EPS131, Introduction to Physical Oceanography and Climate

Section 4: waves, part I

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Contents

1 Inertial oscillations

Inertial oscillations are circular motions of water parcels in the upper ocean occurring in response to a passing storm. We can understand them using an approximation of the momentum balance $F = ma$,

acceleration=pressure gradient+Coriolis+friction+gravity

which neglects all terms except for

acceleration=Coriolis.

The equations are, ★★★

$$
\begin{aligned}
u_t - fv &= 0 \\
v_t + fu &= 0.
\end{aligned}
$$

Substitute the second equation into the first,

$$v_{tt} = -f^2 v,$$

and try exponential solution $v = e^{at}$ to find $a^2 = -f^2$ or $a = \pm if$. The solution is, therefore,

$$v = A' e^{ift} + B' e^{-ift},$$

or, equivalently,

$$
\begin{aligned}
v &= A'(\cos(ft) + i\sin(ft)) + B'(\cos(ft) - i\sin(ft)) \\
&= (A' + B')\cos(ft) + i(A' - B')\sin(ft).
\end{aligned}
$$

Letting $B = (A' + B')$ and $A = i(A' - B')$ this may be written as,

$$v = A\sin(ft) + B\cos(ft).$$

Using $u = -v_t/f$, we therefore also find that

$$u = -A\cos(ft) + B\sin(ft).$$

Now consider specific initial conditions of $v(0) = v_0$ and $u(0) = 0$ to solve for the constants: $A = 0$, $B = v_0$ so that,

$$
\begin{aligned}
v &= v_0\cos(ft) \\
u &= v_0\sin(ft).
\end{aligned}
$$

These are oscillations! What does the trajectory of a fluid particle look like? Let its coordinates be x, y and they satisfy $dx/dt = u$, $dy/dt = v$, or

$$\frac{dx}{dt} = v_0 \sin(ft)$$
$$\frac{dy}{dt} = v_0 \cos(ft).$$

Given initial conditions of $x(0) = x_0$ and $y(0) = y_0$, these equations are integrated [hint: $\int_0^t \frac{dx}{dt} dt = x(t) - x(0)$ and $\int_0^t \sin(ft)dt = -\frac{1}{f}(\cos(ft) - \cos(0))$] to find,

$$x(t) = x_0 + \frac{v_0}{f}(1 - \cos(ft))$$
$$y(t) = y_0 + \frac{v_0}{f}\sin(ft).$$

We can substitute these solutions in the above equations to verify that both equations and initial conditions are indeed satisfied. Note that,

$$(x - (x_0 + r))^2 + (y - y_0)^2 = r^2 = \text{constant}$$
$$r \equiv \frac{v_0}{f}$$

which is the equation for circular motion with a radius $r = v_0/f$. The circle center is at $(x, y) = (x_0 + r, y_0)$, to the right of the initial point for $f > 0$. The larger the initial velocity (excited by the passage of some storm, say), the larger the radius of motion. The frequency of the oscillation/ circular motion is given by $f = 2\Omega \sin\theta$.

Discussion points: rationalizing the location of the center of the circular trajectory; radius as a function of initial velocity and Coriolis parameter; the period of inertial oscillations (in days) at 30N? 20N? 40N?

2 Wave basics

2.1 Definitions

Consider a wave solution for surface elevation,

$$\eta(x, y, t) = \eta_0 \cos(kx + ly - \omega t),$$

and we have,

- wave vector: $\vec{k} = (k, l)$
- wavelength: $\lambda = 2\pi/\sqrt{k^2 + l^2}$ (distance between crests at a fixed time)

- period: $T = 2\pi/\omega$ (time between crests at a fixed location)
- amplitude: η_0

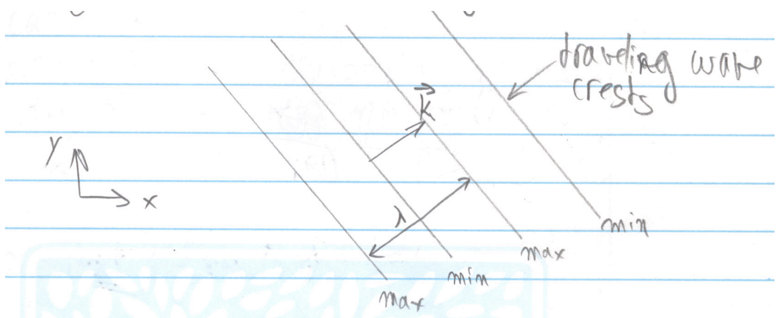


Figure 1:

To see that this wavelength is given by $\lambda = 2\pi/(k^2 + l^2)^{1/2}$, consider the following figure,

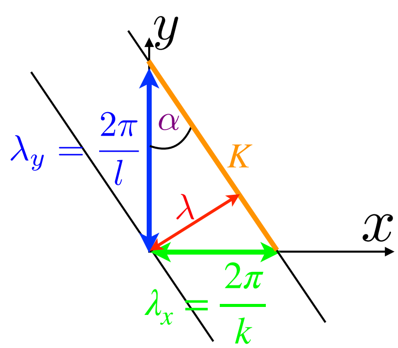


Figure 2:

Use $\sin\alpha = \lambda/\lambda_y = \lambda_x/K$, together with $K = \sqrt{\lambda_x^2 + \lambda_y^2}$ to find that

$$\lambda = \lambda_y \sin\alpha = \frac{\lambda_x \lambda_y}{(\lambda_x^2 + \lambda_y^2)^{1/2}}$$

substituting $\lambda_x = 2\pi/k$, $\lambda_x = 2\pi/l$, we get the desired expression,

$$\lambda = \frac{2\pi}{(k^2 + l^2)^{1/2}}.$$

2.2 Phase velocity

Phase speed is the speed of crests. Consider a fixed y and suppose a given crest is at $x = x_0$ at time t_0 and at $x = x_1$ at time t_1. This implies,

$$\eta(x_0, y, t_0) = \eta(x_1, y, t_1),$$

so that,

$$\eta_0 \cos(kx_0 + ly - \omega t_0) = \eta_0 \cos(kx_1 + ly - \omega t_1).$$

This implies,

$$kx_0 - \omega t_0 = kx_1 - \omega t_1,$$

and therefore that

$$\frac{x_1 - x_0}{t_1 - t_0} = \frac{\omega}{k}.$$

Let $t_1 \to t_0$ and then the LHS is the speed of propagation of the crest, so we found that

$$c_{ph} = \frac{\omega}{k}.$$

We can express this phase velocity in terms of the wavelength λ and period T.

$$c_{ph} = \frac{\omega}{k} = \frac{2\pi/T}{2\pi/\lambda} = \frac{\lambda}{T}.$$

This makes sense, as a wave crest travels a distance equal to the wavelength during one period. In two dimensions, the phase velocity is

$$(c_{ph}^{(x)}, c_{ph}^{(y)}) = \left(\frac{\omega}{k}, \frac{\omega}{l}\right),$$

note that this is *not* a vector.

2.3 Group velocity

Consider two waves of similar (k, ω) traveling together,

$$(k - \delta k, \omega - \delta\omega)$$
$$(k + \delta k, \omega + \delta\omega).$$

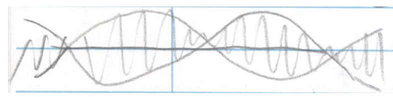


Figure 3:

The surface elevation is then given by

$$\begin{aligned}\eta &= \cos\left[(k-\delta k)x-(\omega-\delta\omega)t\right]+\cos\left[(k+\delta k)x-(\omega+\delta\omega)t\right]\\ &= 2\cos\left[\delta k\, x-\delta\omega\, t\right]\cos\left[kx-\omega t\right],\end{aligned}$$

where we have used

$$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

The phase speed of the envelope, which is the velocity of energy propagation and is termed the group velocity, is,

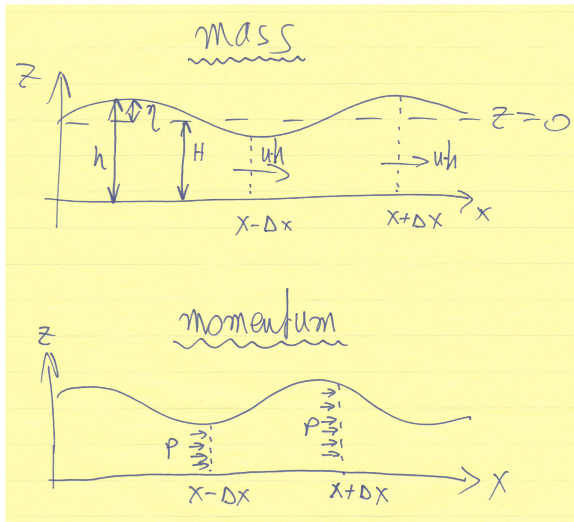
$$\frac{\delta\omega}{\delta k}=\frac{\partial\omega}{\partial k}.$$

In two dimensions, the wave number is a vector, $\vec{k}=(k,l)$, and so is the group velocity,

$$\vec{\mathbf{c}}_g=(c_g^{(x)},c_g^{(y)})=\left(\frac{\partial\omega}{\partial k},\frac{\partial\omega}{\partial l}\right).$$

3 Surface gravity waves without rotation

Our first objective is to find the dispersion relation $\omega(k)$, the relation between wavelength and period, which also tells us about the wave phase and group speeds. We'll derive the dispersion relation for shallow water waves for which the wavelength is much larger than ocean depth; this is relevant for both beach waves and tsunamis. We will also discuss scaling arguments for deriving these relationships for both deep and shallow water waves.



3.1 Shallow water waves scaling argument/ dimensional analysis

We first deduce the dispersion relation merely based on dimensional arguments, without using the equations of motion. The relevant dimensional constants are: H for depth; g for gravity; ω for frequency (2π/period); k is the wavenumber (2π/wavelength). Try writing the phase speed $c = \omega/k$ as function of H, g, ★★★

$$
\begin{aligned}
&[c] = m/s = m^1 s^{-1} = [H]^a [g]^b = m^a (m/s^2)^b = m^{a+b} s^{-2b} \\
&\Rightarrow \quad a + b = 1; \quad 2b = 1 \\
&\Rightarrow \quad a = 1/2; b = 1/2
\end{aligned}
$$

so that $c = \sqrt{gH}$ and therefore $\omega = \sqrt{gH}k$.

- Note that these waves are non-dispersive: different wavelengths travel at the same speed.
- The dispersion relation for these shallow water surface gravity waves also explains why such waves arrive parallel to the coast.
- Tsunamis are also shallow-water waves: wavelength is 1000s km, and the depth of the ocean is 4 km. Their propagation speed is $\sqrt{gH} = \sqrt{10 \times 4000} = 200$ m/s. For comparison, the sound velocity in air is 330 m/s.

3.2 Shallow water 1d mass conservation

Consider a channel of width Δy and height $h(x, t)$, and in it a section from $x - \Delta x$ to $x + \Delta x$. The velocity in the x direction is $u(x, t)$, and there is no velocity in the y direction.

We are assuming the wavelength is much longer than the depth, and therefore that the $u(x,t)$ velocity is depth-independent. Mass conservation for this small section states,

Rate of change of the total mass between $x-\Delta x$ and $x+\Delta x$	=	incoming − outgoing mass flux.

In terms of equations, this is

$$\begin{aligned}\frac{\partial}{\partial t}(h2\Delta x\Delta y\rho_0) &= u(x-\Delta x,t)h(x-\Delta x,t)\Delta y\rho_0 - u(x+\Delta x,t)h(x+\Delta x,t)\Delta y\rho_0 \\ &= \frac{u(x-\Delta x,t)h(x-\Delta x,t) - u(x+\Delta x,t)h(x+\Delta x,t)}{2\Delta x}2\Delta x\Delta y\rho_0 \\ &\approx -\frac{\partial(u(x,t)h(x,t))}{\partial x}2\Delta x\Delta y\rho_0\end{aligned}$$

so that

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0.$$

If $h = H+\eta$, such that H is constant and $\eta \ll H$, $u \ll 1$, we can write $uh = uH + u\eta \approx uH$ and $\partial_t h = \partial_t \eta$ so that

$$\frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0.$$

3.3 Shallow water 1d momentum equation

Consider the momentum budget of the same section:

Mass×acceleration=horizontal pressure force at $(x-\Delta x)$ − that at $(x+\Delta x)$.

The mass of the fluid element being considered is $\rho_0(2\Delta x)\Delta y h(x,t)$. The (linearized) acceleration is $\partial u/\partial t$. Integrating the hydrostatic equation

$$p_z = -g\rho_0,$$

with $z=0$ being the bottom and with the top being at $z=h$, we have

$$p(x,z) = g\rho_0(h-z).$$

Note that the pressure vanishes at the surface, as it should (we are ignoring atmospheric pressure). The total horizontal pressure force acting on a surface in the y-z plane at a point x is then given by

$$\Delta y\int_0^{h(x,t)} p(x,z)dz = \Delta y\int_0^{h(x,t)} \rho_0 g(h-z)\,dz = -\Delta y\rho_0 g(hz - \frac{1}{2}z^2)\Big|_{z=0}^{h} = \Delta y\rho_0 g\frac{1}{2}h^2.$$

We can now write the complete equation,

$$\begin{aligned}\rho_0(2\Delta x)\Delta y h(x,t)\frac{\partial u}{\partial t} &= \Delta y \rho_0 g \frac{1}{2}(h^2(x-\Delta x) - h^2(x+\Delta x)) \\ &\approx -\Delta y \rho_0 g \frac{1}{2}(2\Delta x)\frac{\partial h^2}{\partial x} \\ &= -\Delta y \rho_0 g (2\Delta x) h \frac{\partial h}{\partial x},\end{aligned}$$

so that

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x},$$

or, in terms of η,

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}.$$

3.4 Shallow water 1d wave equation

Combining the above results for the mass continuity and momentum balances, we have ★★★

$$\begin{aligned}\frac{\partial \eta}{\partial t} &= -H\frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial t} &= -g\frac{\partial \eta}{\partial x}.\end{aligned}$$

Take $\frac{\partial}{\partial t}$ of the first equation, take $\frac{\partial}{\partial x}$ of the second equation, multiply the second by H and subtract the second from the first to find,

$$\frac{\partial^2 \eta}{\partial t^2} = gH\frac{\partial^2 \eta}{\partial x^2},$$

which is the wave equation!

To find a solution, try $\eta = \eta_0 \cos(kx - \omega t)$ to find that this solves the equation only if,

$$\omega^2 = gHk^2 \quad \Rightarrow \quad \omega = \pm\sqrt{gH}k.$$

This is the dispersion relation, $\omega(k)$. We can now also calculate the phase velocity (which in this case is equal to the group velocity) to find $c = \omega/k = \pm\sqrt{gH}$, which is the same result found using scaling arguments above. The $\pm$ sign corresponds to two waves traveling in opposite directions.

3.5 Particle trajectories

Consider the motion of a single water parcel as a wave passes by. This trajectory is helpful to analyze because it provides intuition on the wave motion. Let us first find the velocity $u(x,t)$ from the above solution for the surface height $\eta(x,t)$. Use the momentum equation,

$$u_t = -g\eta_x = gk\eta_0 \sin(kx - \omega t),$$

so that therefore the solution for the velocity and surface elevation is,

$$\eta = \eta_0 \cos(kx - \omega t)$$
$$u = (gk/\omega)\eta_0 \cos(kx - \omega t).$$

so that maximum parcel velocity occurs at the crests and is directed in the same direction (that is, has the same sign) as the phase velocity $c = \omega/k$, and the parcel velocity at the troughs is in the opposite direction (because the cosine is negative there). See the following schematic figure.

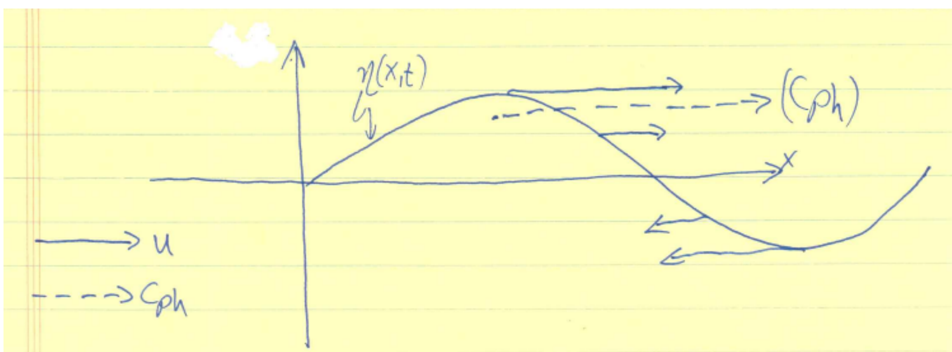


Figure 4:

For a water parcel starting at x_0, we can use

$$\frac{dx}{dt} = u(x,t) \approx u(x_0,t) = (gk/\omega)\eta_0 \cos(kx_0 - \omega t),$$

and we can integrate this to find

$$x(t) = x_0 - (gk/\omega^2)\eta_0 \sin(kx_0 - \omega t).$$

This tells us that water parcels oscillate back and forth around their initial location x_0 but *do not* propagate with the wave!

Note that the nondimensional factor appearing in the solution for $x(t)$ may be written as

$$\frac{gk}{\omega^2} = \frac{gk}{gHk^2} = \frac{1}{Hk} = \frac{1}{2\pi}\frac{\lambda}{H} >> 1,$$

where the inequality is based on the shallow water assumption that the wavelength is larger than the depth. This equation, therefore, also implies that the amplitude of parcel motion in the horizontal direction is much larger than η_0, which is the amplitude in the vertical direction. Hence, the parcel motion in shallow water waves is essentially horizontal.

3.6 Wave energy, breaking waves, Tsunamis

Discuss Tsunami and how it is expressed as a low-amplitude wave in an open ocean. Its frequency does not change as it approaches the shallower coastal areas, but its wavelength does. The conservation of energy per wavelength dictates an increase in amplitude near the coast, with catastrophic consequences. This is a good point to discuss breaking waves.

Quantitatively, this discussion requires that we understand wave energy. Consider a 1-d wave motion, where the relevant (linearized) momentum and mass conservation balances are,

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= -H\frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial t} &= -g\frac{\partial \eta}{\partial x}. \end{aligned}$$

multiply the first by $g\eta$ and the second by Hu and sum,

$$\frac{\partial}{\partial t}\frac{1}{2}(Hu^2 + g\eta^2) = -Hg\frac{\partial}{\partial x}(u\eta).$$

Integrating over the entire domain in x, assuming that there are walls on both ends where $u = 0$, we find the (linearized) energy conservation equation,

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}\frac{1}{2}(Hu^2 + g\eta^2) = -Hg(u\eta)\Big|_{x_1}^{x_2} = 0. \qquad (1)$$

the quantity $\frac{1}{2}(Hu^2 + g\eta^2)$ is the energy density (per unit x distance). It is positive, as one might expect energy to be, and the total energy in the domain is conserved as shown by (1). The two terms are kinetic and potential wave energy, correspondingly.

Now, consider a wave approaching a sloping beach,

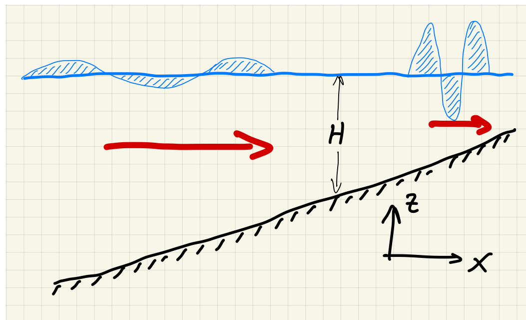


Figure 5:

As the waves approach the beach, slow down due to the smaller depth H. The frequency does not change, so the slowing down of $c = \omega/k$ is achieved by k growing, implying a shorter wavelength. For the energy per wavelength not to change, this means that the amplitude of η and u must increase. At some point, the fluid velocity at the wave crests is faster than the phase velocity, and that leads to wave breaking. Another way to see this is that the number of crests passing through a given location must be constant (crests do not disappear as the wave slows down). Given the slowing down of the crests, this can only happen if the crests are more crowded together, which again implies a shorter wavelength.

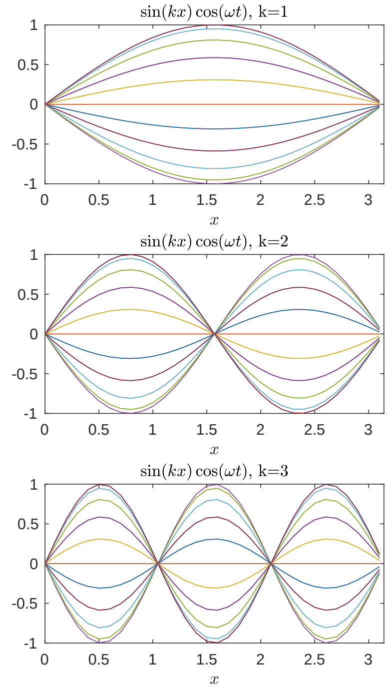
3.7 Standing waves

Consider two counter-propagating superimposed waves of the form $\eta(x,t) = \eta_0 \sin(kx - \omega t) = \eta_0 \sin k(x - ct)$ where $c = \omega/k$, ★★★

$$\eta(x,t) = \eta_0 \sin(k(x - ct)) + \eta_0 \sin(k(x - (-c)t)),$$

The frequencies of the waves are given by $\omega = ck$ and $\omega = -ck$, correspondingly. Thus, they propagate in opposite directions but are identical otherwise. Using the formula for adding sines, we find,

$$\eta(x,t) = 2\eta_0 \sin(kx) \cos(\omega t).$$



This represents a pattern that is fixed in space, of a wavelength $2\pi/k$, that oscillates with a period $2\pi/\omega$. If we think of this as the oscillations of a string of length L fixed at the edges, the string needs to satisfy $\sin(kx) = 0$ at both ends, which implies $\sin(kL) = 0$, and the possible values of k are therefore $kL = n\pi$, $n = 1, 2, \ldots$ These are standing waves! Note the nodal points in space where the oscillation amplitude vanishes. These occur at $kx = \pi$, or, $x = \pi/k = L/n$.

3.8 Tidal resonance

Consider an elongated bay of length L, open to the ocean at one end and closed by a land mass on the other end. The ocean current u at the ocean side oscillates at the tidal frequency, and this excites shallow-water waves that propagate into the bay. The equation and boundary conditions describing this scenario are then,

$$\begin{aligned} &u_{tt} = (gH)u_{xx} \\ &u(x = 0, t) = 0 \\ &u(x = L, t) = u_0 \sin(\omega t). \end{aligned}$$

The waves propagating from the ocean side are expected to be reflected at the other end and lead to opposite-propagating waves and, therefore, to standing waves, so look for a solution of the form,

$$u(x, t) = A \sin(kx) \sin(\omega t),$$

where ω is the tidal frequency and k is to be determined. Substitute this into the equation to find,

$$-\omega^2 A \sin(kx)\sin(\omega t) = -k^2(gH)A\sin(kx)\sin(\omega t)\,,$$

which implies that

$$k = \omega/\sqrt{gH}.$$

At $x = 0$, the boundary condition is satisfied because,

$$A\sin(k\,0)\sin(\omega t) = 0,$$

while at the other end, we have,

$$A\sin(k\,L)\sin(\omega t) = u_0\sin(\omega t),$$

which implies

$$A = \frac{u_0}{\sin(k\,L)} = \frac{u_0}{\sin\left(\omega\,L/\sqrt{gH}\right)},$$

using the value of k calculated before. We, therefore, have the final solution for the velocity field due to forced tides in the bay,

$$u(x,t) = \frac{u_0}{\sin\left(\omega\,L/\sqrt{gH}\right)}\sin\left(\omega\,x/\sqrt{gH}\right)\sin(\omega t)\,.$$

The solution implies that when,

$$\omega\,L/\sqrt{gH} = n\pi$$

the sine in the denominator vanishes, and the amplitude becomes infinite. This is modified to a finite large value when friction is considered, yet this can still explain the large amplitude tides in some bays. We will now see that the large amplitude occurs if the frequency of natural seiches of the bay is equal to the tidal frequency. Because η and u are related through the momentum and continuity equations, a large amplitude velocity also implies large amplitude surface elevation changes, as observed in some locations. To calculate the surface elevation signal, use the mass conservation equation,

$$\eta_x = -u_t/g = -\frac{\omega u_0/g}{\sin\left(\omega\,L/\sqrt{gH}\right)}\sin\left(\omega\,x/\sqrt{gH}\right)\cos(\omega t)$$

from which we find,

$$\eta = \frac{u_0 \sqrt{H/g}}{\sin\left(\omega\, L/\sqrt{gH}\right)} \cos\left(\omega\, x/\sqrt{gH}\right) \cos\left(\omega t\right)$$

and given that for the resonant frequencies we have $\omega/\sqrt{gH} = n\pi/L$, the surface elevation at resonance becomes,

$$\eta = \frac{u_0 \sqrt{H/g}}{\sin\left(\omega\, L/\sqrt{gH}\right)} \cos\left(n\pi x/L\right) \cos\left(\omega t\right).$$

The $n = 1$ case shows that the tidal surface elevation variability is maximal at $x = 0$ and $x = L$ with a nodal point in between.

To interpret the expression for the resonant frequencies, write the resonance condition $\omega\, L/\sqrt{gH} = n\pi$ as,

$$\frac{2L}{\sqrt{gH}} = n\frac{2\pi}{\omega}.$$

The LHS is the time it takes a wave to cross the bay length back and forth. The RHS is n times the tidal period. When this condition is satisfied, the tidal forcing always has the same phase when the traveling wave arrives at the bay opening, allowing it to be repeatedly amplified.

As an example, the length of the Bay of Fundy is 250 km, $g = 10$ m/s^2, take $\omega = 2\pi/(12$ hours$)$, and then for the resonance to occur the depth should be,

$$H = \frac{L^2\omega^2}{n^2\pi^2 g}.$$

This simple explanation is meant to provide only a crude insight into tidal resonance; the details are more complex.

3.9 Deep 1d water waves scaling argument

Consider an option of infinite depth so that the depth is no longer relevant. The relevant dimensional constants are: g for gravity; ω for frequency (2π/period); k is the wavenumber (2π/wavelength). Try writing the frequency ω as function of k, g, ★★★

$$\begin{aligned} [\omega] = 1/s = [k]^a[g]^b = (1/m)^a(m/s^2)^b \\ \Rightarrow \quad a = 1/2; b = 1/2 \end{aligned}$$

so that $\omega = \sqrt{gk}$ and therefore $c_{ph} = \omega/k = \sqrt{g/k}$ and $c_g = \partial\omega/\partial k = \sqrt{g/k}/2$. This turns out to be the exact result obtained if one solves the relevant equations for deep gravity waves. Note that these waves are dispersive: different wavelengths travel at different speeds.

Discuss particle motions in deep gravity waves.

Discuss swell: waves arriving from a remote storm and why a swell is a long (and thus smooth) wave.

3.10 Finite ocean depth and limits of shallow and deep water

If the depth is not assumed to necessarily be very small or very large, the dispersion relation is found to be

$$\omega^2 = gk\tanh(kH).$$

Let's see how this general relation behaves for the limits examined above.

First, in the case of shallow water, $\lambda = 2\pi/k \gg H$, we have $kH \ll 1$ and therefore $\tanh(kH) \approx kH$, so that $\omega^2 \approx gHk^2$ as before.

Next, in the case of very deep water, $\lambda = 2\pi/k \ll H$, we have $kH \gg 1$ and therefore $\tanh(kH) \approx 1$, so that $\omega^2 \approx gk$ as before.

4 Buoyancy oscillations

Consider a fluid element in a stratified ocean displaced by a distance δz. We now allow for acceleration in the vertical dimension so that the vertical momentum budget $F = ma$ takes the form of acceleration balanced by pressure and gravity forces,

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - g\rho.$$

Consider a density perturbation at a level $z + \delta z$ due to a parcel that was lifted from a level z. The density perturbation at the level to which the particle arrives is the density at the level at which the parcel originated, minus that at the level to which it arrived, $\delta\rho(z + \delta z) = \bar{\rho}(z) - \bar{\rho}(z + \delta z)$, or

$$\delta\rho = -\frac{\partial \bar{\rho}}{\partial z}\delta z.$$

Because a stable stratification means that $\partial\rho/\partial z < 0$, if $\delta z > 0$, then $\delta\rho > 0$. This makes sense as a denser fluid parcel moves up into a lighter fluid, creating a positive density anomaly there. Write $\rho = \bar{\rho} + \delta\rho$, plug in the momentum equation and assume the pressure balances the mean density rather than the perturbed density,

$$0 = -\frac{\partial p}{\partial z} - g\bar{\rho}(z).$$

We are left with,

$$\rho_0 \frac{\partial w}{\partial t} = -g\delta\rho.$$

Substitute in this last equation $\delta\rho = -\frac{\partial\bar{\rho}}{\partial z}\delta z$ and $w = \frac{\partial\delta z}{\partial t}$, to find, ★★★

$$\frac{\partial^2}{\partial t^2}\delta z = -\left[\frac{-g}{\rho_0}\frac{\partial\bar{\rho}}{\partial z}\right]\delta z.$$

We define the "buoyancy frequency" to be,

$$N^2 \equiv \frac{-g}{\rho_0}\frac{\partial\bar{\rho}}{\partial z},$$

so that the solution for the displacement of the fluid parcel is,

$$\delta z = A\cos Nt,$$

showing that it oscillates in the vertical direction.

5 Internal shallow water waves

Consider a two-layer model, with the lower layer much thicker and thus assumed to be at rest (Fig. 6).

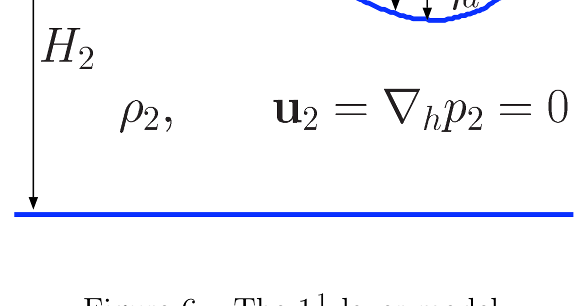


Figure 6: The $1\frac{1}{2}$ layer model

The momentum equations for the two layers are

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}$$
$$\frac{\partial u_2}{\partial t} = -\frac{1}{\rho_2}\frac{\partial p_2}{\partial x}.$$

Assuming the deep velocity vanishes implies that the horizontal pressure gradient is also zero in the lower layer, $\frac{\partial p_2}{\partial x} = 0$. Assuming a hydrostatic vertical momentum balance

$$p_z = -g\rho,$$

and integrating this balance in z, we can write the pressure at a depth z in the upper layer as

$$p_1(x, y, z, t) = g(-z + \eta_s(x, y, t))\rho_1,$$

so that

$$-\frac{1}{\rho_1}\frac{\partial p_1}{\partial x} = -g\frac{\rho_1}{\rho_1}\frac{\partial \eta_s}{\partial x} = -g\frac{\partial \eta_s}{\partial x}.$$

In the lower layer, the pressure is due to an integral of ρ_1 over the depth of the first layer ($h_1 = H_1 + \eta_s - \eta_d$), plus an integral of ρ_2 over the depth range within the second layer, from z to $-H_1 + \eta_d$,

$$p_2(x, y, z, t) = g(H_1 + \eta_s - \eta_d)\rho_1 + g(-H_1 + \eta_d - z)\rho_2$$

so that

$$\begin{aligned}\frac{1}{\rho_2}\frac{\partial p_2}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\rho_1}{\rho_2}g\eta_s + \frac{\rho_2 - \rho_1}{\rho_2}g\eta_d\right)\\ &\approx \frac{\partial}{\partial x}(g\eta_s + g'\eta_d)\end{aligned}$$

where

$$g' \equiv \frac{\rho_2 - \rho_1}{\rho_0}g \approx \frac{\rho_2 - \rho_1}{\rho_2}g.$$

The assumption that the deep horizontal pressure gradient vanishes gives

$$g\frac{\partial}{\partial x}\eta_s = -g'\frac{\partial}{\partial x}\eta_d$$

which, together with the observation that $g' \ll g$ so that $\eta_s \ll \eta_d$, implies

$$g\frac{\partial \eta_s}{\partial x} = -g'\frac{\partial \eta_d}{\partial x} \approx g'\frac{\partial h}{\partial x}.$$

The first equality tells us that the upper surface η_s varies in the opposite direction from the deep interface η_d and at a much smaller amplitude. This means that internal waves have a signal that is seen at the ocean surface and can, therefore, be observed remotely (e.g., from satellites). The second equality, together with the above relations, finally allows us to write the horizontal pressure gradient in the upper layer as a function of the upper layer thickness

$$-\frac{1}{\rho_0}\frac{\partial p_1}{\partial x} = -g'\frac{\partial h}{\partial x},$$

so our momentum and mass conservation equations may be written as

$$\frac{\partial u_1}{\partial t} = -g'\frac{\partial h}{\partial x} \quad (2)$$

$$\frac{\partial h}{\partial t} + \frac{\partial (u_1 h)}{\partial x} = 0 \quad (3)$$

Approximating h by its mean value H_1 in the second equation as we did for the shallow water waves, we finally have

$$\frac{\partial u_1}{\partial t} = -g'\frac{\partial h}{\partial x} \quad (4)$$

$$\frac{\partial h}{\partial t} = -H_1\frac{\partial u_1}{\partial x} \quad (5)$$

from which we can derive the wave equation

$$\frac{\partial^2 h}{\partial t^2} = (g'H_1)\frac{\partial^2 h}{\partial x^2}$$

These are the same equations we had for a single layer of shallow water, with g replaced by g'. The dispersion relation is

$$\omega = \pm\sqrt{g'H_1}k,$$

and the phase and group wave velocities, in this case, are both equal to $c = \pm\sqrt{g'H_1}$. This also shows that internal waves propagate much slower than surface waves. Note also the above relation between the internal wave displacement of the interface between the layers, and the smaller displacement of the surface, providing a surface signature of internal waves.

6 Shallow water waves in the presence of rotation

6.1 Coastal Kelvin waves

Consider a momentum balance that adds the Coriolis force to the balance for shallow gravity waves we examined above. That is, the acceleration is balanced by the Coriolis and pressure forces. The mass conservation statement is similar to that we had for shallow surface gravity waves ($\eta_t + Hu_x$), adding another horizontal dimension. That is, start from linearized shallow water momentum and continuity equation in 2d,

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$
$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

Consider a solution near a coast, as shown in the following schematic figure.

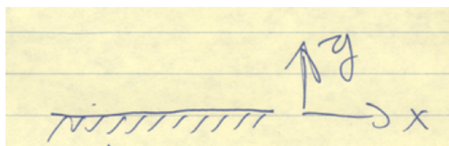


Figure 7: Geometry for coastal Kelvin waves

We look for a solution with $v = 0$ everywhere because we know it must vanish at the coast. The equations become,

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$
$$fu = -g\frac{\partial \eta}{\partial y}$$
$$\frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0.$$

Consider a wave solution,

$$u = \hat{u}(y)\cos(kx - \omega t)$$
$$\eta = \hat{\eta}(y)\cos(kx - \omega t)$$

the first momentum equation gives

$$-\omega\hat{u} = -gk\hat{\eta}$$

or

$$\hat{u} = \frac{gk}{\omega}\hat{\eta}$$

substituting in the second momentum equation we find

$$\frac{\partial \hat{\eta}}{\partial y} = -\frac{f}{g}\hat{u} = -\frac{fk}{\omega}\hat{\eta}$$

and given that $\omega/k = c$, we can write the solution as

$$\hat{\eta}(y) = \eta_0 \exp\left(-\frac{f}{c}y\right)$$

using the first momentum equation and the continuity equation together we find the dispersion relation,

$$c = \omega/k = \pm\sqrt{gH}$$

but given the above structure in y we see that only the positive root is physical and that the wave must travel with the coast to its right in the northern hemisphere. The final solution may, therefore, be written as

$$\begin{aligned}
\eta(x, y, t) &= \eta_0 \exp\left(-\frac{f}{c}y\right)\cos(kx - \omega t) \\
u(x, y, t) &= \eta_0 \frac{g}{c}\exp\left(-\frac{f}{c}y\right)\cos(kx - \omega t) \\
c &= \omega/k = +\sqrt{gH}.
\end{aligned}$$

6.2 Poincare waves

Next, away from a boundary, starting again from

$$\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -g\frac{\partial \eta}{\partial x} \\
\frac{\partial v}{\partial t} + fu &= -g\frac{\partial \eta}{\partial y} \\
\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0
\end{aligned}$$

look for a plane wave solution

$$\begin{aligned}
u &= u_0 e^{i(kx+ly-\omega t)} \\
v &= v_0 e^{i(kx+ly-\omega t)} \\
\eta &= \eta_0 e^{i(kx+ly-\omega t)}.
\end{aligned}$$

substituting this solution into the equations, we find,

$$\begin{aligned} -i\omega u_0 - f v_0 &= -igk\eta_0 \\ -i\omega v_0 + f u_0 &= -igl\eta_0 \\ -i\omega\eta_0 + iH\left(ku_0 + lv_0\right) &= 0. \end{aligned}$$

Write this in matrix form,

$$\begin{pmatrix} -i\omega & -f & igk \\ f & -i\omega & igl \\ ikH & ilH & -i\omega \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ \eta_0 \end{pmatrix} = 0$$

The determinant needs to vanish for a nontrivial solution to exist. Using Wolfram Alpha with,

```
determinant[{{-i*omega,-f,i*g*k},{f,-i*omega,i*g*l},{i*k*H,i*l*H,-i*omega}}]
```

leads to

$$-if^2\omega - igHk^2\omega - igHl^2\omega + i\omega^3 = 0$$

one solution is $\omega = 0$ and then we are left with

$$\omega^2 = f^2 + gH(k^2 + l^2)$$

which is the dispersion relation of Poincare waves. When $f = 0$ this reduces to the usual gravity wave dispersion relation in 2d. With $g = 0$ this becomes the inertial motion dispersion relation.