EPS131, Introduction to Physical Oceanography and Climate

Section 5: Friction

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Contents

1 Damped inertial oscillations

Consider the inertial oscillation problem we did previously, yet with two new friction terms on the RHS,

$$
\begin{aligned}
u_t - fv &= -ru \\
v_t + fu &= -rv,
\end{aligned}
$$

To see why these are referred to as "friction", eliminate the Coriolis terms momentarily

$$u_t = -ru, \quad v_t = -rv.$$

The solution is

$$u = u(t=0)e^{-rt} \quad v = v(t=0)e^{-rt},$$

which is a decay on a time scale of $1/r$, so these friction terms indeed cause the velocities to decay as one might expect. This form may be thought of as representing the effects of *bottom friction* in a shallow ocean, for example, which is proportional to the current velocity along the bottom.

Now reintroduce the Coriolis terms and substitute a solution of the form

$$u = u_0 e^{at}, \quad v = v_0 e^{at}.$$

Substituting this solution into the original equations, we find ★★★

$$au - fv = -ru, \quad av + fu = -rv.$$

The first equation then gives $v = (a+r)u/f$. Substituting in the second equation, we find $(a+r)^2 + f^2 = 0$ so that $a = \pm if - r$. The solution is therefore,

$$u \propto A' e^{ift} e^{-rt} + B' e^{-ift} e^{-rt}$$

and similarly for v. The first exponential factor represents an oscillation and the second a decay, so this is a "damped oscillation". To find a solution that satisfies the initial conditions, it is convenient to write this as

$$u = A\cos(ft)e^{-rt} + B\sin(ft)e^{-rt}$$

The original equations may be used to calculate v from u now, and the initial conditions for both u and v can be used to find the constants A and B.

HW: solve for specific initial conditions for u, v, and solve for the trajectories to show that this is a decaying oscillation.

2 Ekman transport

Near the ocean's surface (upper 50–100 meters), the horizontal wind friction force is applied to the ocean's surface. Turbulent motions then transfer that horizontal frictional force down. Suppose, therefore, that the momentum budget is,

$$0 = \text{Coriolis} + \text{friction force}.$$

Consider a fluid layer of a thickness Δz not far from the ocean surface. The above balance takes the form,

$$\begin{aligned}-fv\Delta z\rho_0 &= \text{friction at } (z+\Delta z) - \text{friction at } (z)\\ &= \tau^{(x)}(z+\Delta z) - \tau^{(x)}(z) = \frac{\partial \tau^{(x)}}{\partial z}\Delta z,\end{aligned}$$

where $\tau^{(x)}(z)$ is the frictional force in the x direction per unit area (stress), exerted by the layer above a depth z on the layer below the depth z. That is, the stress acts on a surface facing the positive z direction. The frictional force acting on a surface facing down at a depth z is therefore $-\tau^{(x)}(z)$. The force represents *interior friction* due to random molecular/fluid motions between layers. Our two momentum equations, therefore, become,

$$\begin{aligned}-fv &= \frac{1}{\rho_0}\frac{\partial \tau^{(x)}}{\partial z}\\ fu &= \frac{1}{\rho_0}\frac{\partial \tau^{(y)}}{\partial z}.\end{aligned}$$

At the surface, the frictional force must be equal to that applied by the wind, $\tau^{(x)}(z=0) = \tau^{(x),\text{wind}}$. At the bottom of the ocean's upper layer, at a depth of, say, 100 meters, turbulence is much weaker, and we can assume that the frictional stress vanishes, $\tau^{(x)}(z=100m) = 0$. We can now integrate the last set of equations to find the total horizontal transport driven by the wind within the upper layer of the ocean,

$$M^{(y)} = \int_{-100m}^{0} v\,dz = \int_{-100m}^{0} \frac{-1}{f\rho_0}\frac{\partial \tau^{(x)}}{\partial z}\,dz = \frac{-1}{f\rho_0}\tau^{(x)}\Big|_{-100m}^{0} = \frac{-1}{f\rho_0}\tau^{(x),\text{wind}},$$

so that we have for both transports,

$$\begin{aligned}M^{(x)} &= \frac{1}{f\rho_0}\tau^{(y),\text{wind}}\\ M^{(y)} &= -\frac{1}{f\rho_0}\tau^{(x),\text{wind}}.\end{aligned}$$

These are referred to as "Ekman transports" and are seen in this last equation to be exactly perpendicular to the wind! (To the right of the wind in the northern hemisphere where $f > 0$). The units of the Ekman transports are volume per time per unit distance in the direction perpendicular to the transport, or m^2/s. ★★★

These Ekman transports driven by along-shore winds are responsible, for example, for the California coastal upwelling. This upwelling brings cold, nutrient-rich deeper water to the surface, making this area and a few similar relatively small ocean regions (Benguela, Peru/Chile, Canary) very productive fisheries.

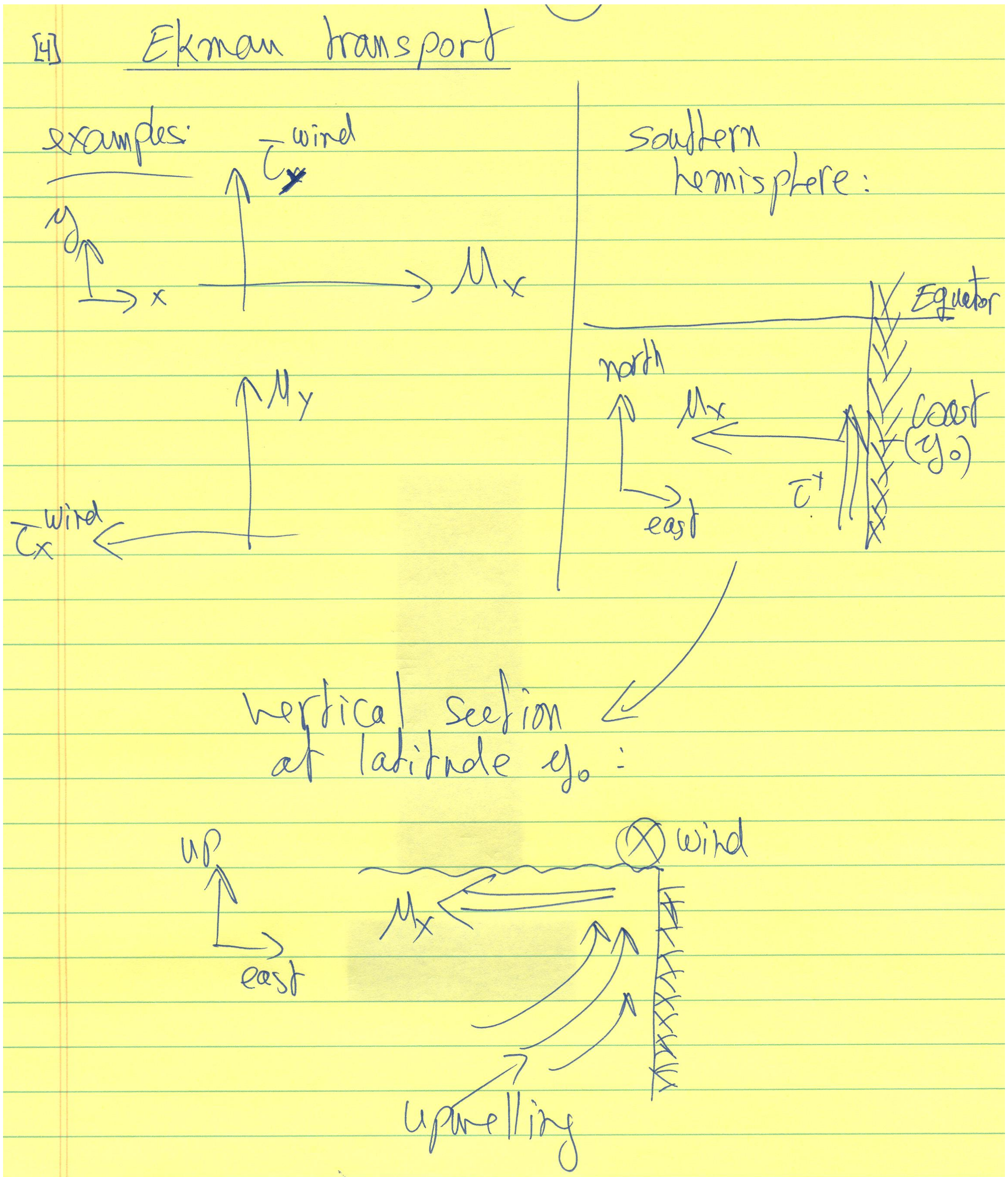


Figure 1: Ekman transport schematics

3 Ocean-interior vertical friction

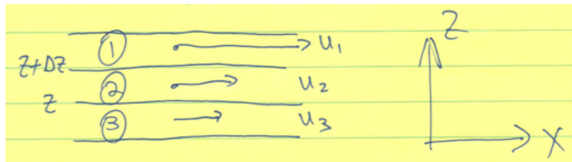


Figure 2: Three vertical ocean layers, we are interested in the frictional forces on the middle one.

Consider three moving "layers" of fluid as shown in Fig. 2. The total friction felt by layer (2) is due to friction from above and from below. The friction is due to random Brownian motion moving molecules with fast u-velocity from layer (1) to (2) and, therefore, dragging layer (2) faster forward. We can write,

$$\text{friction on (2) by (1) at } z+\Delta z = \tau(z+\Delta z) = \text{constant} \times (u_1 - u_2) \approx \mu \left.\frac{\partial u}{\partial z}\right|_{z+\Delta z}$$
$$\text{friction on (2) by (3) at } z = -\tau(z) = -\text{constant} \times (u_2 - u_3) \approx -\mu \left.\frac{\partial u}{\partial z}\right|_{z}.$$

The minus sign in front of τ in the second equation is because the stress is defined to be on a layer facing the positive z direction, and we need the force on a layer facing the negative z direction here. The proportionality constant μ is known as the *dynamic viscosity*. The net force on the layer marked (2) is the sum of those, which is proportional to $\partial^2 u/\partial z^2$. Some special cases to consider:

- Constant velocity, $u(z) = \text{constant}$: $\partial u/\partial z = 0$, $\partial^2 u/\partial z^2 = 0$: no frictional force on any layer.
- Constant shear, $u(z) = az$: $\partial u/\partial z = a = \text{constant} \neq 0$, $\partial^2 u/\partial z^2 = 0$: the frictional force is equal at the top and bottom of each layer, no net frictional force on any layer.

We conclude that the stress in the x direction on a given vertical layer (say layer 2 in the above schematic), which we previously termed $\tau^{(x)}(z)$, is found here as a function of the velocity to be

$$\tau^{(x)}(z) = \mu \frac{\partial u}{\partial z}.$$

We saw above that the net force in the horizontal x direction on a fluid parcel, due to friction with the layer above and below, is given by $\partial \tau^{(x)}/\partial z$. Substituting the above expression we derived for $\tau^{(x)}$, we find that the net force in the horizontal direction due to vertical friction between fluid layers is $\mu \partial^2 u/\partial z^2$. The net acceleration is, therefore, $(\mu/\rho)\partial^2 u/\partial z^2$, and the ratio of the dynamic viscosity to the density, $\nu \equiv \mu/\rho$, is known as the *kinematic viscosity*.

4 Scale-selective vs non scale-selective friction

We considered two forms of friction. The first crudely represents bottom friction, where the momentum equation expressing the balance of acceleration equal to friction is of the form $u_t = -ru$. For the second form, representing internal friction between ocean layers, the momentum equation of acceleration balanced by friction is of the form $u_t = \nu u_{zz}$. Our objective now is to understand the difference between these two friction forms.

Consider a flow in the horizontal direction, which is oscillatory in z, $u(t = 0, z) = A\cos(kz)$,

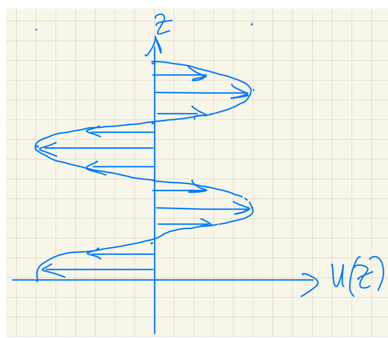


Figure 3: A zonal velocity profile which is oscillatory in depth.

First scenario: non scale-selective friction,

$$\frac{\partial u}{\partial t} = -ru$$

solution is

$$u(z, t) = u(t = 0, z)e^{-rt} = A\cos(kz)e^{-rt}$$

so that the decay rate does not depend on the vertical structure of the flow, in particular not on k.

Second scenario, scale-selective friction,

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial z^2}.$$

Try a solution of the form $u(t, z) = u(t = 0, z)e^{bt} = A\cos(kz)e^{bt}$, substitute in equation to find, ★★★

$$bu(t, z) = -k^2\nu u(t, z),$$

so that $b = -\nu k^2$ and the solution is

$$u(t, z) = A \cos(kz) e^{-\nu k^2 t}.$$

This implies, unlike above, that the decay is faster for larger k, which corresponds to a smaller vertical scale of the flow. Thus smaller scales are dissipated faster, hence the terms "scale-selective" vs "non-scale-selective".

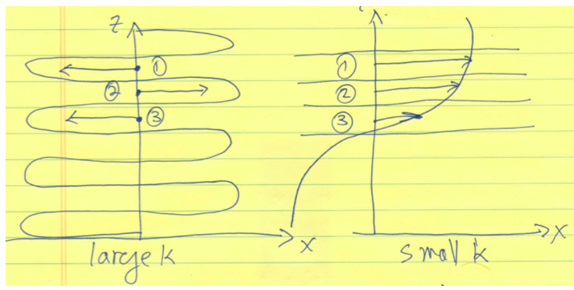


Figure 4: Yet more selective-friction schematics

Why does this make sense intuitively: consider the schematic in Fig. 4, showing the initial conditions for both a large k vs small k. Note that the friction between layer (2) and its neighbors (1,3) is much larger for the large k case, simply because the velocity difference is larger. This leads to the faster decay of the larger k flow. This scale-selective nature of the above friction holds for diffusion in general, given that the diffusion equation for temperature, T, is identical to the above, $\partial T/\partial t = \kappa \partial^2 T/\partial x^2$.

5 Ekman spiral

In section 2, we were only interested in the total transport within the Ekman layer. Consider now the vertical structure of the horizontal velocities driven by the wind. Intuitively, first, the top layer is dragged in the direction of the wind and tends to flow to the right (Northern Hemisphere) of the wind force so that its Coriolis force balances the wind force. However, the top layer senses horizontal friction from below that also partially balances the wind stress, so the top layer flows to the right of the wind but at an angle of less than 90°. The next layer down is dragged by the layer above in the direction given by an angle α to the right of the wind and again flows at some angle to the right of the flow in the layer above. Such a sequence of layers forms a "spiral" of the velocity direction with depth.

Mathematically, let the equations again balance Coriolis with vertical friction, using the scale-selective friction derived above, ★★★

$$-fv = \nu\frac{\partial^2 u}{\partial z^2}$$
$$fu = \nu\frac{\partial^2 v}{\partial z^2}.$$

Take the second derivative of the first equation and substitute the second to find

$$\nu\frac{\partial^4 u}{\partial z^4} = -f\frac{\partial^2 v}{\partial z^2} = -\frac{f^2}{\nu}u,$$

so that the equation to be solved is

$$\frac{\partial^4 u}{\partial z^4} = -\frac{f^2}{\nu^2}u.$$

Substitute an exponential solution, $u = Ae^{bz}$, to find,

$$b^4 = -\frac{f^2}{\nu^2},$$

so that

$$b^2 = \pm i\frac{f}{\nu}.$$

Given that the four square roots of $\pm i$ are

$$\frac{i+1}{\sqrt{2}}, -\frac{i-1}{\sqrt{2}}, -\frac{i+1}{\sqrt{2}}, \frac{i-1}{\sqrt{2}},$$

we can write the four roots for b as,

$$b_{1,2,3,4} = \frac{i+1}{\delta}, -\frac{i-1}{\delta}, -\frac{i+1}{\delta}, \frac{i-1}{\delta},$$

where $\delta = \sqrt{2\nu/f}$, so that the solution for u is

$$u(z) = \sum_{i=1}^{4} A_i e^{b_i z}.$$

The v velocity is then given by

$$v = -\frac{\nu}{f}\frac{\partial^2 u}{\partial z^2} = -\frac{\nu}{f}\sum_{i=1}^{4} A_i b_i^2 e^{b_i z},$$

and we can write this solution as

$$v = -\frac{\nu}{f}\sum_{i=1}^{4} A_i b_i^2 e^{b_i z}.$$

The boundary conditions are that the velocity vanishes at depth as $z \to -\infty$ and that the surface stress is given by

$$\begin{aligned}
\tau^{(x)}(z=0) &= \nu \left.\frac{\partial u}{\partial z}\right|_{z=0} = \frac{1}{\rho}\tau^{(x),\text{wind}} \\
\tau^{(y)}(z=0) &= \nu \left.\frac{\partial v}{\partial z}\right|_{z=0} = \frac{1}{\rho}\tau^{(y),\text{wind}}.
\end{aligned}$$

This surface boundary condition can be understood as follows. Consider a thin layer (say 1 mm) at the ocean surface. If the stress applied by the wind at the surface is different from the interior friction stress given by $\mu\partial u/\partial z$ at a depth of 1 mm, then there is a net force on that layer. Because it is very thin, it has a small mass, and the net force would lead to a large, non-physical acceleration. This is not physical and is not observed. Thus, the interior stress $\mu\partial u/\partial z$ near and at the surface of the ocean needs to be equal to the wind stress.

The boundary conditions at depth, as $z \to -\infty$, imply that $A_3 = A_4 = 0$ as these are the coefficients of the terms that grow exponentially as the depth increases. The remaining terms may be written as,

$$\begin{aligned}
v &= A_1 e^{\frac{i+1}{\delta}z} + A_2 e^{-\frac{i-1}{\delta}z} \\
&= A_1 e^{z/\delta}(\cos(z/\delta) + i\sin(z/\delta)) + A_2 e^{z/\delta}(\cos(z/\delta) - i\sin(z/\delta)) \\
&= (A_1 + A_2)e^{z/\delta}\cos(z/\delta) + i(A_1 - A_2)e^{z/\delta}\sin(z/\delta).
\end{aligned}$$

Letting $A = (A_1 + A_2)$ and $B = i(A_1 - A_2)$, we can then write the solution as,

$$u = e^{z/\delta}(A\cos(z/\delta) + B\sin(z/\delta)),$$

and the remaining two constants, A and B, are determined by the surface boundary conditions, as in the following example.

5.1 Example

Suppose the wind stress is only in the y direction, $\tau^{(x),\mathrm{wind}} = 0$. The above solution leads to

$$u_z = \frac{1}{\delta}e^{z/\delta}(A\cos(z/\delta) + B\sin(z/\delta)) + \frac{1}{\delta}e^{z/\delta}(-A\sin(z/\delta) + B\cos(z/\delta))$$

and therefore

$$\nu u_z(z=0) = \frac{1}{\rho}\tau^{x,wind} = 0 = (A+B)/\delta$$

so that $B = -A$ and

$$\begin{aligned} u &= Ae^{z/\delta}(\cos(z/\delta) - \sin(z/\delta)) \\ v &= -\frac{\nu}{f}\frac{\partial^2 u}{\partial z^2} = Ae^{z/\delta}(\cos(z/\delta) + \sin(z/\delta)) \end{aligned}$$

We can now use boundary condition for v to determine A,

$$\nu v_z = \frac{1}{\rho}\tau^{y,wind} = \nu Ae^{z/\delta}\cos(z/\delta)/\delta\big|_{z=0} = \nu A/\delta$$

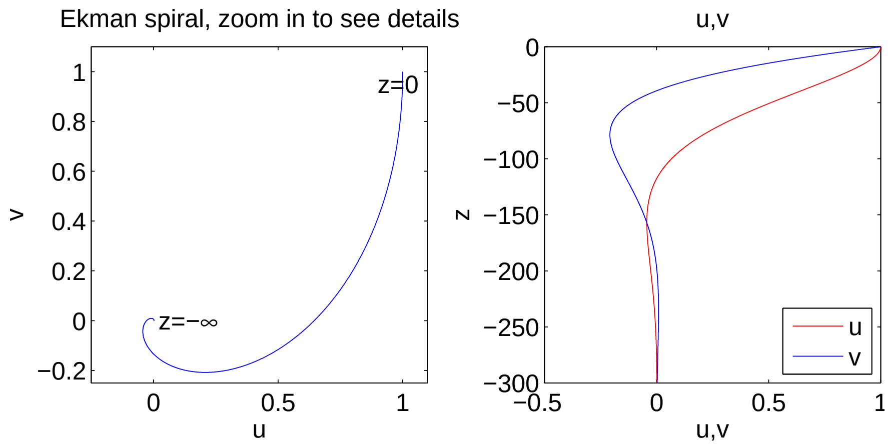
so that

$$A = \frac{1}{2\rho\nu}\tau^{y,wind}\delta$$

and we finally have,

$$\begin{aligned} u &= Ae^{z/\delta}(\cos(z/\delta) - \sin(z/\delta)) \\ v &= Ae^{z/\delta}(\cos(z/\delta) + \sin(z/\delta)). \end{aligned}$$

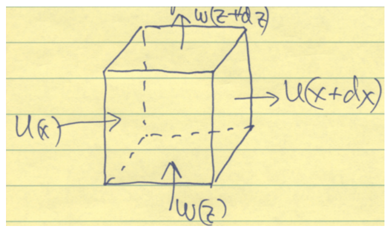
Note that at the surface, $z = 0$, we have $u = v = A$, so the velocity is at $45°$ to the right of the wind, which is in the north direction in this example. And here is the plot of this solution made using the Matlab script `plot_Ekman_spiral.m`.



Ekman spiral, zoom in to see details
z=0
z=−∞
v
u
1
0.8
0.6
0.4
0.2
0
−0.2
0
0.5
1
u,v
z
u
v
0
−50
−100
−150
−200
−250
−300
−0.5
0
0.5
1
u,v

6 Mass conservation/ continuity equation

To derive an equation for Ekman pumping, we need the 3d mass conservation equation for water as an incompressible fluid. Consider an infinitesimal cube through which fluid is passing, and calculate the mass fluxes into and out of the box,



Assuming water is incompressible and the density approximately constant implies that the sum of mass (volume) fluxes into the box must vanish at any time, ★★★

$$\begin{aligned}0 &= [u(x) - u(x+\Delta x)]\Delta y \Delta z + [v(y) - v(y+\Delta y)]\Delta x \Delta z + [w(z) - w(z+\Delta z)]\Delta x \Delta y \\ &= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \Delta x \Delta y \Delta z.\end{aligned}$$

The *continuity equation*, the conservation of mass for an incompressible fluid like water, is, therefore,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

7 Ekman pumping

Another important issue we need to consider concerns the calculation of vertical velocity between the surface layer and the ocean interior due to spatial variations in the wind forcing. Integrate the continuity equation vertically over the mixed layer to find,

$$0 = \int_{-50\ \mathrm{m}}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz + \int_{-50\ \mathrm{m}}^{0} \frac{\partial w}{\partial z} dz$$

so that

$$0 = \frac{\partial}{\partial x} \int_{-50\ \mathrm{m}}^{0} u dz + \frac{\partial}{\partial y} \int_{-50m}^{0} v dz + w(0) - w(-50\ \mathrm{m}).$$

We can set the vertical velocity to zero at the surface (assuming a steady state, linearized). Also, we can use the expressions we derived for the Ekman transport,

$$\int_{-50\ \mathrm{m}}^{0} u dz = M_{(x)} = \frac{\tau^{(y)}}{\rho_0 f}$$
$$\int_{-50\ \mathrm{m}}^{0} v dz = M_{(y)} = -\frac{\tau^{(x)}}{\rho_0 f}.$$

Substituting this into the equation for the velocity at the base of the Ekman layer, $w(-50\ \mathrm{m})$, referred to as the Ekman pumping (units of m/s),

$$w(-50\ \mathrm{m}) = \frac{\partial}{\partial x} \left(\frac{\tau^{(y)}}{\rho_0 f} \right) - \frac{\partial}{\partial y} \left(\frac{\tau^{(x)}}{\rho_0 f} \right) = \mathrm{curl} \left(\frac{\boldsymbol{\tau}}{\rho_0 f} \right).$$