Applied Mathematics 105b:
Ordinary and Partial Differential Equations

Applied Mathematics 105b
(Spring 2010)

Instructors: Eli Tziperman (eli at eps.harvard.edu); Scott Norris (snorris at seas.harvard.edu).

TFs: TBA.

Day & time: MWF, 11

Location: Jefferson 250;

Sections: time and place TBA

1st meeting: Monday, Jan 25.

Office hours: Eli: hours TBA, 24 Oxford, museum building, 4th floor, room 456; Scott: hours TBA, Pierce Hall 312; TFs: see course web page.

Textbooks: Page or section numbers from the relevant textbook for any given lecture are given in the detailed syllabus below.

- **Gr** Greenberg, *Advanced Engineering Mathematics*: our main textbook. ***Any course material not from this textbook will be posted to the course web page as notes***.
- **Kr** Erwin Kreyszig, *Advanced Engineering Mathematics*: very similar contents to Greenberg, somewhat more concise. Will be used occasionally.
- **St** Strogatz, *Nonlinear dynamics and chaos*: will be used for just a few lectures about linear and nonlinear dynamics.
- **Hi** Francis B. Hildebrand Advanced Calculus for Applications (2nd Edition).

Supplementary materials: Additional materials from several additional textbooks and other sources, including Matlab programs used in class, may be found here. Follow links below for the specific source material for each lecture. In order to access these materials from outside the Harvard campus, you'll need to use the VPN software which can be downloaded from the FAS software download site.

This document: http://www.seas.harvard.edu/climate/eli/Courses/APM105b/2010spring/detailed-syllabus-apm105b.pdf

Announcements Last updated: November 15, 2009.
Feel free to write call or visit us with any questions.

1 Outline

Ordinary differential equations: power series solutions; special functions; eigenfunction expansions. Algebra and calculus of vectors. Elementary partial differential equations: separation of variables and series solutions; Introduction to dynamical systems, nonlinear dynamics and chaos. Introduction to numerical methods for solving ordinary and partial differential equations.
Note: Applied Mathematics 105a and 105b are independent courses, and may be taken in any order.

2 Administrative

Prerequisites: Applied Mathematics 21a and 21b, or Mathematics 21a and 21b.
Sections: Regular times for sections will be scheduled at the beginning of the semester. Each TF will hold a weekly section and have weekly office hours. During the sections, the TFs will discuss and expand on the lecture material and solve additional problems. Although these sections are not mandatory, you are encouraged to attend. Occasionally the TFs will explain material covered in the reading material but not in lectures.
Computer Skills: No programming skills are assumed for this class. Some of the demonstrations, sections and homework assignment will be Matlab-based, and as part of the course, students will therefore gain some experience with this package. Students are asked to download and install Matlab on their computers from the FAS software download site.
Homework: Homework assignments will be assigned every Monday, and will be due the following Monday in class unless otherwise noted. It is essential that you actively engage in problem solving using the assigned HW and other problems from the course textbook. Continuously practicing the lecture material on a weekly basis via such problem solving is the only way to become comfortable with the subjects covered in the course.
Grading: Homework: 25%; hour exams (two midterms, possibly given in the evening): 30%; final: 45%;
Readings: Occasionally we will assign reading material from the textbook or other sources. This material will complement the lectures and is therefore an important part of the course.

3 Syllabus

Follow links to see the source material and Matlab demo programs used for each lecture.

1. INTRODUCTION, OVERVIEW here.

2. FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS. downloads

   (a) Introductory remarks (**Gr**§1.1), definitions: ordinary differential equations (ODEs), order of ODEs, system of ODEs, partial differential equations (PDEs), a 'solution', linear vs nonlinear (**Gr**§1.2).

(b) Motivation: applications of first order ODEs.

i. Radioactive decay (**Gr**§2.3.2, p 39-end of example 3, p 41).
ii. Population dynamics, logistic growth (**Gr**§2.3.3, p 41-42).
iii. Mixing in a tank (**Gr**§2.3.4, p 42-43).
iv. Earth's energy balance notes.
v. In sections: Compound interest (**Gr**, exercise 16, p 46);

(c) Geometric/ dynamical systems approach

i. Graphical approach (given $\dot{x} = f(x)$, plot $f(x)$ as function of x and find direction of flow and steady states); fixed points; stability: graphic approach (**St**§2.1, pp 16-18, including Figure 2.2.1)
ii. Linearized stability analysis (above notes or **St**§2.4, p 24-25, to the end of Example 2.4.1);

(d) Linear first order ODEs: $y' + p(x)y = q(x)$.
Homogeneous case ($q(x) = 0$, **Gr**§2.2.1 to and including example 1). Inhomogeneous case: integrating factor method (**Gr**§2.2.2 to example 3). Time permitting: variation of parameter (**Gr**§2.2.4 to eqn 38).

(e) Nonlinear 1st order ODEs.

i. Equations of separable form (**Gr**§2.4.1 and examples 5, 6 in §2.4.3);
ii. Exact differential equations and integrating factors (**Gr**§2.5.1-2.5.2);
iii. Bernoulli, Riccati, d'Alembert-Lagrange, Clairaut: in sections (**Gr**§2.2 exercises 9, 11, 13, 14);

(f) Existence and uniqueness (**Gr**§2.4.2: theorem 2.4.1 on p 49 and example 4 on pp 51-52).

(g) Higher order nonlinear ODEs that are reducible by a change of variables to first order equations (second order equations lacking one variable, **Hi**§1.12.5, p 36-37, including examples 9, 10).

i. Only y'', y' appear in equation, but not y; transform to $p(x) \equiv y'(x)$.
ii. Autonomous equations (x doesn't appear in the equation explicitly) vs non-autonomous (does appear): reduce order by treating $p = y'$ as a function of y: $dp(y)/dx = (dp/dy)(dy/dx) = p(dp/dy)$.

(h) Matlab: symbolic integration and differentiation using Matlab, obtaining an analytic solution to an ODE, and using ode45 for solving ODEs (`symbolic_diff_and_int.m`, `using_dsolve.m`, and `logistic_using_ode45.m` in downloads directory).

3. SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS. downloads

(a) Introductory remarks: general linear form $y'' + p(x)y' + q(x)y = r(x)$, linear vs nonlinear, homogeneous vs nonhomogeneous (**Kr**§2.1, pp 45-46); superposition principle (**Kr**§2.1 examples 1-3, theorem 1);

(b) Motivation and examples:

i. Damped linear pendulum.

ii. Nonlinear pendulum.

iii. Periodically driven damped nonlinear pendulum and chaos.

iv. Love affairs: a brief introduction, more later.

(c) Initial value problems (IVP), basis, general solution (**Kr** pp 47-50 including example 6); briefly contrast with boundary value problems (BVP, **Gr**§3.3.2 pp 88-89), we'll get back to BVPs later.

(d) Writing a second order ODE as two first order ODEs: define $y' \equiv v$, which leads to $v' + p(x)v + q(x)y = r(x)$, together two first order ODEs.

(e) A brief intro to two-dimensional dynamical system perspective: phase plane, possible nonlinear behaviours. Linearization, linear systems. Love affairs in more details (**St**§5.3 pp 138-140, here, and also my_quiver.m). classification of fixed points, stable and unstable spaces (**St**§5.1: example 5.1.2 p 126-128; §5.2 p 129- eqn 6 on p 131).

(f) Reduction of order using a known solution: if $y_1(x)$ is the known solution, try $y_2(x) = u(x)y_1(x)$ and u' then satisfies a 1st order ODE. (**Kr** pp 50-52)

(g) Constant coefficients: guessing an exponential form, characteristic equation, three cases: two distinct real roots, real double root, complex roots (**Kr** pp 53-57). Example: damped harmonic oscillator (**Kr** pp 63-66; and in section: example 2, p 77).

(h) Differential operators, linearity, factorization (**Kr**§2.3)

(i) Euler-Cauchy (equi-dimensional) equations (**Kr**§2.5 pp 69-71)

(j) Linear independence of solutions, Wronskian (**Kr**§2.6)

(k) Nonhomogeneous linear second order ODEs: general solution, particular solution (**Kr**§2.7 p 78-79, including theorem 2)

i. The method of undetermined coefficients (**Kr** pp 79-82)

ii. The method of variation of parameters (**Kr**§2.10 pp 98-101)

(l) Summary

4. POWER SERIES SOLUTIONS OF SECOND ORDER LINEAR ODES. downloads

(a) Introductory remarks (**Gr**§4.1 to equation 6)

(b) Power series review.

i. Basics (**Gr**§4.2.1, equations 1-6),

ii. Cauchy's convergence (theorem 4.2.1), interval of convergence and radius of convergence (theorem 4.2.2 on p 178),

iii. Manipulation of power series, including differentiation, integration, addition, multiplication (theorem 4.2.3 on p 179 and equations 11-14).

(c) Power series solution method around ordinary (non singular) points

   i. Analytic functions, infinite differentiability and Taylor series (**Gr** pp 180-182, including in particular Fig. 2).

   ii. The power series solution method (**Gr**§4.2.2, from p 182 to example 5, top of p 185; in section: example 6).

   iii. General recipe for obtaining a power series solution to $y'' + p(x)y' + q(x)y = 0$ using Taylor expansion of p, q (**Gr** "closure" pp 189-190).

(d) ODEs with singular coefficients

   i. Motivation for ODEs with singular coefficients:

      A. Bessel equation for the shape of a vertical rotating string (notes)

      B. Vibrations of a circular drum leading to Bessel equation (wikipedia, or locally, class demo).

      C. Diffusion on a sphere and Legendre equation, this is dealt with and derived in detail later, see notes below.

      D. A singular love affair (notes)

   ii. Regular singular points and irregular singular points (definition 4.3.1 on p 194)

   iii. Method of Frobenius: series solution of ODEs around regular singular points (**Gr**§4.3.2 including example 3 and to the end of theorem 4.3.1, pp 195-202). The indicial equation $r^2 + (p_0 - 1)r + q_0$, where $xp(x) = p_0 + p_1 x + \ldots$ and $x^2 q(x) = q_0 + q_1 x + \ldots$. There are three cases for the roots of r as outlined in the theorem. (See also summary and examples of the three cases in **Kr**§5.4, pp 183-187).

   iv. Example of case i

   v. Example of case ii

   vi. Example of case iii

   vii. Summary of Frobenius method.

(e) Using Matlab to find analytic solutions to ODEs (using_dsolve.m)

(f) Legendre's equation $(1-x^2)y'' - 2xy' + \nu(\nu+1)y = 0$ or equivalently $\frac{d}{dx}[(1-x^2)\frac{dy}{dx}] + \nu(\nu+1)y = 0$, for $-1 \le x \le 1$, and Legendre's polynomials $P_n(x)$.

   i. Motivation: diffusion on a sphere and snowball earth (notes).

   ii. Regular singular points at $x = \pm 1$, Frobenius solution and the case of $\nu = n =$integer leading to Legendre polynomials (**Gr**§4.4.1).

   iii. Orthogonality of $P_n(x)$ (**Gr**§4.4.2).

   iv. Recursion (**Gr** p 215 equation 16), relation involving derivatives (equation 17), and value of integral of $\int_{-1}^{1}[p_n(x)]^2$ (equation 18).

(g) Gamma function (**Gr**§4.5.2, pp 223-225).

(h) Bessel equation $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$ and functions

i. $\nu \neq$integer: solving Bessel's equation using the method of Frobenius (**Gr**§4.6.1, pp 230-232).
ii. $\nu =$integer: finding a second linearly independent solution to Bessel's equation (**Gr**§4.6.2-4.6.3 pp 233-236)
iii. General properties of the solutions to Bessel's equation: graphical depiction of the solution (plot_Bessel.m); the case of $\nu = 1/2, 3/2, 5/2, \ldots$ where Bessel functions can be expressed in terms of elementary functions (**Kr** p 196; **Gr** exercise 4.6.5 p 242)
iv. Useful properties of Bessel functions: recursion relations between derivatives and different orders (**Kr** p 196; **Gr** exercise 4.6.4 p 242).
v. Modified Bessel equation (**Gr**§4.6.5 pp 236-238)
vi. Bessel-like equations. Any equation of the form $\frac{d}{dx}(x^a \frac{dy}{dx}) + bx^c y = 0$ can be transformed to a Bessel equation (**Gr**§4.6.6 example 1, p 238 and then equations 46-50 p 239).
vii. Time permitting: Airy equation $y'' + xy = 0$ (**Gr** p 192, problem 9).

- [First midterm hour exam]

5. EIGENVALUE PROBLEMS AND INTRODUCTION TO FOURIER SERIES. downloads

   (a) Introduction

   i. 2nd order boundary value ODEs leading to eigenvalue problems (**Gr**§17.7, p 888, example 1)
   ii. A slightly more advanced example: Graphical determination of the eigenvalues (example 3, p 893).
   iii. Motivation: plucked string and the wave equation: Class demo; write wave equation with no derivation for now, use separation of variables to derive a second order ODE and the boundary value problem from the PDE. Cosine and sine solutions, superposition of solution and Fourier series representation. Leave determination of coefficients to a later stage below (**Kr**§12.2, p 539, Eqn 3 to Eqn 10 on p 542.).

   (b) Fourier series

   i. Even, odd and periodic functions: definitions and some elementary consequences, decomposition of an arbitrary function into even and odd components. (**Gr**§17.2, including examples 1,2).
   ii. Fourier series expansion of a periodic function(**Gr**§17.3.1, p 850-851): Stating the Euler formulas for the Fourier coefficients (Eqn 5a,b,c,d), Fourier convergence theorem (theorem 17.3.1); Definition of uniform convergence: p 874 in **Gr**§17.5 including example 1; Gibb's phenomenon is an example for a non uniform convergence.
   iii. Example: Fourier series of $x^3 - x$. Fourier_convergence.m

iv. Fourier expansion of a square wave and Gibbs phenomena for functions with a jump discontinuity (**Gr**§17.3.1 example 1, p 852-854): constant %9 overshoot, gibbs.m, meaning of convergence (paragraph starting with in view of this overshoot on p 854). Example: representation of the Andes in spectral atmospheric models.
v. Convergence speed: slowness of convergence for square wave ($1/n$, because of discontinuity, top of p 857). More generally: convergence speed as function of the smoothness of the expanded function (p 856-857, from equation 20 to the end of section 1.3.1). Fourier_convergence.m.
vi. Euler formulas: justification (**Gr**§17.3.2, p 857-8, until Eqn 26), including orthogonality of sines and cosines. A graphical interpretation of this orthogonality. Fourier_orthogonality.pdf.
vii. (In sections) Fourier series arising from an ODE: periodically forced oscillator (example 3, pp 859 to end of comment 1, p 861)

(c) Sturm-Liouville 2nd order ODE eigen-problems and generalized orthogonal function expansions

i. Definition of S-L problem: $[p(x)y']' + q(x)y + \lambda w(x)y = 0$ on $a < x < b$, with homogeneous b.c. are $\alpha y(a) + \beta y'(a) = 0$, $\gamma y(b) + \delta y'(b) = 0$, all coefficients are continuous, and $p, w > 0$ on $[a, b]$. Note that b.c. are homogeneous ($= 0$)! λ values which allow non trivial solutions are the *eigenvalues* and the corresponding solutions are the *eigenfunctions*. (**Gr**§17.7.1, pp 887-889 including example 1 to equation 9).
ii. Inner product $\langle f, g\rangle$ (**Gr** p 890, eqn 14 and surrounding paragraph).
iii. The Sturm-Liouville theorem (**Gr** theorem 17.7.1a-d, p 891; including example 2 all the way to eqn 18); Note that part d deals with the important completeness property of the eigenfunctions of the Sturm-Liouville problem.
iv. Non negative eigenvalues (**Gr** theorem 17.7.2, p 893);
v. Self-adjoint nature of SL operator and parallels with eigenvectors/ eigenvalues of a symmetric matrix (**Gr**§17.7.2 pp 897-901, end of example 6). Note the following theorems regarding symmetric matrices (**Gr**§11.3, p 555-557): 11.3.1: eigenvalues are real; 11.3.3: eigenvectors corresponding to distinct eigenvalues are orthogonal; 11.3.4: eigenvectors are a complete orthogonal set to n-space.
vi. Proof of parts of the S-L theorem: (**Gr** pp 898-899).
vii. Brief mention of: (1) periodic S-L problems (**Gr**§17.8, eqn 2, p 906), (2) infinite domain (Schrodinger equation for a 1d harmonic oscillator $\left[-\frac{h^2}{2m}\frac{d^2}{dx^2} + \frac{K}{2}x^2 - E\right]\phi(x)$ with E being the energy eigenvalue, K the spring constant, boundedness requirement at $x \to \pm\infty$ leads to discrete eigenvalues and the solution is Hermite polynomials times a Gaussian $H_n(x)e^{-x^2/2}$); (3) singular S-L problems, and example of which involves the Bessel equation discussed next.

viii. The eigenvalue problem for the Bessel equation (**Gr**§17.8, example 2, pp 908-910).

A. Bring Bessel equation to standard S-L form (**Gr** eqns 46-50 p 239).

B. At singular end points, boundedness requirement replaces the homogeneous b.c.

C. Orthogonality of Bessel functions corresponding to different eigenvalues, note the weight function $w(x) = x$ in the orthogonality condition: $\int_0^L J_n(x)xJ_m(x)dx = \frac{2\delta_{mn}}{L^2[J_1(z_n)]^2}$ (for the $n = m$ case, see exercise 7 for **Gr**§4.6, p 243).

D. Expansion of a general function in terms of Bessel functions .

6. NUMERICAL METHODS FOR ODES downloads

   (a) Advantages and disadvantages of numerical vs analytic solutions, blurring boundaries between the two for complex analytic solutions, numerical sensitivity vs analytic expression for dependence on parameters.

   (b) Euler method, estimates of error and accumulated error (**Gr**§6.2 pp 293-296 to eqn 11; Taylor Formula is derived on p 631, eqns 7a-10); definition of order p method: accumulated truncation error is $E_n = O(h^p)$; choosing a sufficiently small h (p 297); Matlab demo: euler_error.m.

   (c) Leap-frog (mid-point) method (**Gr**§); improved Euler method, also known as Runge-Kutta method of second order (**Gr**§6.3.2, p 302-304).

   (d) Introduction to numerical (in)stability: instability due to existence of a second solution (**Gr**§6.5 example 1, p 323), numerical_instability_2nd_solution.m; instability due to the scheme itself with leap-frog (mid-point) as an example (example 2, p 324), numerical_instability_leap_frog.m.

7. VECTOR CALCULUS AND TENSORS. downloads

   (a) Introduction and review

      i. Scalars, vectors, vector addition, scalar multiplication between a vector and a scalar (**Gr**§9.2, pp 412-414)

      ii. Two and three dimensional scalar fields (temperature) and vector fields (wind, heat flux).

      iii. Dot and cross products (**Gr**§14.2, pp 683-686)

      iv. Cartesian coordinates (**Gr**§14.3 to end of example 1, pp 687-690)

v. Einstein index notation and summation convention: sum over repeated indices:

$$\vec{a}\cdot\vec{b} = \sum_{i=1}^{3} a_i b_i \equiv a_i b_i; \quad (\underset{\sim}{A}\underset{\sim}{B})_{ik} = \sum_{j=1}^{3} A_{ij}B_{jk} \equiv A_{ij}B_{jk};$$

$$\vec{a} = \sum_{i=1}^{3} a_i \vec{e}_i \equiv a_i \vec{e}_i; \quad \vec{a}\cdot\vec{b} = (a_i\vec{e}_i)\cdot(b_j\vec{e}_j) = a_i b_j \vec{e}_i\cdot\vec{e}_j = a_i b_j \delta_{ij} = a_i b_i$$

vi. Kronecker delta tensor (**Gr** p 427, eqn 22)

vii. Levi-Civita tensor/ permutation symbol, cross produces, relation to Kronecker delta, to determinants, misc identities notes.

(b) Vector calculus: div $\nabla\cdot\vec{a}$, grad $\nabla\phi$, curl $\nabla\times\vec{a}$.

i. Divergence: definition using a general closed-surface integral over a vector field at the limit of the surface becoming infinitesimal. Derivation for a cube-like surface, and the differential operator. (**Gr**§16.3, pp 761-765, including example 1).

ii. Gradient: definition, input and output of div, grad, curl; directional derivative (**Gr**§16.4, pp 766-769 until but not including example 3).

iii. Providing physical intuition for grad and div: temperature field, heat flux vector field, define Laplacian as $div(grad(T)) = \nabla^2 T = T_{xx} + T_{yy} + T_{zz}$ (derive this). vector_calculus_preliminaries.m;

iv. Curl: definition as a cross product between ∇ and a vector field; physical interpretation via the vorticity of a flow field (**Gr**§16.5 p 774-775 to equation 4; for vorticity interpretation, use section 1 of notes_curl_and_vorticity.pdf).

v. Some vector identities (**Gr**§16.6, pp 778-780, until just before example 1; prove equations 4, 12, 13).

(c) Integral theorems: divergence, Stokes, Green's, potential of a vector field

i. Divergence (Gauss) theorem: theorem 16.8.1 $\int_V \nabla\cdot\vec{v}dV = \int_S \hat{n}\cdot\vec{v}dA$,

A. Proof for a rectangle domain, outline of a proof for general domain, relationship with fundamental theorem of integral calculus (**Gr**§16.8, pp 792-end of first paragraph on p 795).

B. Physical interpretation in terms of heat fluxes over the surface and integral over the local heating rate (example 3, p 799 - equation 40, p 801).

C. 2d version (which will be needed for proving Stokes theorem below): $\int_A \nabla\cdot\vec{v}dV = \int_C \hat{n}\cdot\vec{v}ds$ (equation 47, example 5, p 803).

ii. Stokes theorem: $\int_S \hat{n}\cdot\nabla\times\vec{v}dA = \oint_C \vec{v}\cdot d\vec{R}$.

A. Line integrals, interpretation as the work done by a force vector over a path (definition of line integral from **Gr**§16.9.1, eqns 2,3,4, p 810-811; for interpretation, start with constant force case, example 2 and Figs 177-178 in **Kr** p 373; proceed to variable force case **Kr** second half of p 423).

B. Example: circulation and aerodynamic lift of a wing, connection to Bernoulli law (eqns 15,16, to end of §16.9.1 on p 814, including Fig. 3), class demo of Bernoulli law.

C. Theorem 16.9.1 and proof for a flat (2d) surface (**Gr**§16.9.2, pp 814-815).

D. Geometric intuition (only qualitative): notes-stokes-theorem-intuition.pdf

E. Application 1: potential theory. • Consider first work done by friction as an example of path-dependent work. When is the work performed by a vector force path-independent? • Introduce vector fields that are gradients of a scalar field (**Kr** p 407, the single paragraph above theorem 3). • Three equivalent conditions for line integrals over a vector field to be path-independent: (1) the force may be expressed in terms of a potential, $\vec{F} = \nabla f$ where $f(x, y, x)$ is a scalar function; (2) $curl\vec{F} = \nabla \times \vec{F} = 0$; or (3) line integrals over a closed path vanish. Note that 2 and 3 are equivalent by Stokes theorem! (**Kr**§10.2, from p 426 to the top paragraph on p 429).

F. Application 2: fluid flow around a cylinder - only derivation of Laplace equation for the potential and for the stream function (notes, the solution is given below in the PDEs section).

iii. Green's theorem: $\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial Y}\right) dA = \oint_C P\,dx + Q\,dy$

A. This is simply a private case of stokes theorem, **Gr**§16.9.3 pp 818-819, Fig 9.

B. Application: calculating the area of an ellipse (**Kr**§10.4, p 442, example 2);

C. Application: area of a cardioid, (**Kr**§10.4, p 443, example 3). Animation demonstrating the construction of a cardioid from wikipedia.

D. Infinite number of cardioids in the Mandelbrot set: fractal zoom animation from youtube.

(d) Vector calculus in orthogonal curvilinear coordinates

i. Non-Cartesian coordinates

A. Plane polar coordinates, base vectors and expressions for derivatives of the base vectors. (First derivation: **Gr**§14.6.1 p 700-702, eqns 4a,4b,5,10,11,15,16,18a,b; second derivation: eqns 19a,b to end of page 703).

B. Cylindrical coordinates (first paragraph of **Gr**§14.6.2, p 704 and Fig 5 there)

C. Spherical coordinates (**Gr**§14.6.3, p 705-706, to end of eqns 28).

ii. ∇ in non Cartesian coordinates

A. Cylindrical coordinates (**Gr**§16.7.1 p 783 to end of p 785)

B. Example: curl of solid body rotation in cylindrical coordinates (section 2 of notes_curl_and_vorticity.pdf).

C. Spherical coordinates (**Gr**§16.7.2 eqns 27-33), derivations in HW and sections.

D. General curvilinear coordinates (time permitting, **Gr**§16.7.3, p 789-790)

- [Second midterm hour exam]

8. INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS. Including an introduction to both analytic and (briefly) numerical methods. downloads

   (a) Introduction, basics

      i. Basic examples, general solution involving arbitrary functions rather than the arbitrary constants in ODE case (**Kr**§12.1, everything to the end of example 3, pp 535-537).

      ii. Classification of linear second order PDEs in two independent variables (**Gr**§18.2.2, p 946-948 to end of example 3): $Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = f$, (coefficients may be functions of independent variables). Consider the prototypical PDEs and the related physical problems,

         A. $B^2 - AC = 0$: parabolic, example: diffusion equation: $u_t = \alpha^2 u_{xx}$,

         B. $B^2 - AC > 0$: hyperbolic, example: wave equation: $c^2 u_{xx} = u_{tt}$,

         C. $B^2 - AC < 0$: elliptic, example: Laplace equation: $u_{xx} + u_{yy} = 0$.

      iii. Boundary conditions: Dirichlet (u prescribed), Neumann ($\frac{\partial u}{\partial n}$ prescribed), and mixed (**Gr** p 951-952).

   (b) Diffusion equation $u_t = \kappa u_{xx}$

      i. Derivation of the one dimensional time dependent diffusion equation (using the divergence theorem: **Kr**§10.8 example 2, pp 464-465; using the heat budget of a one dimensional infinitesimal element: see notes or (not so good in this case) **Gr**§18.2.3, p 948-949)

      ii. Solving the time dependent 1d diffusion problems using separation of variables

         A. **Gr**§18.3.1, example 1: from eqn 1a,b,c to 28, pp 954-958, and then comment 1, p 959 interpreting the first term in eqn 26 as the steady state solution.

         B. Demonstrate behavior of time dependent solution using diffusion_1d_SL.m

         C. The physics of Cappuccino: use above Matlab script (icase of 4,5) to discuss scale selective character of diffusion. In-class demonstration of stirring and mixing.

         D. Connection to S-L: Note the need to first solve for and subtract the steady solution in order for the resulting S-L problem to be homogeneous as required.

         E. Need to apply the b.c. before the initial conditions, because b.c. lead to S-L problem, which can then be used to expand arbitrary i.c. (**Gr** p 960, comment 4).

         F. Note that the final solution is not a function of time times a function of space, although each term in the expansion is. The form $X(x)T(t)$ would be too restrictive as the initial spatial structure would not be able to change expect overall time dependent scale (comment 6).

G. Verification of the solution (**Gr**§18.3.2, only eqns 49a,b,c and the following single sentence).

H. Example 4, p 967: in sections.

iii. Heat equation on a disk, in cylindrical coordinates (**Gr**§18.3.3, example 5, pp 968-971, including comments 1,2; may need to also use example 1 in section **Gr**§4.6.6 p 238 to show the transformation to the standard Bessel form, and perhaps example 2 in **Gr**§17.8 p 908 for the needed S-L expansion using Bessel functions). Demonstrate behavior of time dependent solution using diffusion_2d_disk_SL.m.

iv. Uniqueness of solutions to Laplace and heat equations (**Gr** exercise 25, p 979). Physically: why the solution would not be unique with flux (Neumann) boundary conditions ($\frac{\partial u}{\partial n}$ prescribed on boundaries).

v. When separation of variables cannot be used: an example of a non separable equation: $\nabla^4 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$, and where the boundary conditions are not separable (**Gr** p 972).

vi. Numerical solution of the diffusion equation in 1d and 2d

A. Euler forward in time, center difference in space, including a (possibly moving) source. Show both 1d and 2d schemes, and demonstrate using diffusion_1d_numerical.m, diffusion_2d_numerical.m.

(c) Wave equation $u_{tt} = c^2 u_{xx}$

i. Stadium waves, derivation of equation. links

ii. (In sections: Derivation of wave equation for a string $u_{tt} = c^2 u_{xx}$ (**Gr**§19.1, pp 1017-1019; or **Kr**§12.2, pp 538-9).)

iii. State without derivation the two spatial dimensions version: $u_{tt} = c^2 \nabla^2 u$.

iv. Separation of variables: factoring the wave equation into two first order PDEs (**Gr**§19.2.1, p 1023-1024, to Eqn 8).

v. Complete the full solution of the plucked string wave problem using Fourier series. Calculate coefficients in Fourier series based on boundary and initial conditions (**Gr**§19.2.1, pp 1024-1026 to Eqn 20).

vi. Traveling wave interpretation (**Gr**§19.2.2 to Eqn 30).

vii. All steps of the solution of the plucked string problem may also be found in a more concise form in **Kr**§12.3, pp 540-544).

viii. Numerical solution of wave equation, compare series and numerical solutions (plucked_string_waves_numerical.m, plucked_string_waves_analytic.m).

ix. Time permitting/ in section: wave equation for shallow water (misc downloads and notes)

x. D'Alembert's solution to the 1d wave equation, general solution of the initial value problem (**Gr**§19.4.1 eqns 1-7 and then example 1, pp 1043-1048).

xi. Vibration modes of a rectangular membrane (**Gr**§19.3 p 1035-1039, end of comment 2); animation of modes.

(d) Laplace's equation $\nabla^2 u = u_{xx} + u_{yy} = 0$ (**Gr** chapter 20, p 1058)

i. Motivation: diffusion equation in 2d at steady state with or without a distributed specified heat source (**Gr**§20.1)

ii. Separation of variables, choosing the "right" sign for the separation variable, superposition (**Gr**§20.1, example 1, skip comment 1, include comment 2)

iii. Choosing the right sign of the separation variable, discussion after comment 3 on p 1062. see also discussion in the closure note on p 1066 for the case in which there are nonhomogeneous b.c. which require expansion at more than one side of the domain.

iv. Just mention briefly: having Newman b.c. (derivative specified) on all sides can lead to no solution: physically this just means that the prescribed heat flux into the domain must sum to zero for a steady state. Exercises 17,18.

v. Laplace's equation in cylindrical geometries $(u_{zz}+)\, u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ (**Gr**§20.3, p 1070)

   A. Definitions of plane polar coordinates ($\partial/\partial z = 0$) vs cylindrical coordinates (contrast §20.3.1 eqn 1 with §20.3.2, eqn 37). We shall refer to both as cylindrical.

   B. The beginning of example 1, p 1070-1071: Laplace equation on a partial section of a disk, only equations (1,2a on full disk,3-9) until the end of the derivation of the general solution.

   C. Example 2: Dirichlet (specified boundary values) on a full disk, using boundedness and periodicity instead of some of the b.c. (p 1073-1076 to eqn 34).

   D. The maximum principle of the Laplace equation. Also, the average value property. Physical interpretation of the maximumprinciple in terms of temperature distribution and diffusive heat fluxes (comment 1 on p 1075-1076, and then text from end of comment 2 to end of §20.3.1).

   E. A brief comment: in actual 3d cylindrical coordinates with z dependence, the separation of variables leads to the Bessel equation in r because of the u_{zz} term.

vi. Solution of 2d flow around a cylinder. Equations were derived previously during the vector calculus section, see notes there, and see also the solution from wikipedia there, for both the potential and the stream function (Wikipedia or local copy)

vii. Laplace's equation in spherical coordinates and Legendre polynomials (**Gr**§20.3.3, p 1081-1083);

   A. Coordinates are (ρ, ϕ, θ), where θ is longitude-like, and ϕ latitude-like, except that it vanishes at the north pole. Laplace equation is

$$\nabla^2 u = \frac{1}{\rho^2}\left[\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}\right] = 0$$

B. Considering heat diffusion on a spherical earth, forced by solar radiation, cooled by long-wave radiation. Assume “zonally-symmetric”: no dependence on longitude θ, and also domain limited to the surface itself, so no dependence on radius. (notes)

9. DYNAMICAL SYSTEMS, NONLINEAR DYNAMICS AND CHAOS. downloads

   (a) Nonlinear 1st order odes: dynamical systems perspective

      i. Reminder: geometric approach to 1d systems.
      ii. What’s a bifurcation; Saddle node bifurcation (**St**§3.1, p 45-47, to end of example 3.1.1).
      iii. Example: global climate energy balance and snowball earth, hysteresis (notes).
      iv. Transcritical bifurcation, super critical and sub critical (**St**§3.2).
      v. Pitchfork bifurcation: super-critical and sub-critical. Bead on a rotating hoop, higher order nonlinear terms and hysteresis (**St**§3.4).

   (b) Two dimensional systems, phase plane

      i. Linear systems, classification of fixed points, stable and unstable spaces (**St**§5.1: example 5.1.2 p 126-128; §5.2 p 129- eqn 6 on p 131).
      ii. (Time permitting:) Hopf bifurcation, sub- and super-critical (**St**§8.2 p 248-253; image and Hopf_bifurcation.m).
      iii. Nonlinear systems, phase plan (**St**§6.1 p 145-146); linearization (**St**§6.3 p 150-eqn 2 on p 151).
      iv. Example: rabbits and sheep (**St**§6.4 p 155-159).
      v. Limit cycles (**St** Fig. 7.0.1 p 196, Example 7.1.1 p 197-198).

   (c) Chaos

      i. Introduction, history: Poincare, Lorenz.
      ii. Brief mention of Poincare-Bendixson theorem: need at least three dimensions for chaos (**St**§7.3 p 203-204).
      iii. Lorenz equations (motivation: water wheel, **St**§9.1, p 302-304 and Figs 9.1.1,2,3; also: convection); contraction of phase volumes (p 312-313); briefly: dissipative vs conservative systems, Hamiltonian chaos; character of the solution time series and in phase space, sensitivity to initial conditions java applet; limited predictability; (**St**§9.3 p 317-end of example 9.3.1 p 323)
      iv. Definition of chaos, strange attractors (p 323-324).
      v. Fractals: cantor set (**St**§11.2 p 401-402), von Koch curve (Fig 11.3.1 p 405), fractal box dimension (**St**§11.4 p 409-410 including dimensions of cantor set, von Koch curve and example 11.4.2).
      vi. Poincare sections, maps vs continuous dynamical systems, why maps can display such rich behavior (**St**§8.7 p 278 - to upper paragraph of p 278)

vii. Chaos in the logistic map; transition to chaos via the period doubling route to chaos in the logistic map (**St**§10.2 p 353-357).

10. REVIEW