors for finding the instability mechanism: linearize, solve eigenvalue problem, substitute spatial structure of eigenvalue into equations and see which equations provide the positive/ negative feedbacks; results for THC problem (e.g.,

Tziperman et al., 1994, section 3): destabilizing role of  $v'\nabla\bar{S}$  and stabilizing role of  $\bar{v}\nabla S'$ ; difference in stability mechanism in upper ocean ( $v'\nabla\bar{S}$ ) vs that of the deep ocean (where  $\nabla\bar{S} = 0$  and  $\bar{v}\nabla S'$  is dominant); for temperature, also ( $v'\nabla\bar{T}$  is more important, but from the eigenvectors one can see that  $v'$  is dominated by salinity effects; GCM verification and the distance of present-day THC from stability threshold: Figs 4,5,6 from Tziperman et al. (1994); Fig 3 from Toggweiler et al. (1996); Figs 1, 2, 3 from Tziperman (1997).

#### 4.5 (Time permitting:) More on stochastic variability

- As a preparation for the rest of this class: the derivation of diffusion equation for Brownian motion following Einstein's derivation from Gardiner (1983) section 1.2.1; next, justify the drift term heuristically; then, derivation Fokker-Planck equation from Rodean (1996), chapter 5; Note that equation 5.17 has a typo, where the LHS should be  $\frac{\partial}{\partial t} T_{\tau}(y_3|y_1)$ ; Then, first passage time for homogeneous processes from Gardiner (1983) section 5.2.7 equations 5.2.139-5.2.150; 5.2.153-5.2.158; then the one absorbing boundary (section b) and explain the relation of this to the escape over the potential barrier, where the potential barrier is actually an absorbing boundary, with equations 5.2.162-5.2.165; Note that 5.2.165 from Gardiner (1983) is identical to equation 4.7 from Cessi (1994); Next, random telegraph processes are explained in Gardiner (1983) section 3.8.5, including the correlation function for such a process; Cessi (1994) takes the Fourier transform of these correlation functions to obtain the spectrum in the limit of large jumps, for which the double well potential problem is similar to the random telegraph problem.

Next, back to Cessi (1994) section 4: equation 4.4 (Fokker-Planck), 4.6 and Fig. 6 (the stationary solution for the pdf); then the expressions for the mean escape time (4.7) and the rest of the equations all the way to end of section 4, including the random telegraph process and the steady probabilities for this process;

Finally, from section 5 of Cessi (1994) with the solutions for the spectrum in the regime of small noise (linearized dynamics) and larger noise (random telegraph); For the solution in the small noise regime (equation 5.3), let  $y' = y - y_a$  and then Fourier transform the equation to get  $-i\omega\hat{y}' = -V_{yy}\hat{y} + \hat{p}'$  where hat stands for Fourier transform; then write the complex conjugate of this equation, multiply them together using the fact that the spectrum is  $\mathcal{S}_a(\omega) = \hat{y}'\hat{y}'^\dagger$  to get equation 5.5; Show the fit to the numerical spectrum of the stochastically driven Stommel model, Figure 7;

- Zonally averaged models and closures to 2d models (Dijkstra, 2000, section 6.6.2, pages 282-286, including technical box 6.3); Atmospheric feedbacks (Marotzke, 1996)?

## 5 Dansgaard-Oeschger and Heinrich events

Downloads [here](#).

Item numbering in the following outline correspond to file numbering in downloads directory.

- 1 **Introduction, observed record of DO events:** IRD, Greenland warming events.
- 2 **Winton model:** Convection, air-sea and slow diffusion: Relaxation oscillations/ Thermohaline flushes/ “deep decoupling” oscillations (Winton, 1993, section IV).
- 3 **THC flushes and DO events:** DO explained by large amplitude THC changes (Ganopolski and Rahmstorf, 2001); from this paper, show figs 1,2,3,5: hysteresis diagrams for modern and glacial climates demonstrating the ease of making a transition between the two THC states in glacial climate; time series of THC during DO events; these oscillations are basically the same as Winton’s deep decoupling oscillations and flushes (pp 427-428 including Fig. 7 in Winton (1993), see also under THC variability).
- 4 **Alternatively, sea ice as an amplifier of small amplitude THC variability:** Preliminaries: sea ice albedo and insulating feedbacks; volume vs area in present-day climate (i.e., typical sea ice thickness in Arctic and Southern Ocean); simple model equation for sea ice volume (eqn numbers from Sayag et al., 2004): sea ice melting and formation (18), short wave induced melting (19 and 3rd term on rhs in 20), sea ice volume equation (20); climate feedbacks: insulating feedback (3), albedo feedback (19).  
  
Fig. 1 from Li et al. (2005) AGCM experiments (again Heinrich\_slides.pdf). Again sea ice as an amplifier of *small* THC variability (Kaspi et al., 2004); Possible variants of the sea ice amplification idea: Stochastic excitation of THC+sea ice leads to DO-like variability (Fig. 5 in Timmermann et al., 2003); similarly, self-sustained DO events with sea ice amplification in Loving and Vallis (2005), including figures on pages 16, 22, 24 in pdf.
- 5 **Precise clock behind DO events? Stochastic resonance?** First, Figs. 1,2 from Rahmstorf (2003). (Time permitting: clock error, triggering error and dating error); is it significant, or does the fact that we are free to look for a periodicity for which some “clock” might fit the time series makes it more likely for the time series to seem as if it is driven by a precise clock? Next, stochastic resonance: (Alley et al., 2001): consider a histogram of waiting time between DO events (Fig. 2) and find that these are multiples of 1470, suggesting stochastic resonance as a possible explanation. The bad news: no clock, (Ditlevsen et al., 2007), see their Fig. 1 and read their very short conclusions section.
- 6 **DO teleconnections:** Observations: Wang et al. (2001) show a strong correlation of Hulu cave in China with Greenland ice cores (Fig. 1). Similarly, Denton and Hendy (1994) show a correlation of Younger Dryas and glaciers in New Zealand. What is the mechanism? (1) Atmospheric teleconnections via atmospheric Rossby wave teleconnections. (2) MOC/THC teleconnections: weaker NADW import to Southern Ocean due weakening of MOC (Manabe and Stouffer, 1995); MOC/THC seesaw (Broecker (1998), and [this](#)). (3) teleconnections via

ocean wave motions e.g., Johnson and Marshall (2004) Fig. 4 and discussion around eqns 1,2,10): essentially equatorial and coastal Kelvin waves propagate anomalies due to changes in convection rate very fast from the north-west Atlantic and spread the information along eastern boundaries, and then slower Rossby waves transmit the information to the ocean interior on a decadal time scale. This mechanism also allows for inter-basin exchanges.

## 7 Heinrich events:

- a. Observations: IRD layers, cold climate during events, precursor events, synchronous collapses.
- b. MacAyeal's binge-purge mechanism: MacAyeal (1993)a's argument that external forcing is not likely to play a role (section 2, eqns 1-5, p 777);
- c. Difficulties with this argument: Moulins and collapse of ice shelf due to melt water accumulation in cracks leading to elimination of buttressing and acceleration of ice streams.
- d. Heuristic argument for the time scale (section 5, p 782, eqns 19-25, note that lhs of eqn 23, the b.c at the ground, should be  $\hat{\theta}_y(0, t)$ ).
- e. Equations for the more detailed model of MacAyeal (1993)b (from slide 22 of Heinrich\_slides.pdf or Kaspi et al., 2004), and numerical solution for temperature and ice height (slides), explaining all temperature maxima and minima of temperature during the cycle.
- f. An alternative: marine ice sheet instability, MISI. See slides, plus the following derivation of the grounding line flux formula from appendix B of Hindmarsh (2012). Let  $u$  be the along-stream velocity, in the  $x$  direction. The general rate of strain tensor is,

$$\dot{\epsilon}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

while Glenn's law relates it to the stress in a simple experiment as,

$$\dot{\epsilon}_{xz} = A(T) \tau_{xz}^n$$

this is generalized to a general relation between the two tensors,

$$\tau_{ij} = A(T)^{-\frac{1}{n}} E^{\frac{1}{n}-1} \dot{\epsilon}_{ij}$$

where the second invariant of the rate of strain tensor is used

$$E = \left( \frac{1}{2} \sum_{i,j=1}^3 \dot{\epsilon}_{ij} \dot{\epsilon}_{ij} \right)^{1/2}$$

assuming only variations along-stream, can neglect other derivatives,

$$\tau_{xx} \approx A(T)^{-\frac{1}{n}} \dot{\epsilon}_{xx}^{\frac{1}{n}-1} \dot{\epsilon}_{ij} = A(T)^{-\frac{1}{n}} \dot{\epsilon}_{xx}^{\frac{1}{n}}$$

Using as the vertically integrated momentum equation a balance between bottom friction and driving stress,

$$Hu_t = 0 = -\rho_i g H H_x - C|u|^{\frac{1}{\ell}-1} u$$

The mass conservation equation, with  $a$  as the surface mass accumulation, is

$$a = \partial_x(Hu) = H_x u + H u_x$$

assuming a small accumulation rate,

$$u H_x \approx -H u_x \approx -H \dot{\epsilon}_{xx}$$

using the last equation, substitute  $H_x = H \dot{\epsilon}_{xx}/u$  in the momentum equation to find,

$$0 = \rho_i g H^2 \dot{\epsilon}_{xx} - C|u|^{\frac{1}{\ell}-1} u^2$$

use Glenn's law to express this in terms of stress instead of rate of strain,

$$0 = \rho_i g H^2 A(T) \tau_{xx}^n - C|u|^{\frac{1}{\ell}+1}$$

use the boundary condition at the grounding line that the longitudinal stress there is equal to the total ice pressure minus total water pressure. The floating condition is  $\rho_i H = \rho_w b$  so that  $b = \mu H$ ,  $\mu = \rho_i/\rho_w$ . The pressure difference is calculated as

$$\int_0^H g \rho_i z dz - \int_0^b g \rho_w z dz = \dots = \frac{1}{2}(1 - \mu) \rho_i g H^2$$

and the boundary condition is therefore,

$$H^2 \tau_{xx}|_{\text{grounding line}} = \frac{1}{2} \gamma H^2$$

where  $\gamma = (1 - \rho_i/\rho_w) \rho_i g H$ , and we find

$$0 = \rho_i g H^2 A(T) \frac{1}{4^n} \gamma^n \rho_i^n g^n H^n - C|u|^{\frac{1}{\ell}+1}$$

therefore the velocity at the grounding line is,

$$u = \left( \frac{\rho_i g}{C} A(T) \frac{1}{4^n} \gamma^n H^{n+2} \right)^{\ell/(\ell+1)}$$



and the volume flux is  $Hu$ , or,

$$Q_g = H \left( \frac{\rho_i g \gamma^n A(T)}{4^n C} H^{n+2} \right)^{\ell/(\ell+1)}$$

Using  $n = \ell = 3$ , the power of the thickness is  $(n+2)\ell/(\ell+1) = 3.75$ .

**8 Relation between DO and Heinrich events:** do major DO follows Heinrich events? Bond cycle (image on course web page is from [here](#))? Do Heinrich events happen during a cold period just before DO events? When the ice sheet model is coupled to a simple ocean-atmosphere model (slide 21), can get the response of the climate system as well (slides 27,28,29) via a THC shutdown (cold event) followed by a flush (warm event).

**9 Synchronous ice sheet collapses?** slides (33-47).

## 6 Glacial cycles

Downloads [here](#).

### 6.1 Overview and basics

First, briefly: glacial cycle phenomenology, main questions (from SIS intro, slides 1-7)

Next, basic ingredients that will later be incorporated into different glacial theories (lecture 8 from WH notes, unless otherwise noted),

1. energy balance and albedo feedback
2. ice sheet dynamics and Glenn's law (section 6.2 below)
3. parabolic ice sheet profile (section 6.2.2 below)
4. accumulation and ablation (mass balance) as function of ice sheet height, equilibrium line
5. ice streams, calving
6. dust loading and enhanced ablation
7. temperature-precipitation feedback
8. shallow ice approximation (section 3.1, p 222, Schoof and Hewitt, 2013, and section 6.2.3 below)
9. isostatic adjustment
10. Milankovitch forcing (section 6.3 below)
11. geothermal heating

### 6.2 Ice dynamics preliminaries

#### 6.2.1 Intro to ice dynamics

Rate of strain definition from Kundu and Cohen (2002) sections 3.6 and 3.7, pages 56-58; stress and deviatoric stress and stress-rate of strain relationship and Glenn's law from section 2.3, pages 13-15 of Van-Der-Veen (1999).

## 6.2.2 Parabolic profile of ice sheets

First simple version assuming ice is a plastic material, from WH notes p 101. Then the more accurate expression shown by the solid line in Fig 11.4 in Paterson (1994): the following derivation is especially sloppy in dealing with the constants of integration, and very roughly follows Chapter 5, p 243 eqns 6-10 and p 251, eqns 18-22 from Paterson (1994),

$$\dot{\epsilon}_{xz} = \frac{1}{2} \frac{du}{dz} = A\tau_{xz}^n = A(\rho g(h-z) \frac{dh}{dx})^n$$

integrate from  $z = 0$  to  $z$ , and use the b.c.  $u(z = 0) = u_b$ ,

$$u(z) - u_b = 2A(\rho g \frac{dh}{dx})^n \frac{(h-z)^{n+1}}{n+1} - 2A(\rho g \frac{dh}{dx})^n \frac{h^{n+1}}{n+1}$$

Let  $u_b = 0$  (no sliding) and average the velocity in  $z$ ,

$$\begin{aligned} \bar{u} &= (1/h) \int_0^h dz 2A \left( \rho g \frac{dh}{dx} \right)^n \frac{(h-z)^{n+1}}{n+1} - 2A \left( \rho g \frac{dh}{dx} \right)^n \left( \frac{h^{n+1}}{n+1} \right) \\ &= \frac{2A}{(n+1)} \left( \rho g h \frac{dh}{dx} \right)^n h \left( \frac{1}{n+2} - 1 \right) \\ &= -\frac{2A}{(n+2)} \left( \rho g h \frac{dh}{dx} \right)^n h. \end{aligned} \quad (1)$$

Next we use continuity, assuming a constant accumulation of ice at the surface,  $d(h\bar{u})/dx = c$  which implies together with the last equation

$$cx + K_1 = h\bar{u} = -\frac{2A}{(n+2)} \left( \rho g h \frac{dh}{dx} \right)^n h^2 = K_2 \left( h \frac{dh}{dx} \right)^n h^2$$

where ablation is assumed to occur only at the edge of the ice sheet at  $x = L$ . The last eqn may be written as

$$(K_3x + K_4)^{1/n} dx = h^{2/n+1} dh$$

and solved using boundary conditions of  $h(x = 0) = H$  and  $h(x = L) = 0$  to obtain

$$(x/L)^{1+1/n} + (h/H)^{2/n+2} = 1.$$

This last equation provides the better fit to obs in Paterson Fig 11.4 (also shown in WH notes).

### 6.2.3 Shallow ice approximation

Let  $(s, b)$  be the ice surface and ice bottom heights. Then, the momentum eqns, Glenn's law, the top and bottom boundary condition, and mass conservation equations are,

$$\begin{aligned} 0 &= -p_x + \frac{\partial \tau_{xz}}{\partial z}, & p_z &= -\rho g, \\ u_z &= 2A |\tau_{xz}|^{n-1} \tau_{xz} \\ u(z=b) &= 0, & \tau_{xz}(s) &= 0, & p(s) &= 0 \\ s_t + \partial_x q &= a, & q &= \int_b^s u dz \end{aligned}$$

Note that in the  $x$ -momentum equation, the time rate of change, nonlinear advection terms and the Coriolis term are all negligible. The stress term on the rhs depends on the velocity via the dependence of the stress on the range of strain. acceleration Integrating the hydrostatic eqn from the surface, one finds

$$p(x, z) = (s(x) - z)\rho g.$$

Substitute in the momentum equation to find,

$$\frac{\partial \tau_{xz}}{\partial z} = \rho g s_x$$

Integrate using the zero stress condition at the top,  $z = s$ ,

$$\tau_{xz} = \rho g (z - s) s_x$$

substitute in Glenn's law,

$$u_z = 2A (\rho g)^n (z - s)^n |s_x|^{n-1} s_x$$

integrate from the bottom to  $z$  to find  $u$ , using b.c of zero velocity at bottom,

$$u(z) = \frac{2A}{n+1} (\rho g)^n [(z - s)^{n+1} - (b - s)^{n+1}] |s_x|^{n-1} s_x$$

and then again from bottom to top to find  $q$ ,

$$\begin{aligned} q &= \frac{2A}{n+1} (\rho g)^n \left[ \frac{1}{n+2} (b - s)^{n+2} - (b - s)^{n+2} \right] |s_x|^{n-1} s_x \\ &= -\frac{2A}{(n+2)} (\rho g)^n (s - b)^{n+2} |s_x|^{n-1} s_x. \end{aligned}$$

The mass conservation equation can therefore be written as a nonlinear diffusion equation for the ice surface elevation,  $s$ , with a diffusion coefficient  $D$  that depends on  $s$ ,

$$s_t - \partial_x(Ds_x) = a,$$

$$D = \frac{2A}{(n+2)}(\rho g)^n(s-b)^{n+2}|s_x|^{n-1}.$$

### 6.3 Milankovitch

- sea\_ice\_switch\_and\_pacing\_glacial\_cycles.pptx slide 6; Paillard (2001) Fig 2.
- Muller and MacDonald (2002) Chapter 2: sections 2.1 (only until and not including 2.1.1); 2.2 (only 2.2.1-2.2.5).
- Some additions: precession effect is anti-symmetric with respect to seasons and hemispheres, and annual average of precession vanishes at each latitude; precession has no effect when eccentricity is zero (circular orbit); Paillard (2001) p 328: “energy received at a given latitude and between two given orbital positions (for example, between the summer solstice and the autumnal equinox) does not depend on the climatic precession. However, the time necessary for the Earth to move between these two orbital positions (for example, the length of the summer season) does change with climatic precession. The insolation, defined as the amount of energy received per unit time, therefore changes with climatic precession, but only through the lengths of the seasons.”
- Obliquity does effect annual mean at a given latitude, but not global average. Larger obliquity leads to more radiation at the poles in summers, but still none at winter, so more generally high latitude annual insolation depends on obliquity.
- Paillard (2001) Figs. 3,4,5
- Animations and images in web page by Peter Huybers (use Safari), [people.fas.harvard.edu/~phuybers/Inso](http://people.fas.harvard.edu/~phuybers/Inso)
- Integrated insolation: this is left to the end of the glacial cycle mechanism discussion.

### 6.4 Glacial cycle mechanisms

**General outline:**

1. Temperature-precipitation feedback (Ghil 94)
2. Elevation-desert effect (Ghil 94)
3. Adhemar’s model, Croll’s model (Fig. 1, Paillard, 2001),

4. Milankovitch: a correct calculation of orbital parameters and insolation, and realization that it's the summers that matter.
5. Calder's model (eqn on p. 332 and Fig. 9 of Paillard, 2001),
6. Imbrie and Imbrie (eqn on p. 333 and Fig. 10 of Paillard, 2001),
7. Paillard's model (sec 3.3, eqn on p. 339 and Figs 12 and 13 of Paillard, 2001),
8. Le-Treut and Ghil (1983) and the 100 cycle as a difference tone of insolation's 19k and 23k frequencies, due to nonlinear glacial dynamics. Then Rial (1999) with taking the idea even further, (both from WH notes).
9. Stochastic resonance
10. 100 kyr from a collapse of the ice sheets due to geothermal heat build up and induced basal melting (Huybers and Tziperman, 2008)
11. Saltzman "Earth system" models
12. Discussion: (A) we are trying to merely fit the record or explain the mechanism? (B) a reminder that a useful glacial theory must be falsifiable. . .
13. The shallow ice approximation plus orbital forcing Pollard (1982)
14. Sea ice switch
15. Phase locking, and a discussion of how a good fit to ice volume does not imply a correct physical mechanism; difference between locking to periodic and quasi-periodic forcing
16. Huybers' integrated insolation: The 41kyr problem, positive degree days as a motivation for integrating insolation beyond a threshold corresponding to melting point; cancellation of precession because summer intensity and duration are exactly out of phase (Kepler Laws). All explained in Huybers-integrated-insolation.pptx slides 5,7,9,10.

## 6.5 CO<sub>2</sub>

1. Show glacial CO<sub>2</sub> time series, and then use sections 1-4 of the [notes](#) to explain the basics of the carbonate system and its response to various inputs/ changes.
2. The 3 box model of Toggweiler (1999) showing how CO<sub>2</sub> is determined by the Southern Ocean biology and can change in response to change in mixing between surface water and NADW below (section 5 in [notes](#), following Toggweiler's 1999 paper)

## 7 Pliocene climate

Phenomenology (Molnar and Cane, 2007; Dowsett et al., 2010) and relevant proxies for temperature, productivity, upwelling, CO<sub>2</sub>, etc. Pliocene as nearest analogue of future warming (although only for the equilibrium response). Observations of global temperature, permanent El Nino and warming of mid-latitude upwelling sites.

Possible mechanisms for permanent El Nino: FW flux causing a collapse of the meridional density gradient and therefore meridional ocean heat flux, which leads to a deepening of the thermocline; Hurricanes and tropical mixing; opening of central American seaway; movement of new Guinea and changes in Pacific-Indian water mass exchange; atmospheric superrotation, possibly driven by strengthening MJO. Mechanisms for warming of upwelling sites.

Details of all of the above in Pliocene powerpoint presentation.

## 8 Equable climate

Downloads [here](#).

Earth Climate was exceptionally warm, and the equator to pole temperature difference (EPTD) exceptionally small, during the Eocene (55Myr ago), when continental configuration was not dramatically different from present-day. Many explanations have been proposed, and we will briefly survey some.

First, Phenomenology and relevant proxies from slides.

### 8.1 Equator to pole Hadley cell

(Farrell, 1990). The idea briefly: Angular momentum conservation leads to large zonally-averaged zonal velocity  $\bar{u}$  in the upper branch of the Hadley cell: The angular momentum equation is derived from the zonal momentum equation in spherical coordinates, where, in the absence of forcing or friction we have

$$\frac{DM}{dt} = \frac{1}{\rho} \frac{\partial p}{\partial \lambda}$$
$$M = (\bar{u} + \Omega r \cos \theta) r \cos \theta.$$

so that in an axisymmetric system the angular momentum is conserved. Consider a parcel of air which starts at the surface near the equator with  $\bar{u}(\text{equator}) = 0$  and then rises and travels poleward in the Hadley cell. We find from angular momentum conservation that

$$M(\text{equator}) = M(\theta),$$
$$\Omega r^2 = (\bar{u} + \Omega r \cos \theta) r \cos \theta$$
$$\bar{u}(\theta) = \Omega r \left( \frac{1}{\cos \theta} - \cos \theta \right) = \Omega r \frac{\sin^2 \theta}{\cos \theta}$$
$$\bar{u}(30) = (6,300,000 \times 2\pi / (24 \times 3600)) \times (1/\cos(30) - \cos(30)) = 132m/sec.$$

The resulting large  $\bar{u}_z$  is balanced via thermal wind by strong  $T_y$ , leading to a large EPTD (eqn 1.5 in Farrell, 1990). To break this constraint, can dissipate some angular momentum, reduce  $f$  (as on Venus), or increase the tropopause height  $H$  which appears in the solution for the edge of the Hadley cell.

- Start with the frictionless theory from Vallis (2006), highlighted parts of sections 11.2.2-11.2.3.
- Then cover highlighted parts of Farrell (1990), which is based on an extension of Held and Hou (1980) to include dissipation, based on Hou (1984).



## 8.2 Polar stratospheric clouds (PSCs)

PSCs are formed in the lower stratosphere (15-25 km) at temperatures below  $-78^{\circ}\text{C}$ . They are optically thick and high, and therefore have a significant greenhouse effect. Water ice, which is a major ingredient of PSCs, is formed in the stratosphere via methane ( $\text{CH}_4$ ) oxidation. Methane, unlike water vapor, is able to get past the tropopause “cold trap” given its freezing point of  $-182^{\circ}\text{C}$ . Sloan et al. (1992) proposed that PSCs may have contributed to equable climate conditions, and Kirk-davidoff et al. (2002) suggested a positive feedback that would enhance the formation of PSCs in a warm climate. Details follow.

- **Preliminaries 1.** QG potential vorticity, EP fluxes, transformed Eulerian mean equations (Vallis).
- **Preliminaries 2.** Planetary wave forcing of the stratosphere: Topographically forced vertically propagating planetary waves, conditions on mean zonal flow that lead to vertical propagation (Vallis).
- **Zonal stratospheric circulation and seasonal limits on upward wave propagation.** (Vallis) SW absorption near the summer pole leads to a reversed temperature gradient in the summer hemisphere: i.e.,  $T_y < 0$  in southern hemisphere during Jan, and  $T_y > 0$  in northern hemisphere during July. There is no short wave radiation in the polar areas in the winter hemisphere (northern Jan, southern July), and the temperature gradient is not reversed. During the southern winter (July), therefore,  $T_y > 0$  in the southern hemisphere, and therefore  $u_z \propto \rho_y / f_0 \propto -T_y / f_0 > 0$ . Using  $u = 0$  at top of stratosphere we get  $u < 0$  (easterlies) during summer (Jan) in the southern hemisphere stratosphere. Similarly,  $u < 0$  (easterlies) during summer (July) in northern hemisphere. During winter the stratospheric zonal winds are westerlies. Note that stationary Rossby waves cannot propagate from the troposphere into easterlies, and they can therefore only reach the stratosphere in the winter hemisphere.
- **Brewer-Dobson stratospheric circulation:** (Vallis) The zonally averaged Transformed Eulerian Mean (TEM) momentum balance derived above is  $-f_0 \bar{v}^* = \overline{v'q'}$ . Assuming the potential vorticity flux is down-gradient (which is equatorward in the northern hemisphere because the gradient is dominated by  $\beta$ ), the rhs is negative, so that the mean flow  $\bar{v}^* > 0$  is poleward. In the southern hemisphere, the rhs is positive  $f_0$  is negative, so  $\bar{v}^* < 0$  is poleward again. The B-D circulation warms the pole and cools the equator in the stratosphere (Vallis eqn 13.89):  $N^2 \bar{w}^* = \frac{\theta_E - \theta}{\tau}$ , together with positive  $w^*$  in tropics and negative in polar areas forced by poleward B-D meridional flow. This leads to  $\theta < \theta_E$  (cooling!) at the equator (where  $\bar{w}^* > 0$ ) and  $\theta > \theta_E$  (warming!) at the pole (where  $\bar{w}^* < 0$ ).
- **Feedback between EPTD, vertically propagating planetary waves, Brewer-Dobson stratospheric circulation and PSCs:** (Kirk-davidoff et al., 2002) warmer climate means weaker tropospheric EPTD, this leads to weaker mean tropospheric zonal winds and weaker synoptic scale motions (which are, in turn, created via baroclinic instability of the mean winds and meridional temperature gradient). Both of these factors weaken the production of vertically

propagating Rossby waves (forced by mean zonal winds interacting with topography, and by synoptic motions). As a result, there is a weaker Eliassen-Palm flux  $EP$  into the stratosphere, a weaker convergence  $\nabla \cdot EP = \overline{v'q'}$ , and therefore a weaker B-D circulation, and as a result a colder pole and warmer equator. The colder pole allows more PSCs to develop. This, in turn, further weakens the EPTD in troposphere, providing a positive feedback.

- **However:** Korty and Emanuel (2007) show that the BD circulation could strengthen because additional zonal wavenumbers can penetrate into the stratosphere in a warmer climate with a weaker EPTD. Furthermore, nearly all studies of future climate scenarios project a stronger BD circulation, probably because of an upward shift of the critical layer and therefore of the wave breaking altitude resulting from the eastward acceleration and upward movement of the subtropical jets. This allows both the resolved (planetary) and parameterized (internal) waves to penetrate higher, and to drive a stronger BD circulation (Butchart, 2014).

### 8.3 Hurricanes and ocean mixing

The idea: Hurricanes are a heat engine, when described as such they can be shown to get stronger in warmer climates. This leads to stronger ocean mixing, and therefore to stronger ocean MOC. The stronger meridional heat flux by the ocean MOC reduces equator to pole temperature gradient and explain equable climate.

- **Maximum possible hurricane strength** (from Kerry's [web page](#)): rate of kinetic energy *input* per unit area into the hurricane is roughly  $G = \epsilon C_k \rho V_s L (q_0^* - q_a)$ , where  $V_s$  is the max surface wind speed,  $\epsilon$  efficiency in translating enthalpy to K.E.,  $q_a$  the atmospheric surface specific humidity,  $q_0^*$  the saturation humidity based on SST,  $L$  is the latent heat of evaporation,  $\rho$  the air density and  $C_k$  the bulk coefficient for evaporation. The rate of energy dissipation per unit area is given by  $D = C_D \rho V_s^3$ . To calculate the efficiency  $\epsilon$ , need to consider first a Carnot cycle.
- Entropy reminder: consider a container with fluid, divided into two equal parts with temperatures  $T_H > T_C$ . Removing the divider, temperature will eventually be homogenized to  $(T_C + T_H)/2$ . During the process, the infinitesimal change in entropy due to the transfer of an infinitesimal amount of heat  $dQ$  between the two systems leads to a gain  $dQ$  for the cold system and a loss of  $dQ$  for the hot system (gain of  $-dQ$ ), thus the entropy change is

$$dS = \frac{dQ}{T_C} + \frac{-dQ}{T_H} = \frac{T_H - T_C}{T_H T_C} > 0.$$

so the increase in entropy is because temperature flows from the hot reservoir to the cold one.

- **Carnot cycle:** three first pages from [wikipedia article](#), including description of four steps, calculation of efficiency. See also [on-line animation](#).

- **Hurricane as a Carnot cycle:** (Figure 1 in Emanuel, 1991). (1) Equivalent to isothermal expansion: air acquires heat (in the form of moisture from evaporation) as it flows along the surface toward the center at a temperature  $T_H$ . (2) Adiabatic expansion as the parcel goes up in the slanted eye wall, doing work on the environment. (3) Isothermal compression: hurricane releases the heat to the environment (via radiation) while mixing out of the convective plumes at the top of the storm and descending and therefore compressing at a temperature  $T_C$ . (4) adiabatic compression while descending back to the surface.
- **Calculating the efficiency  $\varepsilon$  of a hurricane:** The *efficiency* of work done (kinetic energy generation) in a Carnot cycle in terms of the temperatures of the warm and cold reservoirs involved is  $\varepsilon = (T_H - T_C)/T_H$ . For Hurricanes,  $T_H = SST$  is temperature of the heat source (the ocean surface).  $T_C$  is the average temperature at which heat is lost via radiation by the air parcels at the top of the storm. The taller a hurricane is, the lower the temperature  $T_C$  at its top and thus, the greater the thermodynamic efficiency. For a typical hurricane,  $\varepsilon \approx 1/3$ .
- **Estimating maximum wind speed in a hurricane:** Setting dissipation equal to generation ( $G = D$ ), we get  $V_s^2 = \varepsilon L(q_0^* - q_a)C_k/C_D$ . Assuming the atmosphere to be 85% saturated, and the ratio of the two bulk coefficients to be about one, we get

$$\begin{aligned}
 V_s^2 &= \varepsilon \times L \times 0.15 \times q_0^* \\
 &= \frac{SST - T_0}{SST} L 0.15 q_0^*(SST) \\
 V_s &= \left( \frac{1}{3} 2260 \times 10^3 0.15 \times \frac{20}{1000} \right)^{1/2} = 47 \text{ m/s} = 169 \text{ km/hour}
 \end{aligned}$$

units check:

$$m^2 s^{-2} = \{none\} \frac{J}{kg} \{none\} \frac{kg}{kg} = m^2 s^{-2}$$

Note that this is exponential in temperature, because of the Clausius-Clapeyron relationship.

- **Connection to ocean mixing and equable climate:** Finally, assuming that stronger hurricanes lead to stronger ocean mixing, and this to stronger MOC (Bryan, 1987). Stronger MOC means warmer poles (Emanuel, 2002).
- **Unfortunately,** consequences on SST and on EPTD of strongly enhanced tropical ocean mixing in an idealized coupled model seem small (Figs. 4, 11, 12 of Korty et al., 2008).

## 8.4 Convective cloud feedback

The idea: tropospheric clouds over the arctic during polar night should have a very strong greenhouse effect, but no cooling albedo effect. Ice free Arctic can support atmospheric convection and convective clouds, and these clouds then protect the surface from cooling too rapidly, and maintain

the ice-free Arctic during polar night. This is a positive feedback between the convection and the surface temperature. With ice free Arctic, high latitude continents are more likely to remain above freezing, as observed. The feedback also dramatically reduces equator to pole temperature gradient especially during winter.

- **Importance of clouds for Arctic Energy balance:** back-of-the-envelope calculation for Arctic, comparing options for warming the Arctic, from Powerpoint.
- **Results of box model:** hysteresis and multiple equilibria, from Powerpoint. Our objective is to derive a simple analytic model describing the feedback and resulting hysteresis.
- **Atmospheric convection preliminaries:** Derivation of the conservation of Moist Static Energy (and, optional, moist adiabatic lapse rate which is needed in the followings but perhaps not critical), beginning of section 6.1 from my notes based on Marshall and Plumb (2008). Then using MSE to diagnose convective stability, section 7.3 from my notes.
- **Analytic two-level model:** section 2 of Abbot and Tziperman (2009) demonstrating the multiple equilibria and hysteresis analytically.
- **Supporting results from more complex models:** SCAM, IPCC, SP-CESM, from Powerpoint presentation (e.g., Abbot and Tziperman, 2008; Abbot et al., 2009).

## 9 Data analysis tools

For observations and model output (hands-on practice in sections). [supporting material](#).

- EOFs
- SVD analysis of two co-varying fields
- Composite analysis
- Linear inverse models, POPs
- Spectral analysis

## 10 Review of nonlinear dynamics concepts covered in the course

### 1. One-parameter bifurcations:

(a) 1d: saddle node,

$$\dot{x} = \mu - x^2$$

transcritical,

$$\dot{x} = \mu x - x^2$$

pitchfork: super-critical and sub-critical.

$$\dot{x} = \mu x - x^3$$

$$\dot{x} = \mu x + x^3 - x^5$$

(b) Hysteresis in the case of two back-to-back saddle node bifurcations, and in the case of a subcritical pitchfork.

(c) 1d bifurcations in higher dimensional systems

(d) 2d: Hopf, super-critical

$$\dot{r} = \mu r - r^3$$

$$\dot{\theta} = \omega + br^2$$

and sub critical.

$$\dot{r} = \mu r + r^3 - r^5$$

$$\dot{\theta} = \omega + br^2$$

(e) Hysteresis in subcritical Hopf.

2. Nonlinear phase locking via analysis of flows on a circle, firefly and flashlight.

3. Two-parameter bifurcation: quasi-periodicity route to chaos in circle map.

$$\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi} \sin(2\pi\theta_n)$$

4. Relaxation oscillation, analysis via nullclines of the Van der Pol oscillator,

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

5. Saddle node bifurcation of cycles.

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