bifurcations in 1d systems

what is a bifurcation: a qualitative change in the nature of the solution of a dynamical system as a parameter(s) is varied.

* example: $\dot{x} = \mu - x^2$. steady state: $\dot{x} = \mu - x^2 = 0$

for $\mu < 0$, there are no steady states

$\mu = 0$ 1 steady state, $x = 0$

$\mu > 0$ 2 " " $x = \pm\sqrt{\mu}$.

* when can we expect a bifurcation:

Implicite function theorem:

consider eqn $\vec{f}(\vec{x}, \mu) = 0$, where: $\vec{f}$, $\vec{x}$ are n-dim.

let $f(\bar{x}, \bar{\mu}) = 0$, & view this as an implicit eqn for $\vec{x}(\mu)$ satisfying $f(\vec{x}(\mu), \mu) = 0$. if $J = \left[\frac{\partial f_i}{\partial x_j}\right]$ (Jacobian) is non singular $[\det(J) \neq 0]$, & f is smooth etc, then $\vec{x}(\mu)$ is unique & smooth around $\bar{x}(\bar{\mu})$.

$\Rightarrow$ given $\dot{\vec{x}} = \vec{f}(\vec{x}, \mu)$, if $J = \left\{\frac{\partial f_i}{\partial x_j}\right\}$ is not singular [in one dim: $\partial f/\partial x \neq 0$], we cannot have any bifurcation, as the solutions to $\dot{\vec{x}} = 0$ are unique at $\vec{x}, \mu$.

* in the above example:

$\dot{x} = \mu - x^2$. $J = \frac{\partial f}{\partial x} = -2x = 0$ at $x = 0$, where indeed there is a bifurcation point, there

classifying bifurcation points:

is done based on the # & type of zero eigenvalues of $J \equiv D_x f \equiv [\partial f_i / \partial x_j]$.

*** The simplest (co-dimension one) possibilities are**

(i) J has a single zero eigenvalue

(ii) J has two complex conjugate imaginary eigenvals. (J is real, so it can only have conjugate pairs.)

(real Jordan form)

in (i), $J = \begin{bmatrix} 0 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A} \end{bmatrix}$ where all eigen vals of A have nonzero real part.

in (ii), $J = \begin{bmatrix} \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A} \end{bmatrix}$ — " —

*** consider now the different possibilities in case (i):**

- First, all of those possibilities may be demonstrated using generic 1d systems: $\dot{x} = f(x, \mu)$
- all of those will have $\frac{\partial f}{\partial x} = 0$ at bifurcation pt.

$\Rightarrow$ near bif. pt: (x_0, μ_0)

$$f(x,\mu) = f(x_0, \mu_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial \mu}(\mu - \mu_0)$$

$$+ \frac{\partial^2 f}{\partial x \partial \mu}(x - x_0)(\mu - \mu_0) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x - x_0)^2 + \dots$$

spse. with no loss of generality that $(x_0, \mu_0) = (0, 0)$. We are interested in the bifurcation of fixed pts, where $\dot{x} = f(x, \mu) = 0$, so we also have $f(x_0, \mu_0) = 0$, So, near bif. pt.:

(#) $$f(x,\mu) = \underset{①}{f_\mu \cdot \mu} + \underset{②}{f_{x\mu} \cdot \mu x} + \underset{③}{\tfrac{1}{2} f_{xx}\, x^2} + \underset{④}{\tfrac{1}{2} f_{\mu\mu}\, \mu^2} + \underset{⑤}{\tfrac{1}{6} f_{xxx}\, x^3} + \dots$$

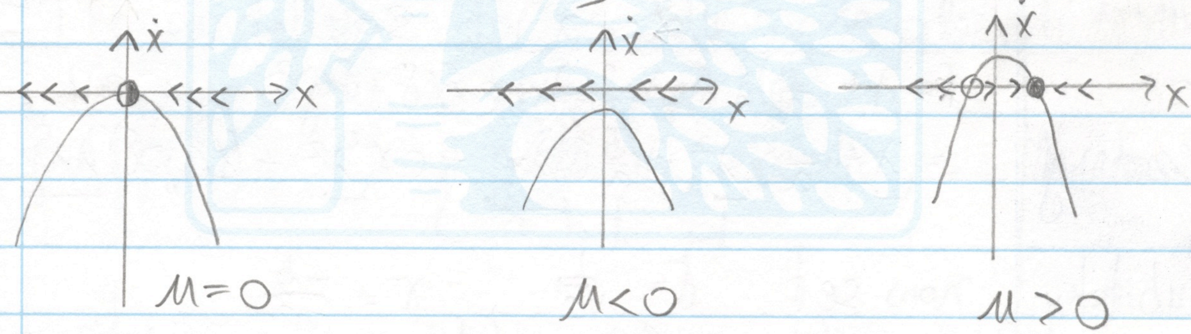
Depending on which coeffs in above expansion vanish, we can get 3 types of bifurcations.

Let us consider 3 specific examples of (#) which demonstrate these 3 types of 1d bifurcations:

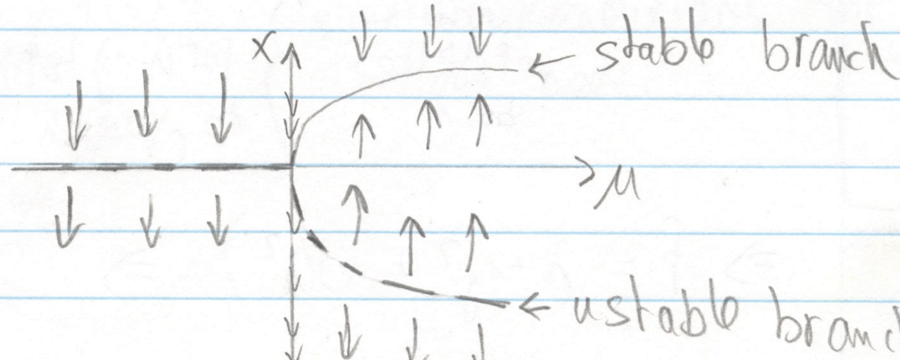
Saddle - node bifurcation

$$\left.\begin{aligned} \dot{x} &= f(x,\mu) = \mu - x^2 \\ \dot{x} &= 0 \Rightarrow x = \pm\sqrt{\mu} \end{aligned}\right\} \Rightarrow$$ fixed pts only for $\mu > 0$, there are two way to plot this:

for a given value of μ ①



combining all values of μ into one plot ②



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transformation to normal form of saddle-node bifurcation

suppose term ② is not zero, so that we can write near the non-hyperbolic fixed point with $\underbrace{f_\mu \neq 0,\ f_{xx} \neq 0}$:

$$f(x,\mu) \approx a\mu + b\mu x + cx^2 + \cdots \qquad (1)$$

transform from (μ, x) to (λ, y) by writing

$$\left.\begin{array}{l}\lambda = \lambda(\mu)\\ y = y(\mu, x)\end{array}\right\} \qquad \begin{array}{l}\mu = \alpha\lambda\\ x = \gamma\lambda + \delta y\end{array} \qquad (2)$$

where α, γ, δ are constants to be determined.

subst (2) in (1):

$$f = a(\alpha\lambda) + b(\alpha\lambda)(\gamma\lambda + \delta y) + c(\gamma\lambda + \delta y)^2$$

to get the normal form for a saddle node (which we expect because $f_\mu \neq 0, f_{xx} \neq 0$)

set $\alpha = \frac{1}{a}$, so that

$$f = \lambda + \left(\frac{b}{a}\right)\lambda(\gamma\lambda + \delta y) + c(\gamma\delta + \delta y)^2$$

$$= \underbrace{\lambda} + \frac{b}{a}\gamma\underbrace{\lambda^2} + \left(\frac{b}{a}\delta\right)\underbrace{\lambda \cdot y} + c\gamma^2\underbrace{\lambda^2} + (2c\gamma\delta)\underbrace{\lambda y} + (c\delta^2)\underbrace{y^2}$$

now set $\delta = \frac{1}{\sqrt{c}}$, $\gamma = \frac{-b}{2ac}$

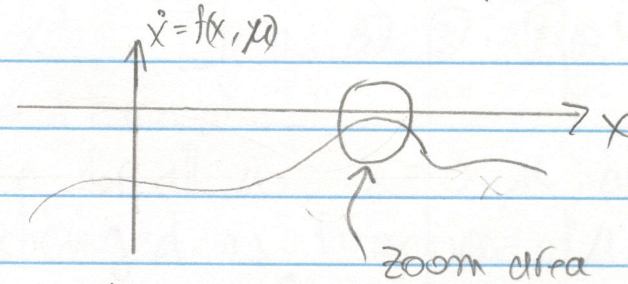
δ: for coeff of y^2 to be 1

γ: for $(\lambda \cdot y)$ terms to vanish

that's the allowed form according to Golubitsky and Schaeffer Vol 1.

$$\Rightarrow f = \lambda + y^2 + O(\lambda^2) \Rightarrow \text{saddle node} \ldots$$

note that saddle node bifurcation is due to a combination of terms 1 & 3 in ⊛ on p. 47.
adding term 2 would not change the qualitative picture⊛, which is like this for a general $f(x,\mu)$:



near the zoom area,
if f_μ & $f_{xx} \neq 0$, the behavior as μ changes is qualitatively the same as $f = \mu - x^2$.
so, the form $\dot{x} = \mu - x^2$ is representative of all saddle node bifurcations & is called "normal form".

The conditions for saddle-node bif. at (x_0, μ_0) are:

1 – $f(x_0,\mu_0) = 0$ { fixed point
2 – $\frac{\partial}{\partial x} f(x_0,\mu_0) = 0$ { non hyperbolic, so it can be a bif. pt.

3 – $\left.\frac{\partial f}{\partial \mu}\right|_{x_0,\mu_0} \neq 0$
4 – $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0,\mu_0} \neq 0$
} specific conditions for saddle node.

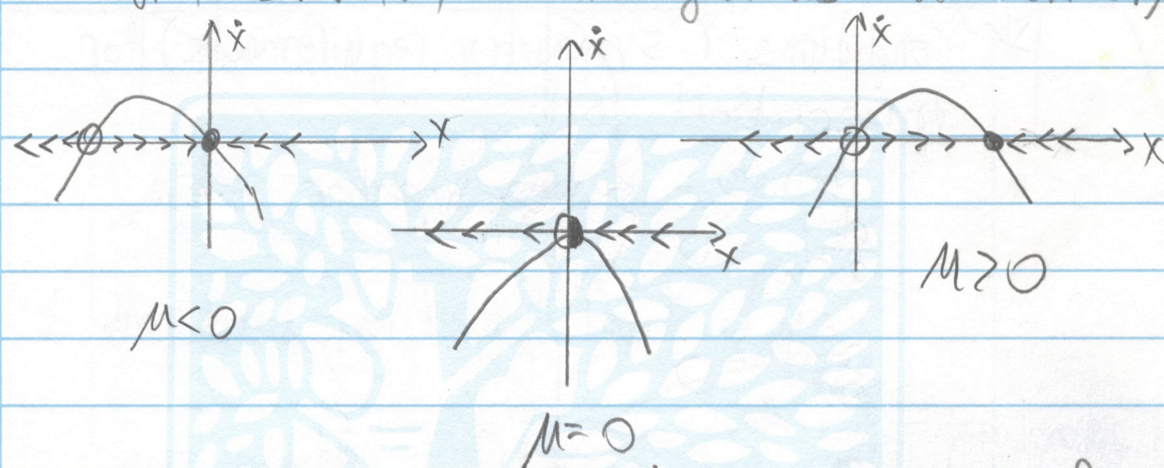
The next example of a 1d bifurcation is:

Transcritical bifurcation:

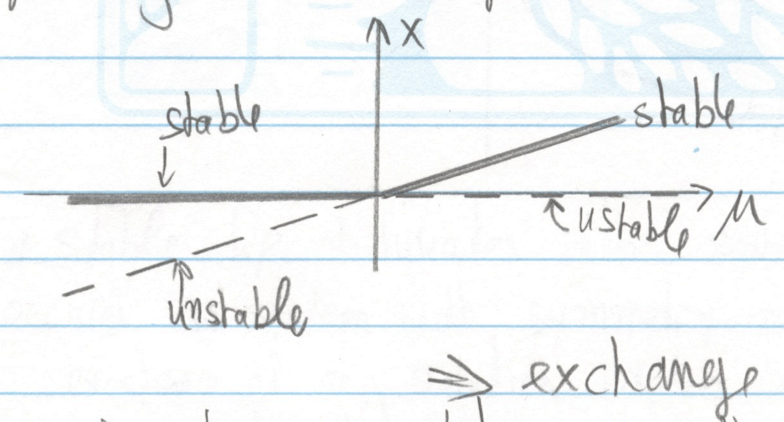
The normal form is

$$\dot{x} = f(x,\mu) = \mu x - x^2 \quad \text{[terms ②, ③ in ⑦, p. 47]}$$

- bif. point is at $(\mu, x) = (0,0)$.
- steady states are, for all μ: $x = 0, \mu$, but their stability is changed as function of μ:



plotting only fixed points as funct of μ:



$\Rightarrow$ exchange of stability.

zero is always a solution, so this bifurcation may be expected in systems for which a solution is known to always exist, & that it's stability may change as function of some parameter.

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example: logistic map.

$$x_{n+1} = \mu x_n (1 - x_n)$$

* conditions (symmetry requirements) for transcritical bif:

$$\left.\begin{aligned} f(x^*) &= 0 \\ \left.\frac{df}{dx}\right|_{x^*} &= 0 \end{aligned}\right\} \text{non hyperbolic f.p.}$$

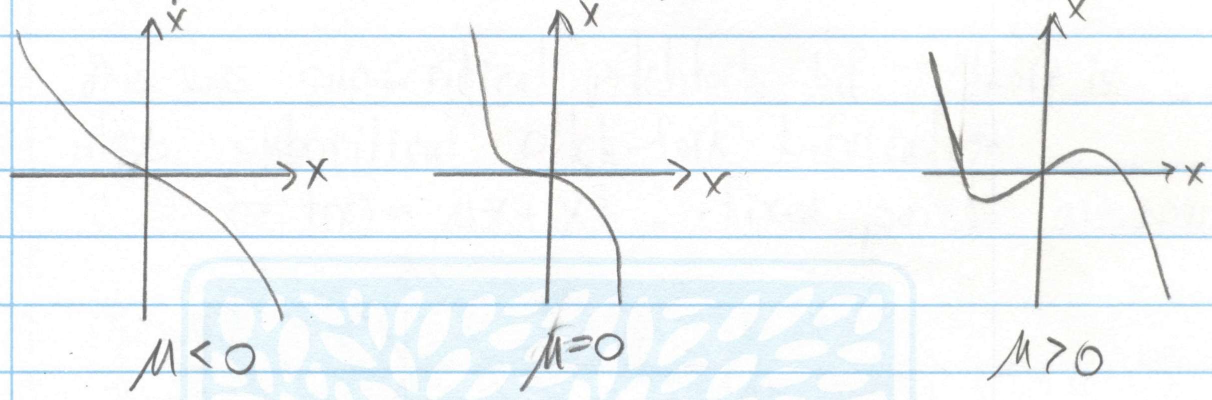
$$\left.\begin{aligned} &\left.\frac{\partial f}{\partial \mu}\right|_{\mu_0} = 0 \\ &\frac{\partial^2 f}{\partial \mu \partial x} \neq 0, \quad \frac{\partial^2 f}{\partial x^2} \neq 0 \end{aligned}\right\}$$

Pitchfork bifurcation.

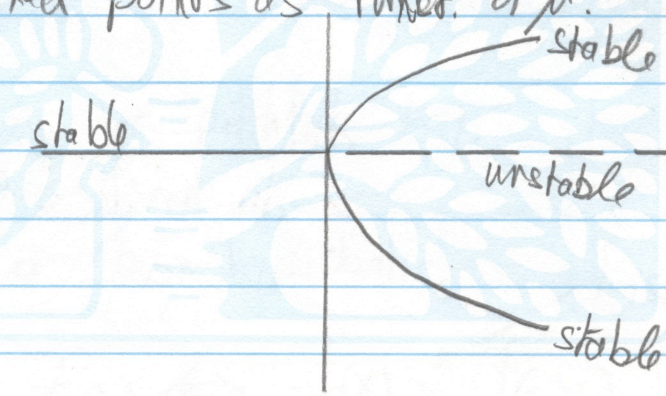
the normal form is

$$\dot{x} = \mu \cdot x - x^3 = f(x) \quad [\text{terms } 2, 5 \text{ in } (7), \text{ p. } 47]$$

$$f(x^*) = 0 \Rightarrow x^* = \{0, \pm\sqrt{\mu}\} \ \mu > 0; \quad x^* = \{0\} \ \mu < 0.$$



& the fixed points as funct. of μ:



looks like pitchfork (pdf$_1$)

* 1 stable f.p. bifurcates into 2 stable + 1 unstable.
* occures in system with symmetry in which the existence of one solution implies that of another.
→ e.g. looking at the oceanic circulation on both sides of the equator as function of some forcing parameter (strength of wind)
→ e.g. spatial symmetry, where eq'ns are invariant to $x \to -x$.

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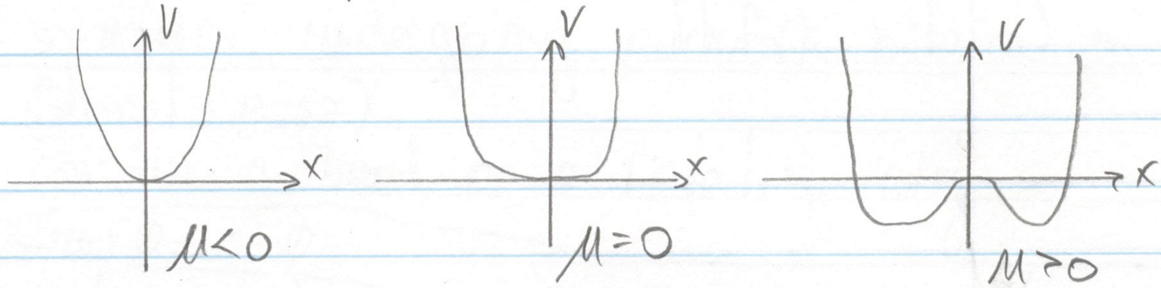
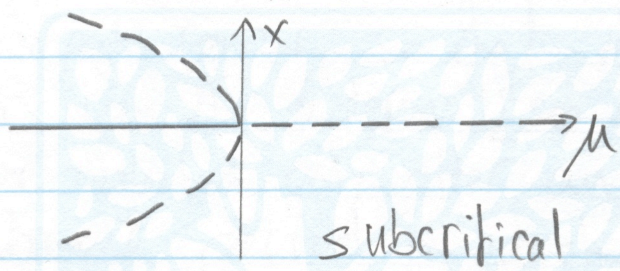
in terms of a potential: $-\frac{dV}{dx} = f(x) = \dot{x}$



this was <u>supercritical</u> pitchfork bif., there is also subcritical pitch-fork bifurcation:

$\dot{x} = f(x) = \mu x + x^3$. fixed points are now:



subcritical pitchfork bifurcation

* <u>conditions for pitchfork bif:</u>

$f(x^*)=0$ fixed pt.

$\frac{df}{dx}=0$ non hyperbolic

f is odd

$\frac{\partial^3 f}{\partial x^3} \neq 0 \Leftrightarrow -f(x) = f(-x)$ [because then $\dot{x}=f(x)$ is invariant under $x \to -x$]

$\frac{\partial^2 f}{\partial x^2} = 0 \leftarrow$ [otherwise f is not odd].

$\frac{\partial^2 f}{\partial x \partial \mu} \neq 0$

$\frac{\partial f}{\partial \mu} = 0$

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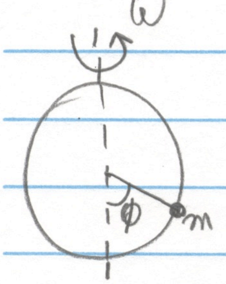


Example of a physical system with a symmetry, undergoing pitchfork bifurcation (Strogatz p. 62)

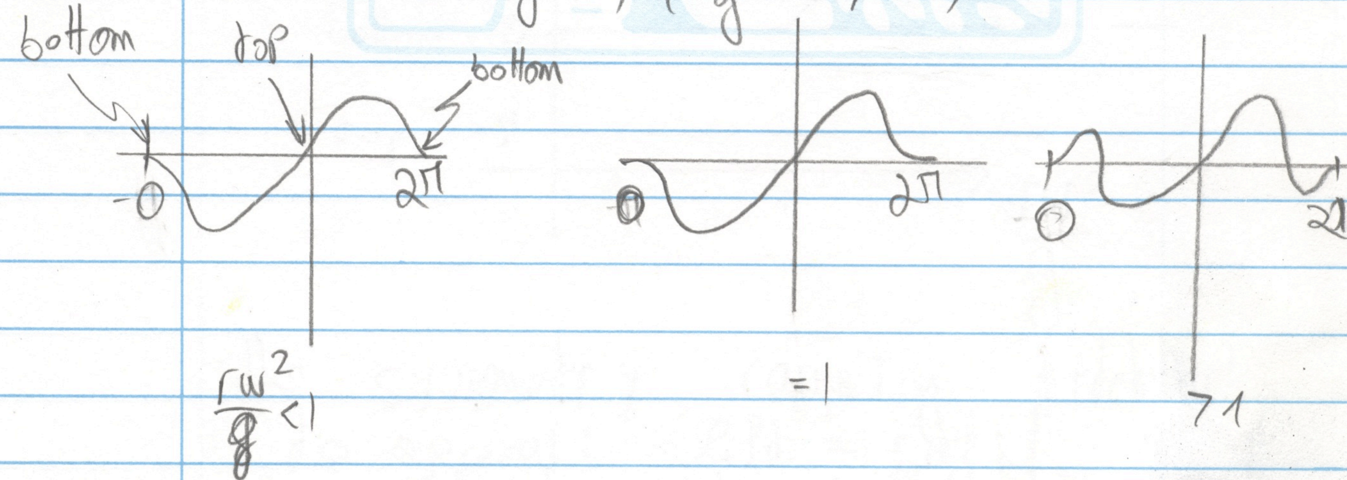
consider a bead on a rotating ring:

$$\underbrace{mr\ddot{\phi}}_{\text{acceleration}} = \underbrace{-b\dot{\phi}}_{\substack{\text{friction}\\\text{proportional to velocity}}} \underbrace{- mg\sin\phi}_{\text{gravity}} + \underbrace{mr\omega^2\sin\phi\cos\phi}_{\text{centrifugal force}}$$

neglect $mr\ddot{\phi}$ by assuming this to be an "overdamped" regime, so that the $\ddot{\phi}$ term is much smaller than the $\dot{\phi}$ term [see Strogatz p. 66-68, $\ddot{\phi}$ may be neglected only after an initial brief adjustment period in which it is large].

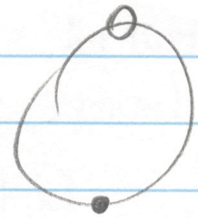
$$\Rightarrow \quad b\dot{\phi} = -mg\sin\phi + mr\omega^2\sin\phi\cos\phi$$

$$= mg\sin\phi\cdot\left(\frac{r\omega^2}{g}\cos\phi - 1\right)$$

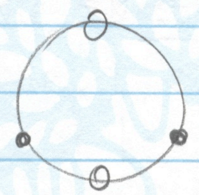


physically:

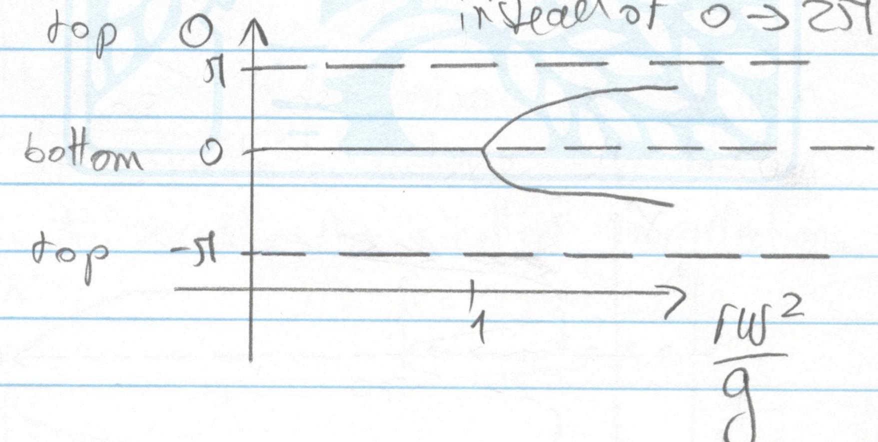
weak rotation $\Rightarrow$ only $\begin{cases} \text{stable bottom,} \\ \text{unstable top} \end{cases}$:



larger rotation: unstable top & bottom, two stable mid-pts, balancing centrifugal & gravity forces:



bifurcation curve (plotted from $-\pi \to \pi$ instead of $0 \to 2\pi$. same thing...)

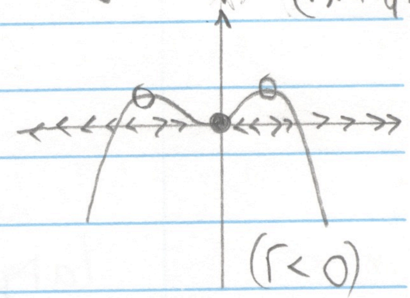


the symmetry causing pitchfork to appear: left = right.

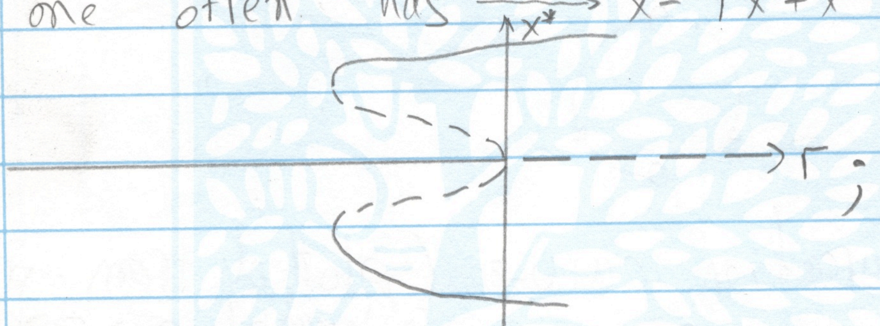
changing $\phi \to -\phi$ leaves eq'n unchanged.

hysteresis in subcritical pitchfork bifurcation:

the equation $\dot{x} = rx + x^3$ has diverging solutions. in terms of the potential function: $V = -(rx^2 + \frac{1}{4}x^4)$



in physical systems where diverging solutions are not acceptable, yet where the symmetry $x \to -x$ is perserved, one often has $\longrightarrow$ $\dot{x} = rx + x^3 - x^5$:



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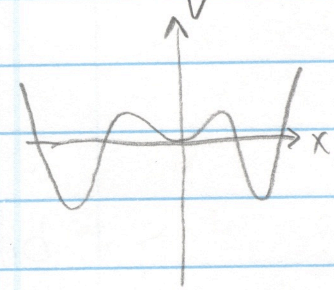
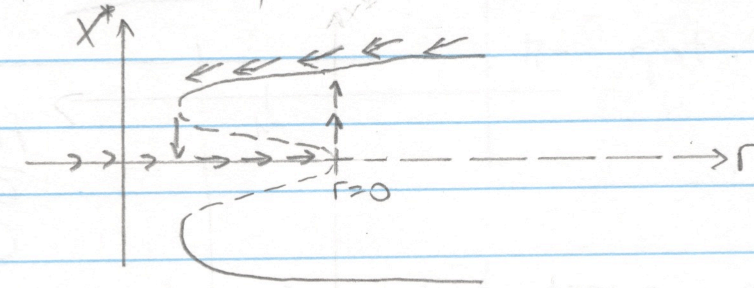


now, as r varies slowly, so that at any given moment the system is always at equilibrium (f.p.):



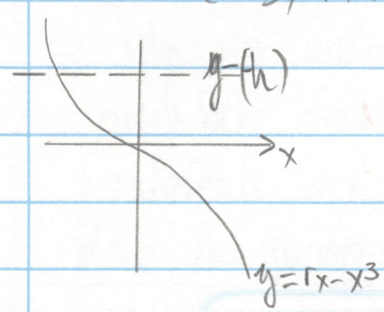
$\Rightarrow$ existence of multiple equilibria allows a hysteresis as function of r.

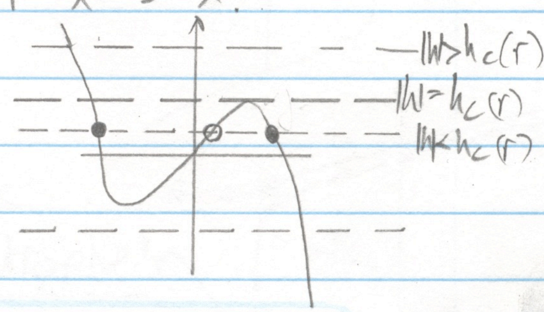
Imperfections (symmetry breaking) & Catastrophes

consider a perturbation h to a pitchfork bif:

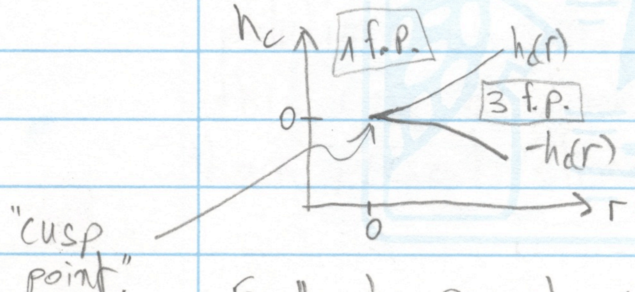
$$\dot{x} = h + rx - x^3$$

no more symmetric under $x \to -x$.

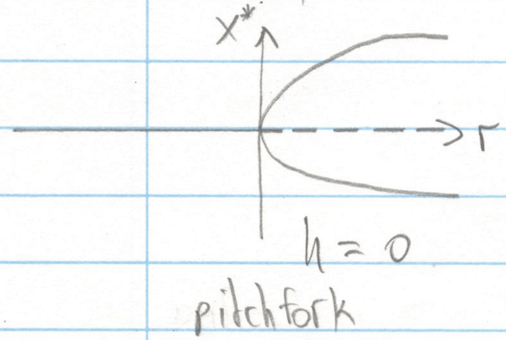


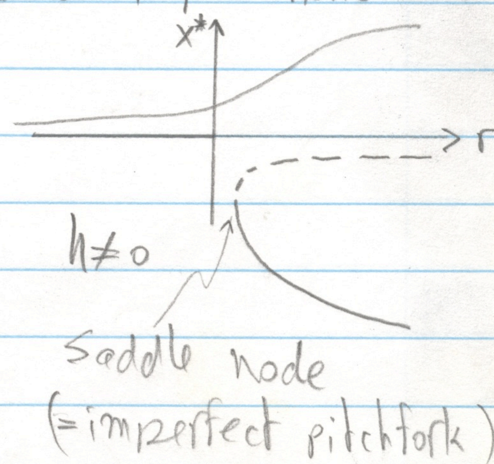


fixed pts are crossings of $y=-h$ & $y = rx - rx^3$
$\Rightarrow$ can have 1, 2, 3 f.p. as h varies. Find $h_c(r)$
explicitly: at $h = h_c$, $\frac{d}{dx}(rx - x^3) = 0 \Rightarrow x_{max} = \sqrt{\frac{r}{3}}$
$\Rightarrow h_c(r) = r x_{max} - x_{max}^3 = \frac{2}{3} r \sqrt{\frac{r}{3}}$, similarly for x_{min}...

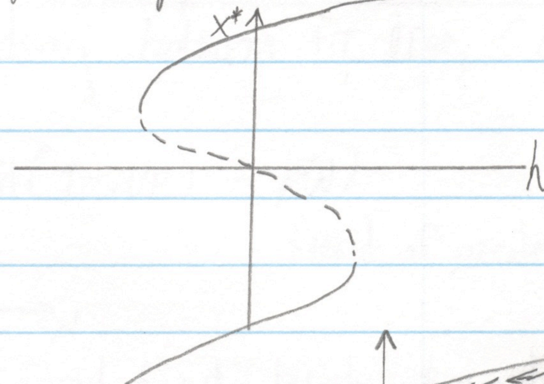


Finally, to see why this is called imperfections:

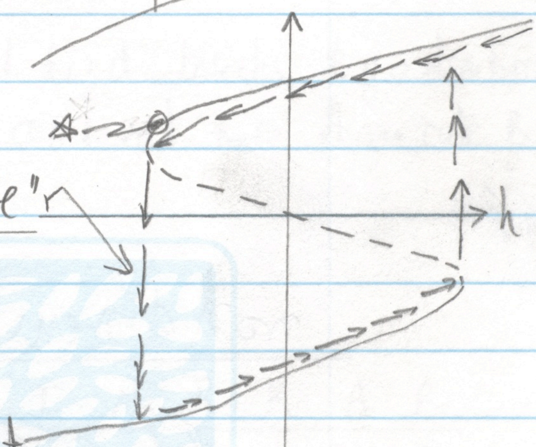




what about catastrophes: plot f.p. as function of h, for a given r:



⇒ like in subcritical pitchfork, can have hysteresis as h varies. also, the jump is "catastrophe"



* example: climate system,
$X \equiv$ global-averaged temperature,
h = incoming solar radiation.
* = location of present-day climate
⇒ a small decrease in solar radiation (due to atmc polution blocking radiation) ⇒ a catastrophe, in this case a climate state of over 100°c colder than present climate, snow-covered earth....

non dimensionalize w/o pi theorem (strogatz, p75)

let $x = N/A$

$$\Rightarrow \frac{A}{B}\frac{dx}{dt} = \frac{R}{B} A x\left(1 - \frac{Ax}{K}\right) - \frac{x^2}{1+x^2}$$

let $t = \tau \cdot \left(\frac{A}{B}\right) \Rightarrow \frac{d}{dt} = \frac{B}{A}\frac{d}{d\tau}$

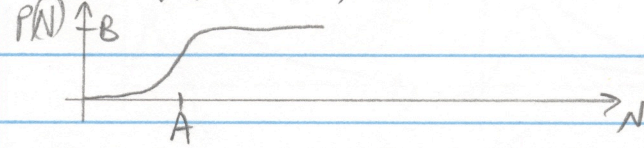
also let: $r = \frac{RA}{B}$, $k = \frac{K}{A}$

$$\Rightarrow \frac{dx}{d\tau} = r \cdot x \cdot \left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2} = f(x).$$

example of bifurcation & catastrophe: insect outbreak

spruce budworm, attacking balsam fir tree, Canada, & eaten by birds:

N = insect population; $\dot{N} = RN(1 - N/K) - p(N)$



$\hat{}$effect of predators.

$p(N)$: if N is small, birds can't find insects, & go elsewhere

if N is too large, birds can't eat more than rate B.

let: $p(N) = BN^2/(A^2+N^2)$

nondimensionalize:

Buckingham's Π theorem:

	T	N_0	R	K	A	B
(kg)	0	1	0	1	1	1
(sec)	1	0	-1	0	0	-1

rank = 2, variables = 6 $\Rightarrow$ nondim parameters = 4

[above, we let $N = N_0 \cdot X$, $t = T \cdot \tau$]. repeating variables: A, B

express others as $[T] = [A/B] \Rightarrow \delta \equiv \frac{TB}{A}$

$[N_0] = [A] \Rightarrow \mu \equiv N_0/A$

$[R] = [B/A] \Rightarrow r \equiv RA/B$

$[K] = [A] \Rightarrow k \equiv K/A$

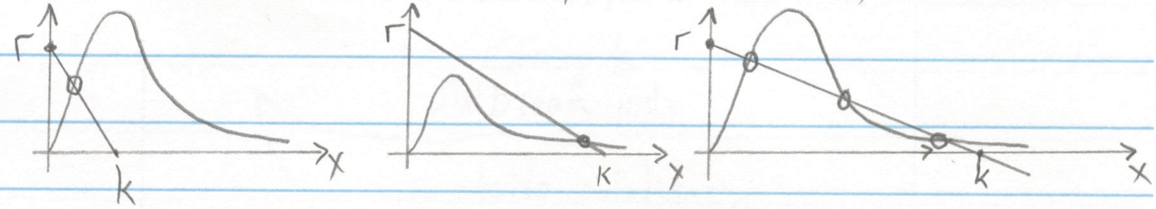
choose two non dim variables $\delta = \mu = 1$

$$\Rightarrow \frac{dX}{d\tau} = \boxed{\dot{X} = rX\left(1 - \frac{X}{k}\right) - \frac{X^2}{1+X^2}} = f(X)$$

fixed points for insect population: occur when

$f(x) = x \cdot \left[r \cdot (1 - x/k) - x/(1+x^2) \right] = 0$

$\Rightarrow$ $x = 0$ or lines $x/(1+x^2)$ & $r(1-x/k)$ cross.

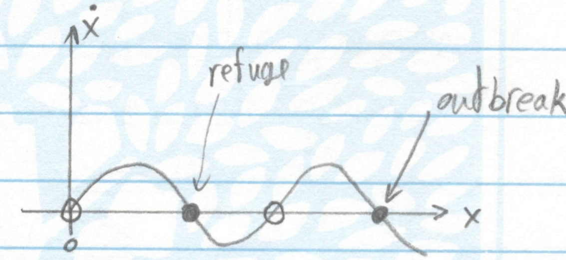


(small r, k) (larger r, large k) (larger k)

$\Rightarrow$ either 1, 2 or 3 fixed points = steady state populations <u>in addition</u> to f.p. at $x = 0$

<u>stability:</u>

[large k case:]



refuge: small # of insects,
outbreak: large " " " ...

bifurcation curve: transition from 1 to 3 solutions as k varies occurs via saddle-node bifurcation, a bit like in imperfect pitchfork. can find the critical $k(r)$ or $r(k)$ where this happens using two eq'ns:

(a) f.p. $\Rightarrow$ $r(1 - x/k) = x/(1+x^2)$

(b) saddle node: two lines have same slope

$$\Rightarrow \frac{d}{dx}\left(r(1-x/k)\right) = \frac{d}{dx}\left(x/(1+x^2)\right)$$

$$\Rightarrow -\frac{r}{k} = (1-x^2)/(1+x^2)^2$$

$\Rightarrow$ solve (a), (b) for $r(x)$, $k(x)$ and plot as

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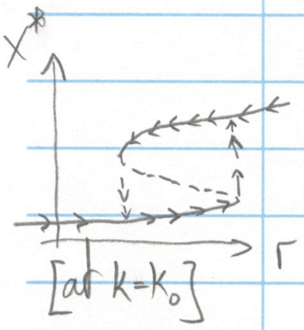
function of x: $r = 2x^3/(1+x^2)^2$

$k = 2x^3/(x^2-1)$.

because $k>0$, we plot $r(x)$, $k(x)$ for $x>1$



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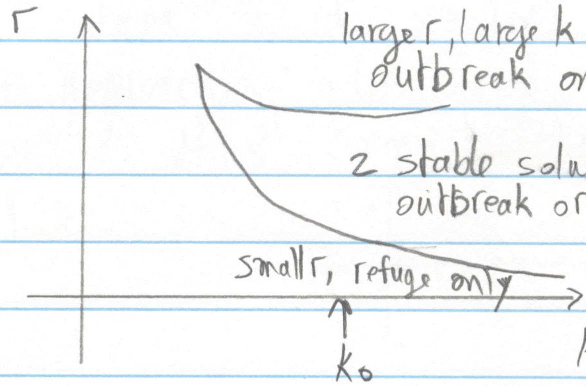
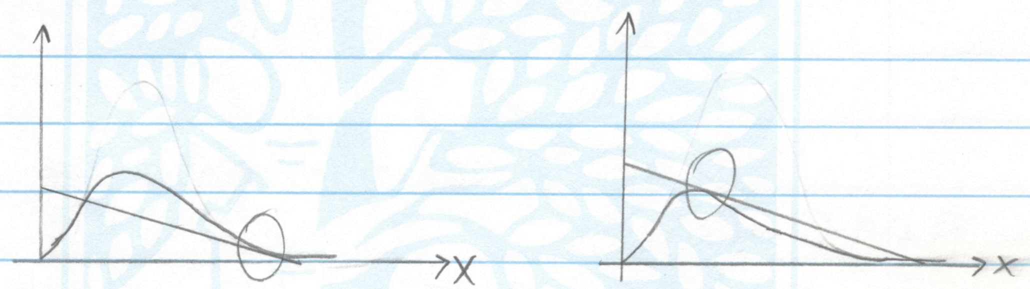
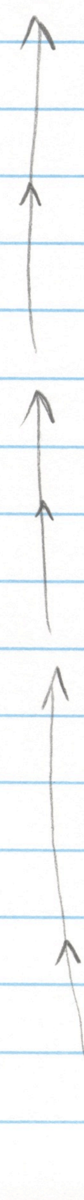


two lines on bif diagram correspond to two possible saddle node bifurcations:





hysteresis

can show that as forest grows, if s is average tree size, carrying capacity $\propto s \Rightarrow K = k' \cdot s$.
also, the critical population above which birds start looking for insects depend on a critical density per leaf area: $A = A' \cdot s \Rightarrow r = \frac{RA'}{B} \cdot s$, $k = \frac{k'}{A'}$

$\Rightarrow$ r increases with tree size $\Rightarrow$ can have a hysteresis phenomenon as r varies.
for insect population