EPS131, Introduction to Physical Oceanography and Climate

Section 3: Horizontal Circulation I: The Coriolis Force

Dept of Earth and Planetary Sciences, Harvard University

Eli Tziperman

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Contents

1 Geostrophy - introduction

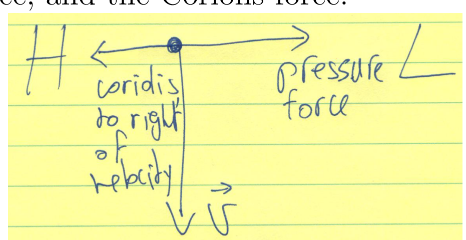
Geostrophy is an approximation of the horizontal momentum balance represented by $\mathbf{F} = m\mathbf{a}$ where the LHS is the force and $\mathbf{a}$ is the (vector) acceleration. The full momentum balance for a fluid element in the ocean is given by

$$m\mathbf{a} = \text{pressure force} + \text{Coriolis} + \text{gravity} + \text{friction}.$$

At a steady state, currents do not accelerate, and therefore $\mathbf{a} = \partial \mathbf{u}/\partial t = 0$ (more formally, we also need to assume that currents are not too strong so that nonlinearities can be neglected, more on that later). Gravity is acting in the vertical direction and can be ignored when dealing with the horizontal direction, and friction is weak in most of the ocean, leaving us with the geostrophic balance,

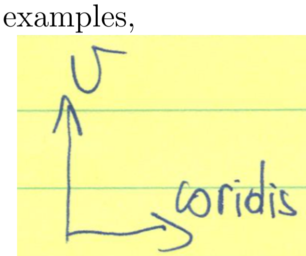
$$0 = \text{pressure force} + \text{Coriolis}.$$

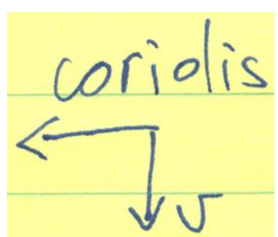
Here is an example of this balance, showing a high pressure, low pressure, the direction of the velocity, the pressure force, and the Coriolis force:



2 Geostrophy - derivation of equations

From the following two examples,





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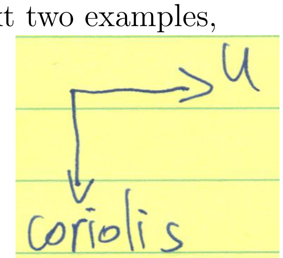
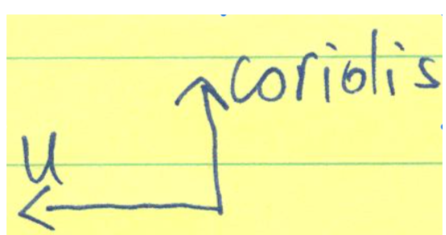
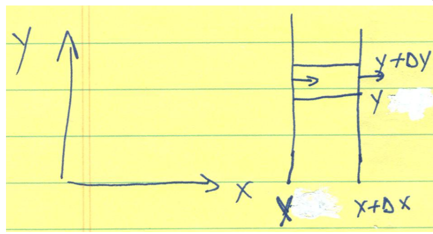
we deduce that the Coriolis force in the x-direction (eastward) is given by $\Delta x \Delta y \Delta z \rho f v$, while from the next two examples,





we deduce that the Coriolis force in the y-direction (northward) is given by $-\Delta x \Delta y \Delta z \rho f u$.

The pressure force in the x direction is derived considering a slice of fluid,



Noting that the net forced on the slice is given by the difference in the pressure force applied to both of its sides,

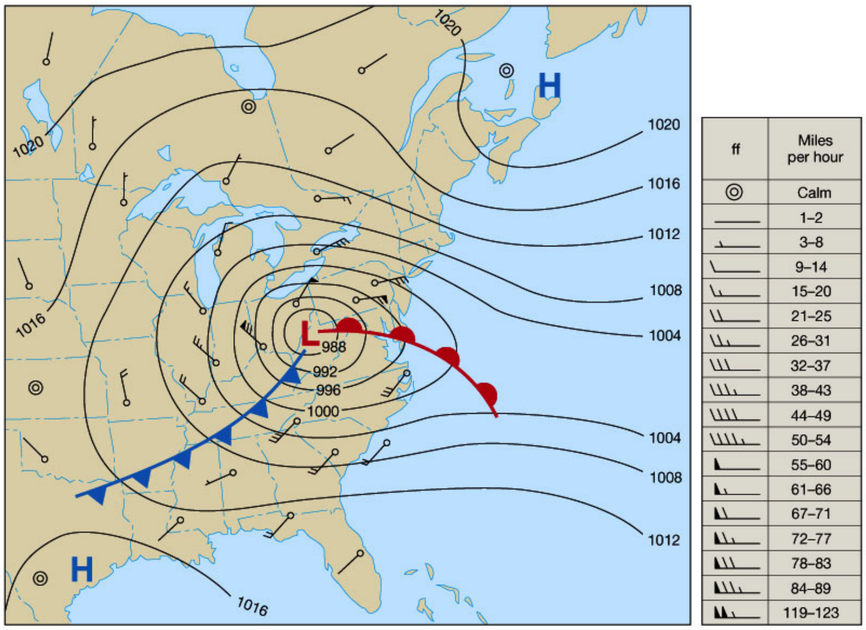
$$\text{pressure force} = p(x)\Delta y \Delta z - p(x+\Delta x)\Delta y \Delta z$$
$$= -\frac{\partial p}{\partial x}\Delta x \Delta y \Delta z.$$

Combining the Coriolis and pressure forces in the x direction derived above, and then repeating in the y direction, moving the Coriolis force to the LHS, and dividing by the density, we find the geostrophic equations, ★★★

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad x \text{ (east) direction}$$
$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad y \text{ (north) direction}$$

3 Example of geostrophic balance: weather map

Consider the relationship between the pressure low on a weather map and its corresponding wind velocity.



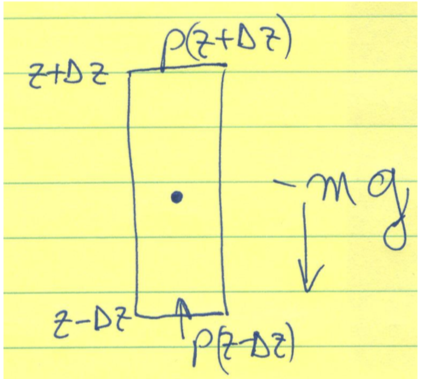
Our objective is to calculate the wind just west of the low center, on the 992 hPa contour. Start from the geostrophic equation, ★★★

$$
\begin{aligned}
v &= \frac{1}{f\rho}\frac{\partial p}{\partial x} \approx \frac{1}{f\rho}\frac{\Delta p}{\Delta x} \\
&\approx -\frac{1}{10^{-4}s^{-1} \times 1kg/m^3}\frac{1000 - 988\, mb}{250\, Miles} \\
&= -\frac{1}{10^{-4}s^{-1} \times 1kg/m^3}\frac{1200\, pascals}{400,000\, m} \\
&= -\frac{1}{10^{-4}s^{-1} \times 1kg/m^3}\frac{1200\, kg\, m^{-1}\, s^{-2}}{400,000\, m} \\
&= 30m/s = 67\, \text{miles per hour.}
\end{aligned}
$$

This is a reasonable estimate compared with the wind velocity shown on the weather map. We ignored friction, which does play a significant role near the surface, introducing the angle between velocity directions and pressure contours on the map, as we will see later.

4 The hydrostatic balance

Consider the vertical component of the momentum balance $F = ma$, acting on a fluid element in a resting ocean. The relevant forces are gravity and pressure, as follows



The *upward* gravity force is $-mg = -\Delta x \Delta y (2\Delta z)\rho g$. The net *upward* pressure force on the fluid element is that acting from below minus that acting from above,

$$\begin{aligned}\text{net upward pressure force} &= [p(z-\Delta z) - p(z+\Delta z)]\Delta x \Delta y \\ &= \left[\left(p(z) + \frac{\partial p}{\partial z}(-\Delta z)\right) - \left(p(z) + \frac{\partial p}{\partial z}(\Delta z)\right)\right]\Delta x \Delta y \\ &= \frac{\partial p}{\partial z}(-2\Delta z)\Delta x \Delta y.\end{aligned}$$

At a steady state, the acceleration vanishes, and the sum of forces must vanish so that

$$0 = \text{pressure force} + \text{gravity},$$

or

$$\frac{\partial p}{\partial z} = -\rho g,$$

which is the hydrostatic balance.

5 Boussinesq approximation: why is it OK to replace ρ by ρ_0 in $-\frac{1}{\rho_0}\nabla p$ but not in $-g\rho$?

Density varies in the ocean roughly in the range $1024 < \rho < 1045$, so it seems that replacing it always with a constant may not be a bad approximation.

However, write the density as a sum of a constant reference density, a horizontal average of the remaining density, and a variable part,

$$\rho(x, y, z, t) = \rho_0 + \bar{\rho}(z) + \rho'(x, y, z, t),$$

where

$$\begin{aligned} \rho_0 &= 1025 \\ \bar{\rho}(z) &= \frac{1}{\text{area}} \int\int (\rho(x, y, z, t) - \rho_0)\, dx\, dy. \end{aligned}$$

Next, use the hydrostatic equation

$$p_z = -g\rho$$

to divide the pressure as well into two components that satisfy,

$$\begin{aligned} \frac{\partial}{\partial z}\overline{p} &= -g(\rho_0 + \bar{\rho}(z)), \\ \frac{\partial}{\partial z}p' &= -g\rho' \end{aligned}$$

The sum of these two equations is the original hydrostatic equation. The first of these can be solved, resulting in,

$$\overline{p}(z) = p_a + \int_z^\eta g(\rho_0 + \bar{\rho}(z'))\, dz'$$

where $z = 0$ is the ocean surface when there are no currents, $z = \eta(x, y)$ is the sea surface height when there are currents, and p_a is the atmospheric pressure at the ocean surface. Note that in the last equation, when $z = \eta$, the integral vanishes, and the pressure is equal to the atmospheric one, p_a, as expected.

Note that $\overline{p}(z)$ is a function of z only. As a result, it does not contribute to the horizontal momentum equations,

$$\nabla_h p = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) p = \nabla_h(\overline{p} + p') = \nabla_h p'.$$

The "dynamical" pressure $p'(x, y, z)$ may, therefore, be smaller than the hydrostatic reference pressure $\overline{p}(z)$, but it is the only part of the pressure that drives the horizontal momentum equations and the horizontal velocities. It must, therefore, be retained. If we neglected the perturbation density ρ' in the hydrostatic equation, the dynamical pressure would also vanish. This implies that the density perturbation must be kept in the hydrostatic equation.

On the other hand, when we calculate the pressure gradient term, replacing the density with the reference density involves a very small correction, so that we can write $-\frac{1}{\rho}\nabla p \approx -\frac{1}{\rho_0}\nabla p$.

We can now drop the primes, so that p and ρ refer to p' and ρ', so that our geostrophic and hydrostatic momentum equations become,

$$
\begin{aligned}
-fv &= -\frac{1}{\rho_0}p_x \\
fu &= -\frac{1}{\rho_0}p_y \\
p_z &= -g\rho.
\end{aligned}
$$

Neglecting density variations except when they affect the vertical momentum balance is referred to as the "Boussinesq approximation".

6 Sea level vs stratification in a geostrophic flow such as the Gulf Stream

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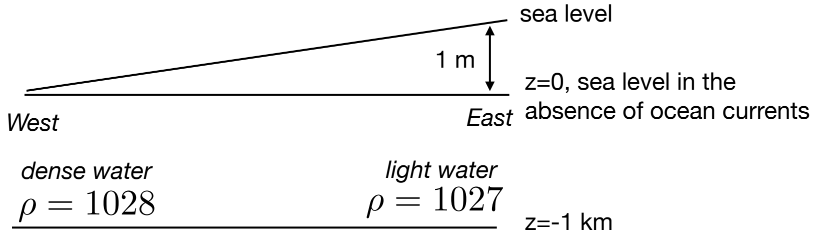
Consider an east-west section across the northward-flowing Gulf Stream, and estimate first the effect of sea-level variations across the stream vs. the effect of density variations, evaluated at a depth of 1 km.



First, evaluate the effect of sea level while assuming that the density is constant, $p(x) = \rho_0 g h(x)$ (the pressure at the surface is assumed zero). This implies that the pressure difference across a horizontal distance Δx across the Gulf Stream is equal to $\Delta p = \rho_0 g \Delta h$. The pressure difference across the Gulf stream is, therefore,

$$\Delta x \frac{\partial p}{\partial x} = \Delta p = \Delta x \rho_0 g \frac{h_2 - h_1}{\Delta x} = \rho_0 \times g \times (1\text{ m}) = 10^4\text{ Pa}.$$

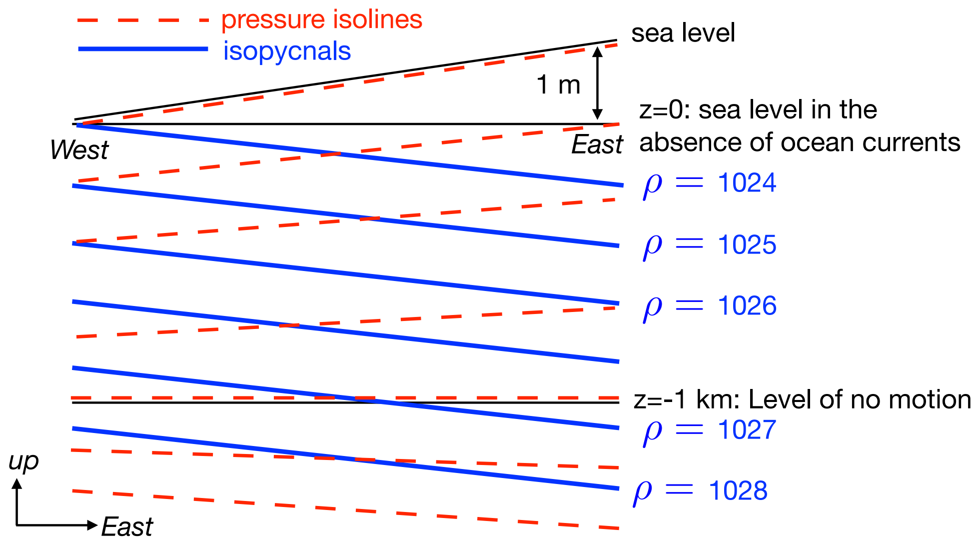
Here, $\Delta h = h_2 - h_1$, and h_1, h_2 are the sea surface heights at the two horizontal locations on two sides of the Gulf Stream.

As for the effects of density variations, evaluated again at a depth of 1 km, using $p(x, z = H) = -\int_0^H \rho(x,z) g\, dz = \int_H^0 \rho(x,z) g\, dz$, and assuming the density is only a function of horizontal location, not of depth. The difference in pressure across the two sides of the Gulf Stream is now the difference between two such vertical integrals, or $\Delta p(z = H) = \int_H^0 \Delta\rho(x,z) g\, dz$. This may be estimated as,

$$\Delta x \frac{\partial p}{\partial x} = \Delta p = \Delta\rho \times g \times H = (1028 - 1027) \times g \times (1\text{ km}) = 10^4\text{ Pa},$$

so that the effects of sea level and density variations on the pressure difference across the two sides of the stream are comparable.

Next, what do these effects look like as a function of depth in a stratified Gulf Stream? Consider the following schematic,



which shows how the density differences gradually weaken and eventually reverse the east-west pressure gradient set near the surface by the sea level height difference. We, therefore, expect the flow, in this case, to be northward in the upper km and southward below.

7 Thermal wind equations and the level of no motion

So far, we derived and discussed the geostrophic and hydrostatic balances, together representing the three-dimensional momentum balance $F = ma$ for a fluid element in the ocean,

$$
\begin{aligned}
-fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
\frac{\partial p}{\partial z} &= -g\rho
\end{aligned}
$$

We now wish to derive equations that allow us to calculate currents from observed temperature and salinity in the ocean. Start by taking the vertical derivative of the first two momentum equations and then using the third,

$$
\begin{aligned}
-f\frac{\partial}{\partial z}v &= -\frac{1}{\rho_0}\frac{\partial}{\partial z}\frac{\partial p}{\partial x} = -\frac{1}{\rho_0}\frac{\partial}{\partial x}\frac{\partial p}{\partial z} = -\frac{1}{\rho_0}\frac{\partial}{\partial x}(-g\rho) \\
f\frac{\partial}{\partial z}u &= -\frac{1}{\rho_0}\frac{\partial}{\partial z}\frac{\partial p}{\partial y} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}\frac{\partial p}{\partial z} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}(-g\rho)
\end{aligned}
$$

which we can then write as the final *thermal wind equations*,

$$
\begin{aligned}
f\frac{\partial v}{\partial z} &= -\frac{g}{\rho_0}\frac{\partial \rho}{\partial x} \\
f\frac{\partial u}{\partial z} &= \frac{g}{\rho_0}\frac{\partial \rho}{\partial y}.
\end{aligned}
$$

Why are these useful? Integrating these equations from some level $z = -H$, which we term the "reference level," we find,

$$
\begin{aligned}
\int_{-H}^{z}\frac{\partial v}{\partial z}dz = v(z) - v(-H) &= \int_{-H}^{z}\frac{-g}{f\rho_0}\frac{\partial \rho}{\partial x}dz \\
\int_{-H}^{z}\frac{\partial u}{\partial z}dz = u(z) - u(-H) &= \int_{-H}^{z}\frac{g}{f\rho_0}\frac{\partial \rho}{\partial y}dz.
\end{aligned}
$$

or,

$$
\begin{aligned}
v(x,y,z) &= v(x,y,-H) + \int_{-H}^{z}\frac{-g}{f\rho_0}\frac{\partial \rho}{\partial x}dz \\
u(x,y,z) &= u(x,y,-H) + \int_{-H}^{z}\frac{g}{f\rho_0}\frac{\partial \rho}{\partial y}dz.
\end{aligned}
$$

Now, if we know the temperature and salinity (from ship or float observations), we can calculate the density ρ, and evaluate the integrals on the RHS. If we also know the reference level velocities $v(x, y, -H), u(x, y, -H)$ or can find a level at which they may be assumed to vanish, then the above equations can be used to calculate the velocity at all levels!

Why couldn't we calculate the velocities based on the geostrophic equations by calculating $\partial p/\partial x$ and $\partial p/\partial y$? Because the pressure depends on both the density and sea surface height, which was not known before the satellite age.

If we do know the sea surface height $h(x, y)$ from satellite altimetry we can find the velocity at any depth in the ocean by using $z = 0$ as our reference level. That is, set $H = 0$. At that depth, the pressure is simply given by $p(x, y, z = 0) = p_a + g\rho_0 h(x, y)$, where $p(z = h) = p_a$ is the atmospheric sea level pressure with we can take to be a constant. To see that, integrate the hydrostatic equation from $z = 0$ to the SSH h,

$$\int_0^h \frac{\partial p}{\partial z} dz' = -\int_0^h g\rho dz' \approx -\int_0^h g\rho_0 dz' = -g\rho_0 h$$

or

$$\int_0^h \frac{\partial p}{\partial z} dz' = p|_{z=0}^h = p_a - p(z) = -g\rho_0 h,$$

which results in $p(x, y, z = 0) = p_a + g\rho_0 h(x, y)$. The surface velocities at $z = 0$ are then given by geostrophy $fu(z = 0) = -(\partial p/\partial y)/\rho_0 = -g\partial h/\partial y$. Writing this for both directions,

$$\begin{aligned} -fv(z = 0) &= -g\frac{\partial h}{\partial x} \\ fu(z = 0) &= -g\frac{\partial h}{\partial y} \end{aligned} \tag{1}$$

We can therefore write the velocities at some arbitrary depth z in the ocean as functions of the density and sea surface height,

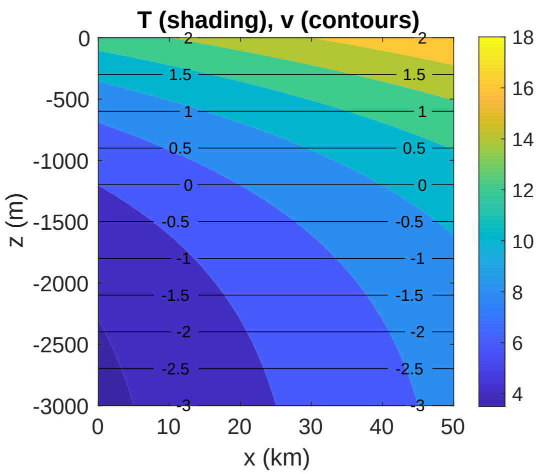
$$\begin{aligned} v(x, y, z) &= \frac{g}{f}\frac{\partial h}{\partial x} - \int_0^z \frac{g}{f\rho_0}\frac{\partial \rho}{\partial x} dz \\ u(x, y, z) &= -\frac{g}{f}\frac{\partial h}{\partial y} + \int_0^z \frac{g}{f\rho_0}\frac{\partial \rho}{\partial y} dz. \end{aligned} \tag{2}$$

7.1 Example

Suppose the temperature and sea surface height are given by,

$$T(x,z) = T_0 + \Delta T(e^{z/H} + \delta \frac{x}{L}),$$
$$h(x) = a\frac{x}{L},$$

where L is the horizontal size of the domain. The following figure shows the temperature to be increasing with x, indicating that ΔT is positive.



Use $\Delta\rho = -\alpha\rho_0\Delta T$, and the northward velocity is therefore given via (2) by

$$v(x,z) = \frac{ag}{fL} + \frac{g\alpha\Delta T\delta}{fL}z.$$

This means that the level of no motion is at

$$z = -\frac{a}{\delta\alpha\Delta T}.$$

8 Dynamic height/topography derived from geostrophy

Start with geostrophy,

$$u(x,y,z) = -\frac{1}{f\rho_0}\frac{\partial p}{\partial y}$$
$$v(x,y,z) = \frac{1}{f\rho_0}\frac{\partial p}{\partial x}$$

In pressure coordinates,

$$\left.\frac{\partial p}{\partial x}\right|_z = \frac{\partial p}{\partial z}\left.\frac{\partial z}{\partial x}\right|_p = -g\rho\left.\frac{\partial z}{\partial x}\right|_p = -\rho\left.\frac{\partial \Phi}{\partial x}\right|_p$$

where $d\Phi \equiv g\,dz$ is the geopotential height, such that

$$\frac{\partial \Phi}{\partial p} = g\frac{\partial z}{\partial p} = g/p_z = -1/\rho.$$

Writing the geostrophic equations in pressure coordinates,

$$u(x,y,p) = -\frac{1}{f\rho_0}\rho\frac{\partial \Phi}{\partial y} \approx -\frac{1}{f}\frac{\partial \Phi}{\partial y}$$
$$v(x,y,p) = \frac{1}{f\rho_0}\rho\frac{\partial \Phi}{\partial x} \approx \frac{1}{f}\frac{\partial \Phi}{\partial x}.$$

Taking the derivative with respect to pressure to find the thermal wind equations in pressure coordinates,

$$\frac{\partial}{\partial p}u(x,y,p) = -\frac{1}{f}\frac{\partial}{\partial y}\frac{\partial \Phi}{\partial p} = \frac{1}{f}\left.\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\right|_p$$
$$\frac{\partial}{\partial p}v(x,y,p) = \frac{1}{f}\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial p} = -\frac{1}{f}\left.\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right|_p$$

Integrate these equations between two pressure levels and take the horizontal derivatives out of the integral (can do this because they are taken at constant pressure),

$$u(x,y,p_2) - u(x,y,p_1) = \frac{1}{f}\frac{\partial}{\partial y}\int_{p_1}^{p_2}\left(\frac{1}{\rho}\right)dp$$
$$v(x,y,p_2) - v(x,y,p_1) = -\frac{1}{f}\frac{\partial}{\partial x}\int_{p_1}^{p_2}\left(\frac{1}{\rho}\right)dp.$$

Let $\alpha \equiv 1/\rho$, and $\delta(S,T,p) \equiv \alpha(S,T,p) - \alpha(35,0,p)$, we define "dynamic height" difference between two pressure levels as

$$\Delta D = \int_{p_1}^{p_2} \delta \, dp,$$

which satisfies,

$$u(x,y,p_2) - u(x,y,p_1) = \frac{1}{f}\frac{\partial}{\partial y}\Delta D$$
$$v(x,y,p_2) - v(x,y,p_1) = -\frac{1}{f}\frac{\partial}{\partial x}\Delta D.$$

This shows that the dynamic height is a stream function for the geostrophic velocities. Numerous oceanographic papers, therefore, analyzed dynamical height maps between any number of pairs of pressure levels, especially assuming that one of the two levels is a level of no motion, where the velocities vanish.