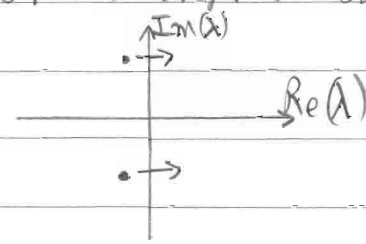
* bifurcations in 2d: [chapter 8, strogatz]

+ As mentioned previously, the 1d bifurcations (saddle-node, transcritical, pitchfork) can occur in higher dimensional systems, along the center manifold. [p. 85-91 in first notebook].
+ There are also bifurcations that are inherently 2d:

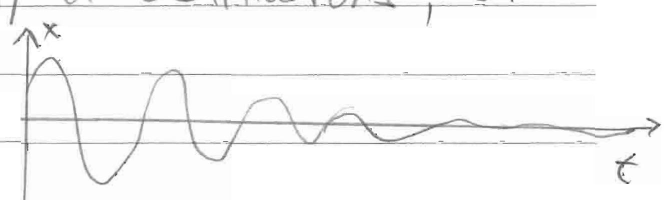
* Hopf bifurcations:

+ bifurcations occur when $Re(\lambda)$ passes through zero. The 1d bifurcations mentioned above occure when a real eigenvalue (in a possibly n-dim system) passes through zero.
+ consider now the case of a complex conjugate pair of eigenvalues:

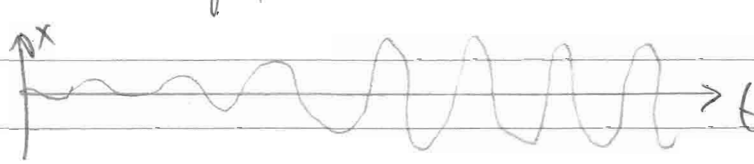


* Super critical Hopf:

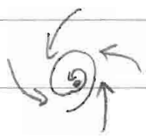
When $Re(\lambda) < 0$, we have damped osillations, or what we called a spiral:

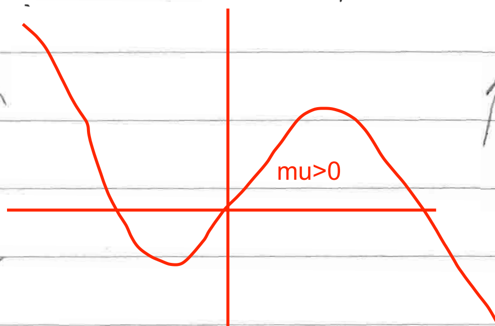


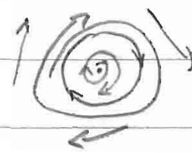
we also saw (weakly nonlinear van der pol), a case where an unstable spiral saturates at a limit cycle.

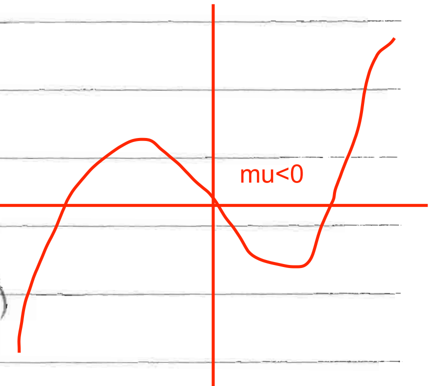


+ Hopf bifurcation in phase space:









$\mu < \mu_c$ $\qquad\qquad\qquad$ $\mu > \mu_c$

+ generic example:

$$\left.\begin{aligned} \dot r &= \mu r - r^3 \\ \dot\theta &= \omega + b r^2 \end{aligned}\right\} \quad (\#)$$

$\mu < 0 \Rightarrow \left\{\begin{matrix}\text{stable} \\ \text{origin}\end{matrix}\right\}$; $\mu > 0 \Rightarrow$ unstable origin.

ω = frequency at $0 < \mu << 1$

b = nonlinear correction to frequency.

+ linearize in cartesian coordinates: $\dot x = f(x,y)$; $\dot y = g(x,y)$.

$$x = r\cos\theta \;;\; y = r\sin\theta. \Rightarrow r^2 = x^2 + y^2$$

$$\Rightarrow \dot x = \dot r\cos\theta - r\dot\theta\sin\theta \;;\; \dot y = \cdots$$

subst this in (#):

$$\dot x = (\mu - [x^2+y^2])x - (\omega + b[x^2+y^2])y$$

$$\Rightarrow \dot x = \mu x - \omega \cdot y + O(x^3, y^3)$$

$$\dot y = \omega \cdot x + \mu \cdot y + O(x^3, y^3)$$

$$\Rightarrow J = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

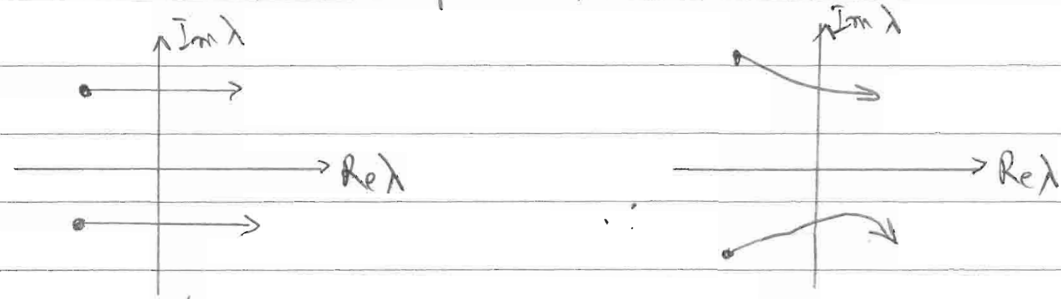
$$|J - \lambda I| = 0 \Rightarrow \underline{\lambda = \mu \pm i\omega}$$

+ conclusions:

1. size of limit cycle is prop to $(\mu-\mu_c)^{1/2}$ for $|\mu-\mu_c| \ll 1$.
2. frequency of limit cycle is $Im(\lambda)$. period is given by $T = 2\pi/Im(\lambda) + O(\mu-\mu_c)$.

+ Also:

1. As μ varies, λ moves horizontally in the complex λ plane in this example. In most cases it does not:



our example　　　　generic case.

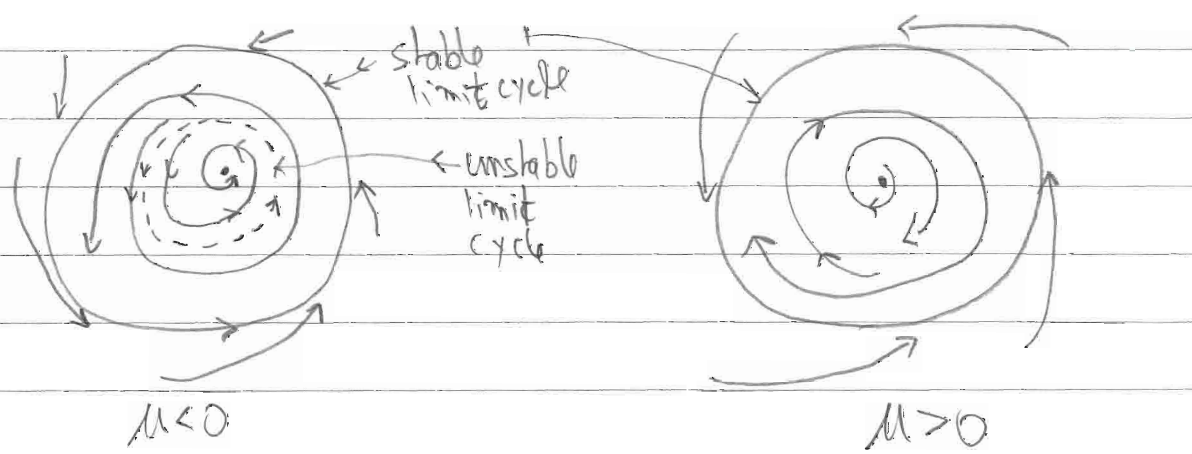
* subcritical Hopf bifurcation:

+ consider $\dot r = \mu r + r^3 - r^5$; $\dot\theta = \omega + b r^2$

new..

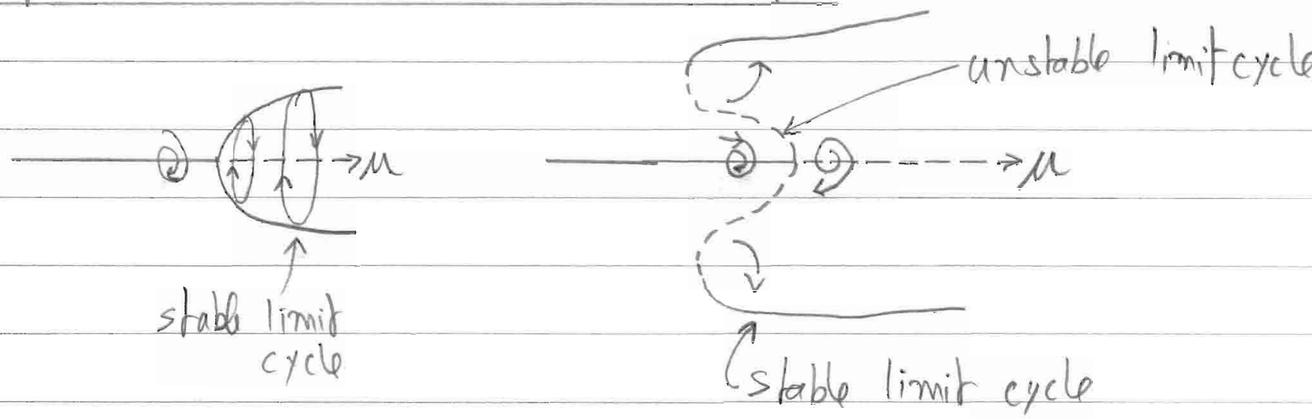
$\Rightarrow$ the r^3 term is destabilizing at the origin.

+ for $\mu<0$: origin is stable, $\mu>0$: unstable



$\mu<0$　　　　$\mu>0$

* <u>super critical vs. subcritical Hopf:</u>



\+ Note that the $-r^5$ term in our model system for subcritical Hopf bifurcation is not typical of all such bifurcations. Beyond μ_c, the system jumps to some distant attractor. may be a fixed pt, infinity, another limit cycle (as with $-r^5$), or chaos.

\+ Note the hysteresis in the sub-critical case, as evident in above figure.

\+ The super- & sub- versions may be differentiated by the appearance of a small, growing like $\sqrt{\mu}$ limit circle in the super-case, vs immediate appearance of finite-amplitude limit cycle in sub-case.