center manifold theorem

(a) First, examples from p. 85 & 87 here (Strogatz p. 242-245).

(b) ⇒ we want to find the equation along the center manifold only, as dynamics on stable & unstable manifolds are simple (that is, exponentially contracting & expanding).

(c) let stable, unstable & center subspaces be spanned by the stable, unstable & zero real part eigenvectors of $\dot{\vec{x}} = \vec{f}(\vec{x}, \mu)$ at a fixed pt x^*. Denote them by E^s, E^u, E^c.

(d) stable manifold, W^s: parallel to E^s at x^*, invariant under $\dot{x} = f(x, \mu)$; unstable manifold W^u: similar. both defined on p 223 (OH 4.3).

(e) objective of center manifold theorem: write $\vec{x} = (\vec{X}, \vec{Y}, \vec{Z})$ such that $X \in W^c$, $Y \in W^s$, $Z \in W^u$. derive an eq'n for X which does not involve Y, Z, hence reducing the problem's dimensionality. E.g. if $\dim(E^c) = 1$, the procedure will result in a 1-d bif problem regardless of $\dim(\vec{x})$.

(f) theorem guarantees that bif of $\dot{x} = f(x, \mu)$ is equivalent to

$$\begin{cases} \dot{X} = \tilde{f}(X) \\ \dot{Y} = -Y \\ \dot{Z} = Z \end{cases}$$

(g) example: Lorenz system, $x^* = (0,0,0)$: Guckenheimer & Holmes pp. 128-130, eq'ns 3.2.5 until just before 3.2.12.

center manifold reduction

*purpose: simplify system of eqns as much as possible while preserving the bif structure.

start with $\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{\mu})$ with a bif at $(\vec{x}_0, \vec{\mu}_0)$.

step 1: transform fixed point to zero:

$\vec{x}(t) = \vec{x}_0 + \vec{y}(t)$, where bif pt is now at $\vec{y} = 0$.

$$\rightarrow \quad \dot{y} = F(\vec{x}_0 + \vec{y}; \mu_0) \qquad (2.1.4)$$

$\vec{\mu}_0$ = parameter values at bif pt.

step 2: Taylor expand:

$$\dot{y} = Ay + F_2(y) + F_3(y) + O(|y|^4). \qquad (2.3.68)$$

where $A = D_{\vec{x}} F(x_0; \mu_0)$; each of the components of the vector F_n is an n-degree polynomial in y_i, the components of $\vec{y}$.

step 3: Arrange eigen values of A such that $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the m eigenvalues with a zero real part. $\vec{p}_1 \ldots \vec{p}_m$ are the corresponding eigenvectors. $\lambda_{m+1}, \ldots, \lambda_n$ have non-zero real part.

step 4: introduce linear transformation $\vec{y} = \underset{\sim}{P} \vec{v}$ where $\underset{\sim}{P} = [P_1, P_2, \ldots, P_n]$ & we find

{this is a rotation in phase space}

$$\dot{v} = \underset{\sim}{J} v + \underset{\sim}{P}^{-1} F_2(\underset{\sim}{P} v) + \underset{\sim}{P}^{-1} F_3(P v) + \cdots \qquad (2.3.69)$$

where $J = \begin{bmatrix} J_c & 0 \\ 0 & J_s \end{bmatrix} = \underset{\sim}{P}^{-1}\underset{\sim}{A}\underset{\sim}{P}$ and (2.3.70)

J_c is $m \times m$ with eigenvalues $\{\lambda_1 \dots \lambda_m\}$ &
J_s is an $(n-m)\times(n-m)$ with eigenvalues $\{\lambda_{m+1}, \dots \lambda_n\}$

$$\begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

Jordan block

[about Jordan form: 1st Jordan form is:
(a) only diagonal & super diagonal are non zero
(b) diagonal values are eigenvalues
(c) one Jordan block per eigenvector, eigenvalues on diagonal are repeated # of times equal to their multiplicity.
(d) size of each Jordan block: is equal to the degree
]

[generalized eigenvectors: repeated eigenvalues does not necessarily imply less than n indep eigenvectors. but if there are less, we need to find generalized eigenvectors: let $\vec{v}_{i-1}$ be an eigenvector. $\vec{v}_i$ is a generalized eigenvector if

$$\underset{\sim}{A}\vec{v}_i = \lambda_i \vec{v}_i + \vec{v}_{i-1} \quad \text{or} \quad (\underset{\sim}{A} - \lambda \underset{\sim}{I})v_i = v_{i-1}$$

note that $(\underset{\sim}{A} - \lambda\underset{\sim}{I})^2 v_i = (\underset{\sim}{A} - \lambda\underset{\sim}{I}) v_{i-1} = 0$

so a generalized eigenvalue is one that satisfies $(\underset{\sim}{A} - \lambda\underset{\sim}{I})^k v = 0$ for a high enough k.]

[transformation to Jordan form: is $\underset{\sim}{P}$ is composed of eigenvectors (& generalized eigenvectors if needed) of A: Jordan form is $\underset{\sim}{P}^{-1}\underset{\sim}{A}\underset{\sim}{P}$]

<u>step 5:</u> we can now separate into two eq'ns that are linearly independent, although nonlinearly coupled:

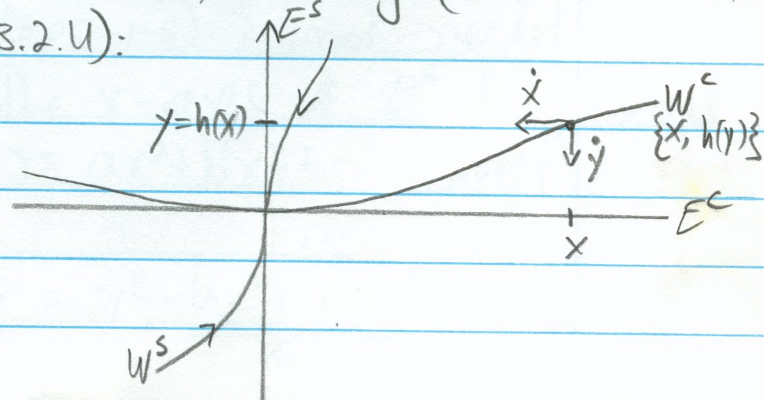
$$\dot{V}_c = J_c V_c + G_2(V_c, V_s) + G_3(V_c, V_s) + \cdots \quad (2.3.71)$$
$$\dot{V}_s = \underset{\sim}{\tilde{J}_s} V_s + H_2(V_c, V_s) + H_3(V_c, V_s) + \cdots \quad (2.3.72)$$

where $V_c \equiv (V_1, \ldots V_m)$; $V_s \equiv (V_{m+1} \ldots V_n)$.

<u>Note</u> that $G_2(0,0) = G_3(0,0) = \cdots G_N(0,0) = 0$ and similarly for $H_N(0,0)$. this is because G_N, H_N are <u>non linear</u> polynomials in V_c, V_s & hence vanish for ($V_c = 0$, $V_s = 0$). (nonlinear polynomials without constant & linear terms!)
<u>Note also</u> that the jacobian of G_N & H_N vanishes at ($V_c = 0$, $V_s = 0$) for the same reason.

<u>step 6:</u> center manifold theorem guarantees that we can write $V_s = h(V_c)$ as a local approximation where h is a polynomial. h is the center manifold, as illustrated by this fig (Guckenheimer & Holmes p. 131, fig 3.2.4):

[Here:
$x \equiv V_c$, $y \equiv V_s$]



As can be seen in this figure, the center manifold is parallel to E_c so that $D_{\vec{x}} h(\vec{x})\big|_{\vec{x}=0} = 0$. clearly also $h(0)=0$, as the manifold starts at the origin. in our notation (rather than figure's) $h(0)=0$ & $D_{\vec{v}_c} h(v_c)\big|_{\vec{v}_c=0} = 0$.

step 7: finding $h(v_c)$: given the above b.c. for h, let $h =$ polynomial in v_c with no linear or constant terms, & find coeffs by substituting $v_s = h(v_c)$ into (2.3.72)

then, the stable manifold eq'n gives us:

$$\dot{v}_c = J_c v_c + G_2[v_c, h(v_c)] + G_3[v_c, h(v_c)] + \cdots \quad (2.3.75)$$

this has the right dimension, but still needs to be transformed to normal form, using normal form theory (more on this later).

Example: (2.20)

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}kxy\\ x^2\end{pmatrix} \quad (2.1.22)$$

because the linear part is already in jordan form, the rotation (steps 1-4) is not needed. clearly E^c is the x-axis & E^s the y-axis. so, let $y = h(x) = ax^2 + bx^3 + \cdots$, subst in eq'n for $\dot{y}$ to find

$$\underbrace{(2ax + 3bx^2)\dot{x}}_{\dot{y}} = x^2 - \underbrace{(ax^2 + bx^3)}_{y} \quad (\#)$$

subst $\dot{x}$ from original eq'n, to get:

$$\dot{x} = Kxy = kx(ax^2 + bx^3)$$

plug this into #:

$$(2ax + 3bx^2 + \cdots)\,kx(ax^2 + bx^3 + \cdots) = (1-a)x^2 - bx^3 + \cdots \quad (2.3.77)$$

equate coeffs $\Rightarrow$ $a = 1$, $b = 0$

$$\Rightarrow \quad y = h(x) \approx x^2 + 0 \cdot x^3 + \cdots$$

subst this into the x-equation (center manifold eq'n), $\Rightarrow$ $\dot{x} = Kx^3 + \cdots$

it is easy to analyse the behavior of this eq'n as function of k, although this was not so clear for the original eq'n!

* Next, discuss normal form theory, using pages 89-91 here.