0d energy balance and snowball bifurcation

A zero-dimensional version of the "Budyko-Sellers" energy-balance model for Earth's temperature

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1 Derivation of energy balance equation

An equation for the earth heat budget has the general form

$$\frac{d}{dt}\text{Heat} = \text{net incoming radiation} - \text{outgoing radiation}$$

Let the incoming radiation from the sun, averaged over Earth's surface, be S watts/m^2. A portion α of this radiation is reflected by the ice, snow, clouds, land and oceans. This portion depends on the amount of ice and snow cover and therefore on the temperature, $\alpha = \alpha(T)$. So only the net incoming radiation is $S(1 - \alpha(T))$. The outgoing radiation is in the form of infrared radiation, or heat, which according to the law of black body radiation is σT^4. Because the earth is not a perfect black body this expression is multiplied by a factor ϵ, the emissivity, that is smaller than 1.

A globally-averaged energy balance equation for the atmosphere may therefore be written as

$$C\frac{dT}{dt} = S(1 - \alpha(T)) - \epsilon\sigma T^4$$

where the lhs is the heat capacity times the time rate of change of the globally averaged temperature; the rhs includes the incoming radiation S multiplied by one minus the albedo (reflectivity, α), and the outgoing long wave (infrared) radiation. At steady state the lhs vanishes. Assuming the albedo is mostly a function of the amount of ice, it makes sense to assume it has a low value at high temperature (no ice), a high value at very low temperature (earth completely frozen), and some linear variation in between

$$\alpha(T) = \begin{cases} \alpha_1 & \text{if } T < T_1 \\ \alpha_1 + (\alpha_2 - \alpha_1)[T - T_1]/[T_2 - T_1] & \text{if } T_2 > T > T_1 \\ \alpha_2 & \text{if } T > T_2 \end{cases} \quad (1)$$

2 Qualitative solution

Steady state is obtained when the lhs vanishes, so that incoming solar radiation is equal to the outgoing long wave radiation. Fig. 1 shows the incoming radiation, outgoing, and their difference. Note that there are three equilibria (middle panel). Two of them are stable and one unstable. The leftmost one is a snowball solution, the righ-most one is a no-ice solution.

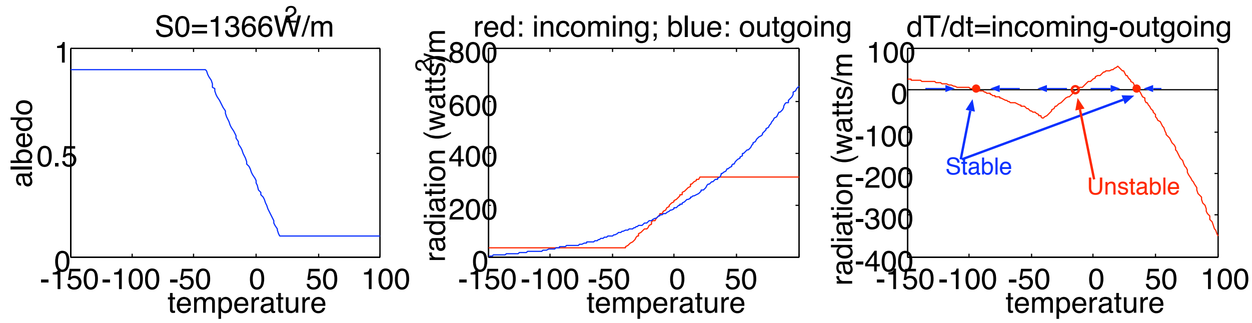


Figure 1: (left) albedo as function of temperature. (center) incoming and outgoing radiation terms in the energy balance model. (right) right hand side of heat balance equation, corresponding to incoming minus outgoing radiation, as function of temperature.

3 Bifurcations and hysteresis

As the strength of the solar input S or the emissivity ε is changed, the two curves in the middle panel of Fig. 1 vertically move with respect to one another. As a result the number of crossing points between them changes. If the blue curve moves down (atmospheric emissivity becomes lower, therefore increasing the greenhouse effect and warming climate, or solar constant S becomes larger), the two lower crossing points representing cold and moderate climates disappear via a saddle node bifurcation, leaving only the warmer climate solution. If it moves up, the two upper crossing points similarly disappear, leaving only the cold (snowball) climate. The number and character of the solutions as function of such a bifurcation parameter may be plotted schematically as Shown in Fig. 2. Note that a hysteresis is implied here.

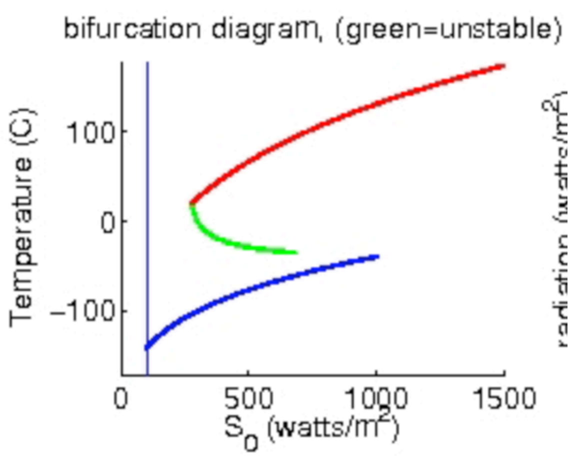


Figure 2: Bifurcation diagram for 0d energy balance model, green curve corresponds to unstable solutions.

The original papers for this are Budyko (1969); Sellers (1969). For geological evidence for Snowball Earth events and some relevant teaching material, see http://www.snowballearth.org/slides.html.

References

M. I. Budyko. The effect of solar radiation variations on the climate of the earth. *Tellus*, 21:611–619, 1969.

W. D. Sellers. A global climate model based on the energy balance of the earth-atmosphere system. *J. Appl. Meteor.*, 8:392–400, 1969.

saddle node bifurcation

2 back-to-back saddle nodes,
hysteresis and multiple equilibria

So, near bif. pt.:

(#) $$f(x,\mu) = f_\mu \cdot \mu + f_{x\mu} \cdot \mu x + \tfrac{1}{2} f_{xx} x^2 + \tfrac{1}{2} f_{\mu\mu} \mu^2 + \tfrac{1}{6} f_{xxx} x^3 + \dots$$
(1) (2) (3) (4) (5)

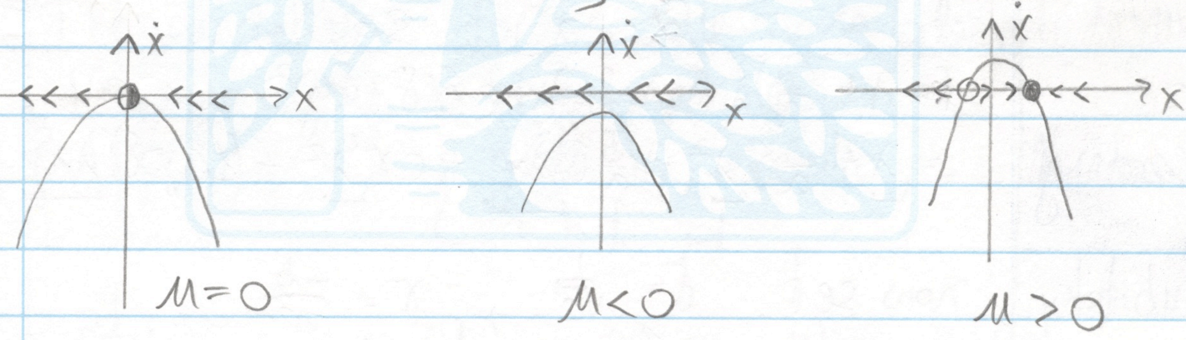
Depending on which coeffs in above expansion vanish, we can get 3 types of bifurcations.

Let us consider 3 specific examples of (#) which demonstrate these 3 types of 1d bifurcations:

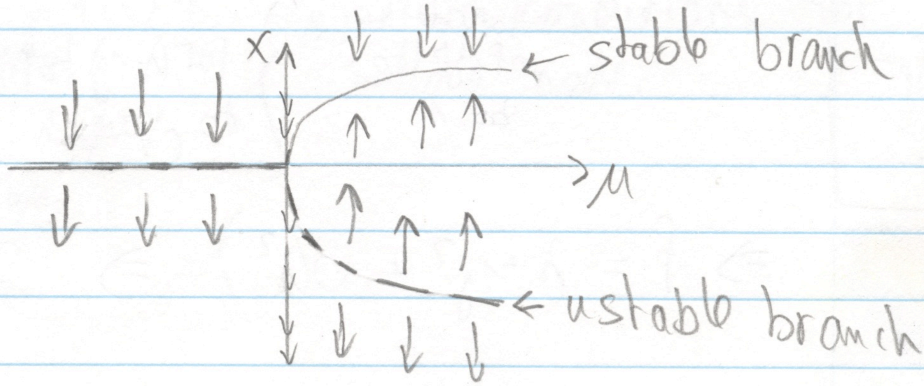
Saddle - node bifurcation

$\dot{x} = f(x,\mu) = \mu - x^2$ } ⇒ fixed pts only for $\mu > 0$,
$\dot{x} = 0 \Rightarrow x = \pm\sqrt{\mu}$ } there are two way to plot this:

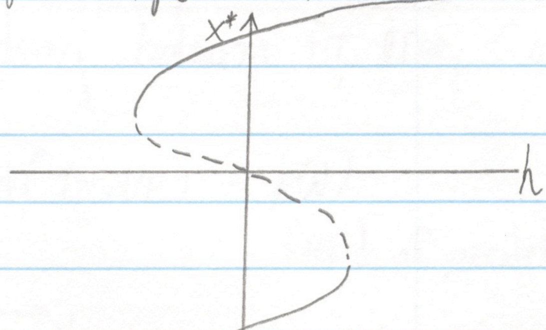
(1) for a given value of μ



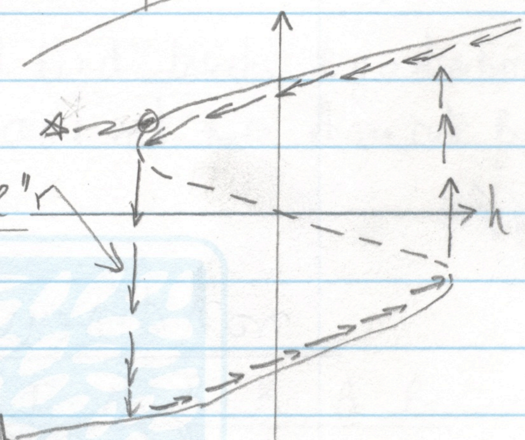
(2) combining all values of μ into one plot



what about catastrophes: plot f.p. as function of h, for a given r:



⇒ like in subcritical pitchfork, can have hysteresis as h varies. also, the jump is "catastrophe"



* example: climate system,
x ≡ global-averaged temperature,
h = incoming solar radiation
☆ = location of present-day climate
⇒ a small decrease in solar radiation (due to atmc polution blocking radiation) ⇒ a catastrophe, in this case a climate state of over 100°C colder than present climate, snow-covered earth....

1&2 level energy balance

lapse rate and emission level

tion scenario known as the "Representative Concentration Pathway (RCP) 8.5" used by the Intergovernmental Panel on Climate Change (IPCC), as well as some more moderate scenarios. The concentrations of other anthropogenic greenhouse gases have also increased, including methane (CH_4), nitrous oxide (N_2O), CFC12, CFC11, and as we will see further increase the anthropogenic greenhouse effect.

The discussion below starts with simple representations of the greenhouse effect allowing us to understand how it leads to warming (section 2.2). We then consider how greenhouse gases work, the role of the wavelength-dependence of the absorption of electromagnetic radiation (heat) by greenhouse gases, a comparison between different greenhouse gases, and the water vapor feedback that further amplifies the effect of anthropogenic greenhouse gases (section 2.3).

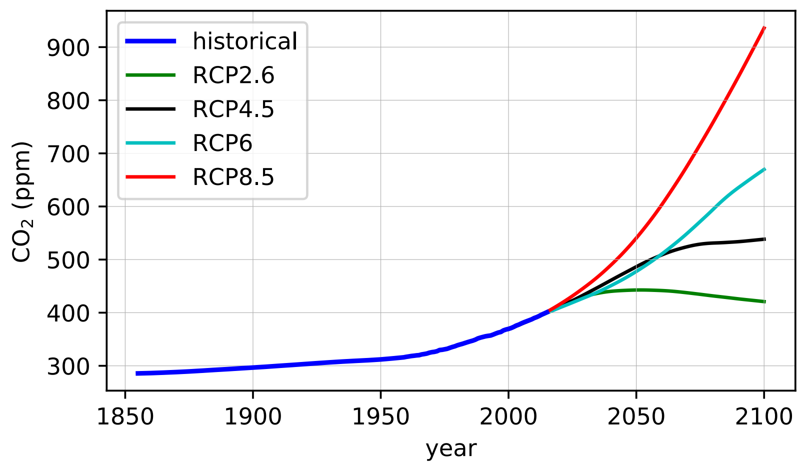


Figure 2.1: Annually-averaged CO_2 concentration, observed and projected according to different RCP scenarios.

2.2 The greenhouse effect

2.2.1 Earth's energy balance

We can estimate the globally averaged surface temperature of Earth based on the balance of incoming radiation from the sun with the outgoing heat escaping to outer space (Figure 2.2a). Consider first what the averaged Earth temperature would have been ignoring the effects of the atmosphere. The incoming solar radiation at the top of the atmosphere is given by the solar constant, $S_0 = 1361$ W/m^2. The total solar radiation encountered by Earth is therefore the solar constant times the cross-section area of Earth, $S_0\pi R_E^2$, where R_E is the Earth radius. Over a day, this radiation is distributed over the entire Earth surface area, $4\pi R_E^2$, so that the radiation per unit area, averaged over the Earth surface and over a day, is the ratio of the total radiation and the total area, or $S_0/4$. The globally averaged surface temperature is denoted T, and the fraction of the incoming solar radiation reflected (by ice, clouds, land, etc), or the "albedo", is denoted by $\alpha \approx 0.3$. A "black body" is one that absorbs all incident radiation, and when it is in thermal equilibrium at a temperature T, it emits a total radiation per unit area of σT^4, where $\sigma = 5.66961 \times 10^{-8}$ W m^{-2}K^{-4} is the Stefan-Boltzmann constant. Assume the outgoing radiation from Earth (escaping heat) to be given by the black body radiation formula, with T being the averaged surface temperature. The electromagnetic radiation from the sun has wavelengths of about 0.25–2 micrometer (μm), and is referred to as "shortwave" radiation, which includes visible light that is characterized by wavelengths of 0.4–0.7 μm. The thermal radiation emitted by Earth is characterized by wavelengths of roughly 5–35 μm and is therefore referred to as "longwave" radiation.

Energy conservation requires that the incoming shortwave radiation from the sun equals the outgoing longwave radiation (Figure 2.2a),

$$\frac{S_0}{4}(1-\alpha) = \sigma T^4,$$

which gives,

$$T = \left(\frac{(S_0/4)(1-\alpha)}{\sigma}\right)^{1/4} = 255\text{K} = -18\text{C} \equiv T_0. \qquad (2.1)$$

This is too cold as at such a temperature the Earth surface would have been frozen, and the actual globally average surface temperature of the Earth is about 14°C (287K), so something is clearly missing. That missing factor is the greenhouse effect of the atmosphere.

2.2.2 The greenhouse effect: a 2-layer model

We now add the effect of the atmosphere, whose temperature is denoted by T_a. To begin, we treat the atmosphere as a single layer and assume that it absorbs heat (longwave radiation) escaping from the surface and then re-emits it both up and down (Figure 2.2b). We write separate energy balances for the surface and for the atmosphere. The atmosphere is not a perfect black body, and only emits a fraction ε of the corresponding radiation of a black body with its temperature. Similarly, it also absorbs a fraction ε

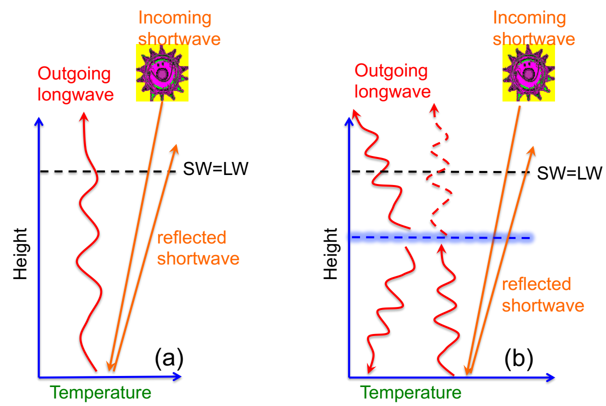


Figure 2.2: Two models of Earth's energy balance. (a) A one-layer model showing the three components of radiation. (b) A two-layer model including the greenhouse effect of the atmosphere.

of the LW radiation from the surface. This fraction is referred to as the LW "emissivity" which is smaller than but close to one. The emissivity, also equal to the absorptivity (the part of incident radiation absorbed by a surface divided by that absorbed by a black body), is a function of the water vapor and CO_2 concentrations among other things, and can be set for preindustrial climate to, say, 0.75. Thus the atmosphere emits LW radiation at a rate of $\varepsilon\sigma T_a^4$ W/m^2 both upward and downward, and as the surface (whose emissivity is closer to one) radiates σT^4 W/m^2, the atmosphere absorbs only a fraction ε of this radiation, with the rest continuing to outer space (wiggly dash upward arrow in Figure 2.2b). Given these, the energy balances for the surface and atmosphere may be written as,

$$\frac{S_0}{4}(1-\alpha)+\varepsilon\sigma T_a^4=\sigma T^4$$
$$\varepsilon\sigma T^4=2\varepsilon\sigma T_a^4. \tag{2.2}$$

The first equation is the energy balance for the surface, showing the input from the sun and from the atmosphere on the LHS and the output on the RHS. The second equation represents the energy balance for the atmosphere, with input LW radiation from the surface on the LHS, and output due to the radiation from the atmosphere on the RHS. The factor 2 on the RHS represents the upward and downward LW emission from the upper and lower surfaces of the assumed single layer atmosphere. Note that the atmosphere is assumed transparent to the shortwave radiation from the sun, a reasonable assumption to first order, so that the solar radiation warms the surface, and the radiation from the surface then affects the atmosphere. Substitute the second eqn in the first,

$$\frac{S_0}{4}(1-\alpha)+\frac{1}{2}\varepsilon\sigma T^4=\sigma T^4,$$

so that,

$$T=\left(\frac{(S_0/4)(1-\alpha)}{\sigma(1-\varepsilon/2)}\right)^{1/4}=T_0(1-\varepsilon/2)^{-1/4}=284\text{K}=13^\circ\text{C}, \quad (2.3)$$

which is reasonably close to the observed global mean surface temperature. Had we neglected the emissivity factor, the solution for $\varepsilon = 1$ is $T = T_0 2^{1/4} = 303\text{K} = 30^\circ\text{C}$, which is too warm. The difference between the energy balances with and without an atmospheric layer taken into account (panels a vs b in Figure 2.2, and eqns 2.1 vs 2.2) demonstrates the greenhouse effect of the atmosphere. It should be noted that the main greenhouse gas which contributes to the LW emissivity/absorptivity of the atmosphere is water vapor, which accounts for most of the greenhouse effect, although with a critical contribution from CO_2.

Finally, getting to the anthropogenic greenhouse effect, we note that an increased concentration of greenhouse gases increases the LW absorptivity/emissivity of the atmosphere, ε, and the solution (2.3) shows that the surface temperature will increase accordingly. It turns out, though, that while the explanation of the natural greenhouse effect of the atmosphere using this 2-layer model is helpful, this is too much of an oversimplification when it comes to the atmospheric response to anthropogenic increase in greenhouse gas concentrations, and this picture is therefore accordingly refined below in the next subsection.

It is worth noting, perhaps, that an actual greenhouse, such as shown in the title image of this chapter, operates differently from the above atmospheric greenhouse effect. The glass allows sunlight in, and while it prevents much of the infrared (LW) radiation from the surfaces inside the greenhouse from escaping due to its optical properties, this is not the main mechanism by which it keeps the air in the greenhouse warm: The glass temperature is not significantly colder than that of the air inside the greenhouse and thus its LW radiation is not much lower than that of the air in the greenhouse. Similarly, while glass is a relatively good insulator, the typical thickness of about 3 mm used for constructing greenhouses cannot significantly reduce conductive (diffusive) cooling through the glass. Instead, the glass keeps the greenhouse warm by preventing air exchanges with the surroundings: surfaces under the glass absorb solar radiation and warm, also warming the interior air that is in contact with these surfaces. The glass, then, prevents the warm air from rising and being replaced by (or mixed with) cooler air as would have happened outside the greenhouse, keeping the interior warm. In the atmosphere, greenhouses gases trap longwave radiation rather than restricting the movement of warm air.

2.2.3 The emission height and lapse rate

The absorption by the atmosphere of longwave radiation emitted by the surface depends on the wavelength of the radiation, as further discussed in section 2.3 below. This differential absorption depends on the concentration of different greenhouse gases in the atmosphere, each of which absorbs at different wavelengths. It turns out that the atmosphere absorbs nearly all photons of longwave radiation emitted from the surface in the wavelength range corresponding to CO_2 absorption. Why is it then that increasing CO_2 concentration is still expected to lead to further warming? One of the main reasons is that the temperature declines with height in the lower few km of the atmosphere, rather than the atmosphere being a single layer with a single temperature as assumed in Section 2.2.2.

The atmospheric temperature decreases linearly with height in the troposphere at a "lapse rate" of approximately 6.5–9.8°C per km (this will be derived in section 7.3), and then increases in the stratosphere (right panel of Figure 3.8) due to absorption of SW solar radiation by the ozone layer there. Most of the radiation emitted from Earth's surface is absorbed by the air above it, which re-emits the radiation both downwards and upwards to the air above it, and so on. At some level in the atmosphere, the overlaying air is thin enough that most of the longwave radiation emitted from that level is able to make it to outer space without being absorbed again. This is the "emission height", and its temperature determines the rate of outgoing longwave radiation from the atmosphere. Increasing the concentration of greenhouse gases moves the emission height further up, as some of the radiation from the previous emission height is trapped by the newly added greenhouse gas molecules above it. Doubling the CO_2 concentration, for example, raises the emission height by about 150 m. Given that there is only weaker absorption above it, the temperature of the emission height at equilibrium, once the climate system adjusted to the changed greenhouse gas concentration, is determined by the balance between incoming shortwave radiation and outgoing longwave radiation.

Figure 2.3 shows a schematic of the temperature profile and emission height before an increase of the CO_2 concentration (a), immediately after an assumed abrupt increase (b), and after the atmospheric temperature

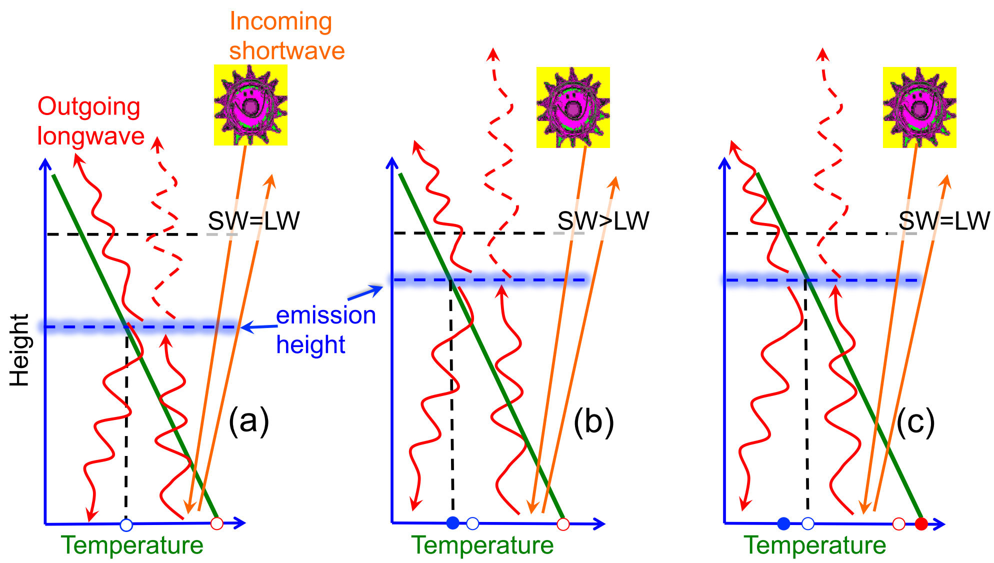


Figure 2.3: The emission height (dash blue horizontal lines) and the lapse rate (solid green) (a) before CO_2 increase, (b) immediately after CO_2 increase but before temperature adjustment, and (c) after the atmospheric temperature adjusts. Red dots at bottom of each panel denote the surface temperature. Blue dots denote the emission temperature. Before the warming and after the CO_2 increase, the emission temperature decreases (filled blue circle). As a result, the outgoing LW radiation decreases and is smaller than the incoming SW radiation. After the warming (panel c), the new emission temperature at the raised emission level is equal to the original emission temperature at the lower original emission level, both denoted by the empty blue circle, thus balancing the incoming SW radiation again.

adjusted (c). Before the increase in greenhouse gas concentration (panel a), the atmosphere emits LW radiation from the emission height to space at a temperature for which the outgoing longwave radiation (OLR) equal to incoming shortwave radiation (as in eqn. 2.1). After the CO_2 increase, the emission height moves upwards, and because of the lapse rate, the temperature that radiates to outer space is now lower, and therefore so is the OLR. Thus, the incoming shortwave radiation is greater than the OLR, causing an energy imbalance that leads to a warming of the atmosphere (panel b). This warming causes the temperature profile to adjust (from the dash green line to the solid green line Figure 2.3c) until the temperature at the new emission height is high enough to put OLR in balance with incoming shortwave radiation once again. The increase in the emission height dominates the response of the energy balance to higher CO_2. This effect is supplemented by a smaller effect due to the increase in downward LW radiation at the surface: The increased atmospheric LW emissivity due to an increased CO_2 implies that more of the downward LW at the surface arrives from lower in the atmosphere, where the temperature is higher. The higher emitting temperature means more LW radiation arriving at the surface, and therefore a surface warming.

The key takeaway is that an increase in greenhouse gas concentration leads to a change in the emission height, and therefore to a warming at the surface. The important factors leading to this warming are that the emission height depends on the CO_2 concentration, and that the atmospheric temperature decreases with altitude within the troposphere. The lapse rate is further discussed and explained in section 7.3.

2.3 Greenhouse gases

2.3.1 Wavelength-dependent black-body radiation

While we have been treating the outgoing radiation so far as a single entity, it is made of different wavelengths of electromagnetic radiation. The Earth emits from the surface approximately like a black body with a temperature of 288K or so (blue curve in Figure 2.4a), while the sun radiates as a black body of a much higher temperature (red curve), where both curves follow Planck's Law,

$$B(\lambda,T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda k_B T}}-1}. \tag{2.4}$$

Here, $B(\lambda,T)$ is the spectral radiance emitted by the black body (measured in power in watts (W), per unit area in m^2 of the emitting surface normal to the direction of propagation, per unit solid angle in steradian, per unit wavelength in m; together, $W{\cdot}m^{-2}{\cdot}ster^{-1}{\cdot}m^{-1}$), at a wavelength λ (m), at thermal equilibrium at temperature T (K). Also, $k_B = 1.38064852\times10^{-23}$ m^2 kg s^{-2} K^{-1} is the Boltzmann constant, $h = 6.62607004\times10^{-34}$ m^2kg/s the Planck constant, and $c = 3\times10^8$ m/s the speed of light,. The integral of the Planck function over all wavelengths yields the total blackbody radiation, σT^4 (W/m^2), used previously, and σ can be expressed in terms of the constants appearing in (2.4).

2.3.2 Energy levels and absorption

A given molecule may be in different energy levels, depending on the energy levels occupied by its electrons, and on which rotation and vibra-