Lecture 2

ENSO toy models

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2.3 A heuristic derivation of a delayed oscillator equation

Let us consider first a heuristic derivation of an equation for the sea surface temperature in the East Pacific, which will be followed by a more rigorous derivation in the following sections. Assume that the East Pacific SST affects the atmospheric heating and thus the central Pacific wind speed. The resulting wind stress, in turn, excites equatorial Kelvin and Rossby ocean waves. These waves affect the East Pacific thermocline depth and hence the East Pacific equatorial SST, and the whole feedback loop may be quantified as follows. Let τ_K and τ_R be the basin crossing times of equatorial Kelvin and Rossby waves, correspondingly. Now, a positive central Pacific equatorial thermocline depth anomaly $h_{eq}(x_c)$ at time $t-\frac{1}{2}\tau_K$ excites an eastward propagating downwelling Kelvin wave at the central Pacific that arrives after about $\frac{1}{2}\tau_K \approx 1$ month to the eastern Pacific and deepens the thermocline there. Similarly, a negative off-equatorial depth anomaly (that is, a shallowing signal of the thermocline) in the central Pacific $h_{off-eq}(x_c)$ at a time $t-[\frac{1}{2}\tau_R+\tau_K]$ ($\frac{1}{2}\tau_R+\tau_K \approx 6$ months) excites a westward propagating Rossby wave at the central Pacific that is reflected off the western boundary as an equatorial Kelvin waves and eventually arrives to the eastern Pacific at time t, shallows the thermocline there and causes cooling of the SST. We add a nonlinear damping term that can stabilize the system, and write an equation for the eastern Pacific temperature that includes the Kelvin wave, Rossby wave and local damping terms as follows

$$\frac{dT(t)}{dt} = \hat{a}h_{eq}(x_c, t-\frac{1}{2}\tau_K) + \hat{b}h_{off-eq}(x_c, t-[\frac{1}{2}\tau_R+\tau_K]) - cT(t)^3$$

where $\hat{a}, \hat{b}, c$ are positive constants. Note that we assume that once the thermocline deepening or shallowing signal reaches the East Pacific it immediately affects the SST there. This actually neglects the SST adjustment time and we will include this time scale in the more rigorous derivation below. Note that because the mean thermocline depth is shallower in the East Pacific than in the West Pacific, a deepening or rise of the thermocline in the East Pacific is able to affect the mixing between cool sub-thermocline waters and surface waters, and thus affect the SST; in the West Pacific, the thermocline is deeper, so that even if it rises somewhat, it is still too deep to affect the SST. Now, the thermocline depth in the equatorial central Pacific is a response to the equatorial central Pacific wind. The off-equatorial thermocline depth in the central Pacific will be shown below to be a response to the wind curl off the equator, which will be shown to be negatively correlated with the wind stress at the equator. We can therefore write the above equation as

$$\frac{dT(t)}{dt} = \bar{a}\tau_{eq}(x_c, t-\frac{1}{2}\tau_K) - \bar{b}\tau_{eq}(x_c, t-[\frac{1}{2}\tau_R+\tau_K]) - cT(t)^3$$

where $\bar{a}, \bar{b}$ are some new proportionality constants. Next, the wind stress in the Central Pacific is a pretty much simultaneous response to the East Pacific SST, so that we can actually write

$$\frac{dT(t)}{dt} = aT(t-\frac{1}{2}\tau_K) - bT(t-[\frac{1}{2}\tau_R+\tau_K]) - cT(t)^3 \qquad (9)$$

where again the constants of proportionality a, b, c are all positive. The first term in this equation provides a positive feedback due to the Kelvin wave, with a short delay of about one month; the second term represents the Rossby wave with a longer-delayed negative feedback, and the last term is a nonlinear damping term. This last equation (9) is the desired delayed oscillator equation for El Nino.

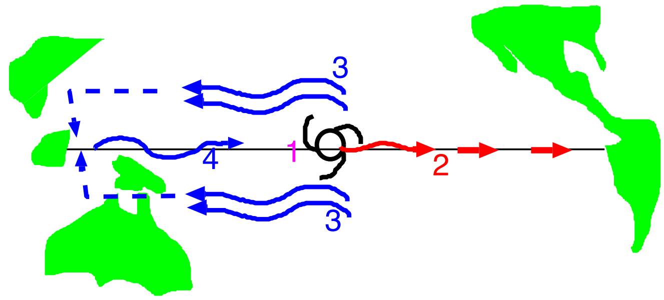

Figure 21: A schematic picture of the delayed oscillator mechanism

Based on the delayed oscillator model, the El Nino cycle may be described as follows (Fig. 21). A wind weakening (1 in Fig. 21)) creates an equatorial warm (downwelling) Kelvin wave (2) that travels to the East Pacific within 1-2 months, where the thermocline deepening induces an SST heating and starts an El Niño event. The SST heating further weakens the central Pacific winds and the event is therefore amplified by ocean-atmosphere instability. The original wind weakening also creates off-equatorial cold (upwelling) Rossby waves (due to the induced changes to the wind curl, as will be shown below) (3) that are reflected from the western boundary as cold Kelvin waves (4), arrive at the eastern boundary about 6 months later and terminate the event.

Note that we ignored reflection at the eastern boundary. Much of the energy of an eastward traveling equatorial Kelvin wave incident on the eastern boundary will be reflected as poleward traveling coastal Kelvin waves and will escape from the equatorial domain. In contrast, westward traveling Rossby waves incidenting on the western boundary are reflected as equatorward traveling coastal Kelvin waves. These, in turn, are reflected eastward at the equator as equatorial Kelvin waves, hence creating an efficient reflection process in which the wave energy ermains in the equatorial strip.

2.3.1 Analysis of the delayed oscillator equation

Battisti [2] and Suarez and Schopf [56] (see also Dijkstra [9] section 7.5.4.2) have used a slightly different delayed oscillator equation, basically ignoring the shorter Kelvin wave delay, which in a nondimensional form is

$$\frac{dT(t)}{dt} = T(t) - \alpha T(t-\delta_T) - T^3(t). \tag{10}$$

Note first that a delayed equation formally has an infinite number of degrees of freedom (it requires an infinite number of initial conditions corresponding to the times from $t = -\delta_T$ to $t = 0$, and is thus equivalent to an infinite number of ODEs). So formally this is not a "simple" equation. Only a few of these degrees of freedom are actually activated in reasonable parameter regimes (as measured by the dimension of the attractor in phase space). The various delayed oscillator equations result in El-Nino like oscillations whose periods may be tuned, by changing the coefficients, to about 4 years (Fig. 22).

Let us analyze the linearized stability behavior of 10. The equilibria of the above delayed oscillator equation are the zero solution, and then one warm solution and one cold solution

$$\bar{T} = 0, \pm\sqrt{1-\alpha}.$$

Considering a perturbation about these steady states by setting $T = \bar{T} + \tilde{T}$ and linearizing, we have

$$\frac{d\tilde{T}(t)}{dt} = \tilde{T}(t)(1-3\bar{T}^2) - \alpha\tilde{T}(t-\delta_T).$$

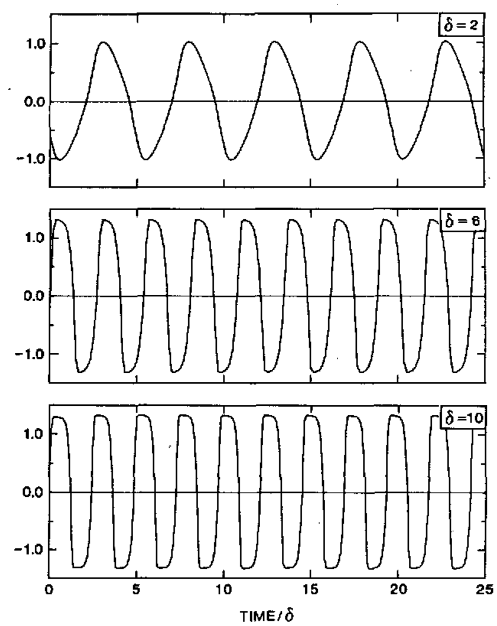

FIG. 4. Behavior of the nonlinear oscillator. (a) $\alpha = 0.75$, $\delta = 2$, (b) $\alpha = 0.75$, $\delta = 6$, and (c) $\alpha = 0.75$, $\delta = 10$. The time axis is scaled in units of the delay.

Figure 22: Results of the delayed oscillator of equation 10, from [56].

Letting $\tilde{T} = e^{\sigma t}$ where $\sigma = \sigma_r + i\sigma_i$, results in the linearized eigenvalue problem

$$\sigma = 1 - 3\bar{T}^2 - \alpha e^{-\sigma \delta_T}$$

(note that this is a complex transcendental equation, with the real and imaginary parts of σ satisfying equations that involve sine and cosine functions) which can be solved for the frequency σ as function of the two nondimensional parameters α and δ_T. It turns out that the zero solution is unstable, with a non oscillatory exponential growth. The two other (warm and cold) equilibria may become oscillatory unstable, as shown in Fig. 23.

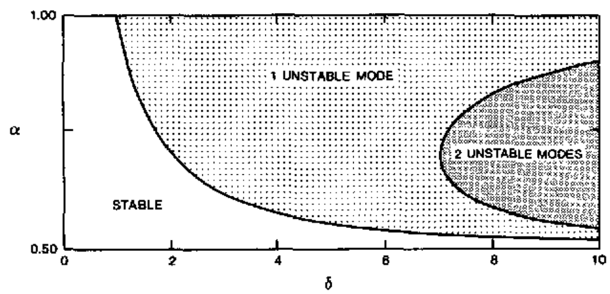

FIG. 2. Neutral stability curves of the outer stationary solution. Parameters lying below the lower line are stable. An infinite number of additional neutral curves exist to the right of the lines shown, but are only found for large δ.

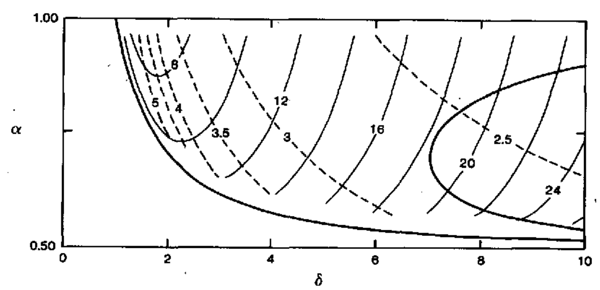

FIG. 5. The fundamental period of the nonlinear oscillator found numerically. The heavy solid lines are the neutral curves of the linear problem, reproduced from Fig. 2. The light solid contours give the period of the oscillation ($2\pi/\sigma_i$), while the dashed contours present the period in multiples of the delay.

Figure 23: Stability and period of the delayed oscillator of equation 10; Suarez & Schopf [56].

The behavior of the unstable modes is not completely simple nor intuitive: the unstable modes appear for larger values of the negative feedback (Rossby term) α, and for larger values of the delay time δ... The period of the oscillatory solutions in the delay model is shown by the light solid lines in Fig. 23, while the dashed contours give the period in multiples of the delay time. The period of the unstable modes is in the range of up to 2-3 times the Rossby delay time. Taking that delay time to be some 8 months, we get a 16-24 months period, which is significantly smaller than the observed period of 48 months. Clearly the period is not a well determined part of the picture, as it is not a robust outcome of this model, and has reasonable values for a fairly small range of model parameters. Other studies [36] also found that the period of ENSO may not be well determined by linearized theories, and may be due to some not understood nonlinear effects.