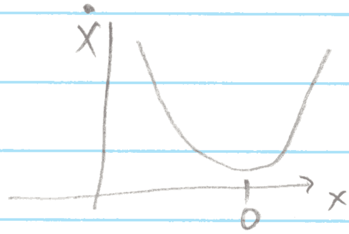
consider $\dot{x} = r + x^2$



the time to pass via the bottle neck is $\int_{-\infty}^{\infty} \frac{dx}{r+x^2} = \frac{\pi}{\sqrt{r}}$ again square root law.

synchronization/nonlinear phase locking:

example: Fireflies

fireflies tend to flash on & off together. why? how? Also, given an external periodic light, fireflies try to follow it, & may not be able to do it if it's too fast & then try again...

a specific model: let Θ be the phase of an external stimulation (flash light) of a constant rate:

$\dot{\Theta} = \Omega$. (flash occurs at $\Theta = 0$).

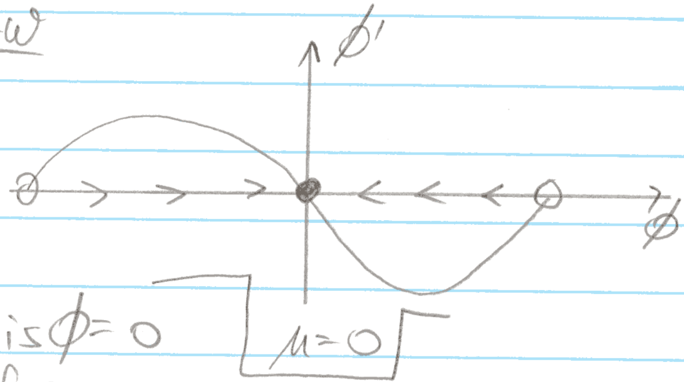
The fire fly increases/decreases it's rate of flashing depending on phase difference from external source:

$$\dot{\theta} = \underbrace{\omega}_{\text{natural rate for fire flies}} + \underbrace{A\sin(\Theta - \theta)}_{\text{adjustment to stimulus.}}$$

The phase diference between the stimulus & the firefly satisfies $\dot{\phi} = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A\sin\phi$

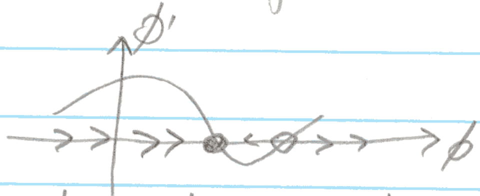
let $\tau = At$, $\mu = \frac{\Omega - \omega}{A}$

$$\Rightarrow \phi' \equiv \frac{d\phi}{d\tau} = \mu - \sin\phi$$



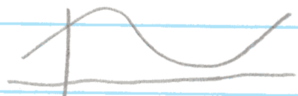
* $\mu = 0 \Rightarrow$ fixed pt is $\phi = 0$
$\Rightarrow$ no phase difference between flashlight & firefly. ($\mu = 0$ means natural firefly freq $\equiv$ external freq $\Omega = \omega$).

* $0 < \mu < 1$



$\Rightarrow$ a constant phase lag, fireflies are "phase locked" to flashlight.

* $1 < \mu$

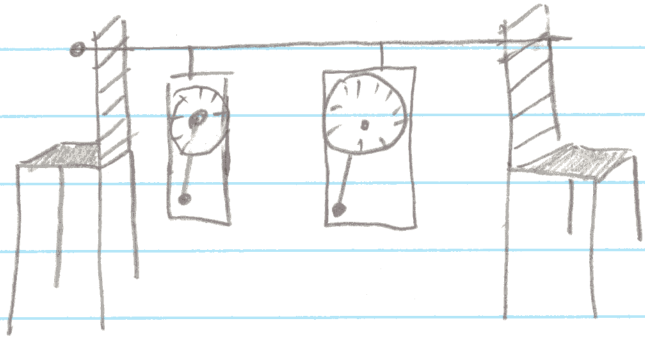


$\Rightarrow$ phase difference $\phi = \Theta - \theta$ varies.
$\Rightarrow$ "phase drift."

note that phase locking: (1) occurs because the nonlinear firefly oscillator can change it's period to fit the external one (Ω); (2) occurs over a specific range of parameters: $\omega - A < \Omega < \omega + A$
= "range of entrainment".
(a linear oscillator cannot change its period!!)

Another example of mode locking (= phase locking = nonlinear resonance)

(1) 17th century Dutch physicist Huyghens: synchronisation between clocks:



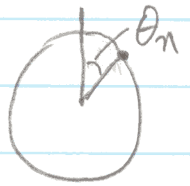
(2) Glacial cycles & Milankovitch forcing

...

Another view at phase locking: The circle map

[Schuster, chapter 6.2]

$$\theta_{n+1} = f(\theta_n) = \theta_n + \Omega + \frac{K}{2\pi}\sin(2\pi\theta_n) \quad \underline{[\text{mod } 1]}!$$



$\Rightarrow$ a simple model for a periodically forced oscillator:

Ω = forcing

$\frac{K}{2\pi}\sin(2\pi\theta_n)$ "=" gravity, nonlinear

$$\left[mr\ddot{\theta} = -b\dot{\theta} + mg\sin\theta + c\sin(\Omega\cdot t)\right]$$

when $K=0$, θ_n rotates uniformly;

* if Ω is rational, θ_n is periodic

e.g. $\Omega = \frac{1}{2} \Rightarrow$ period 2 $\theta_n = 0, \frac{1}{2}, 0, \frac{1}{2} \ldots$

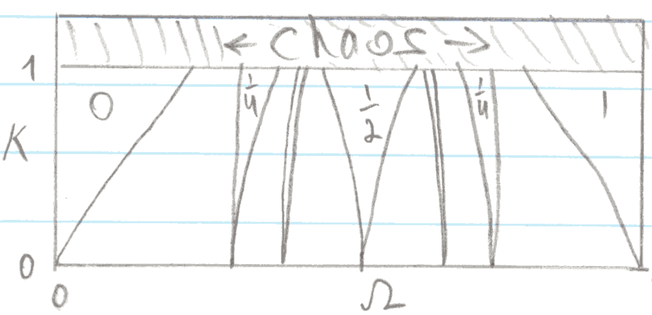
$\Omega = \frac{3}{4} \Rightarrow \theta_n = 0, \frac{3}{4}, 1\frac{1}{2}(=\frac{1}{2}), 1\frac{1}{4}(=\frac{1}{4}), 1(=0).$

$\Rightarrow$ period 4, 3 rotations per period.

* if Ω is irrational, θ_n never repeats, & eventually covers all points on the circle.

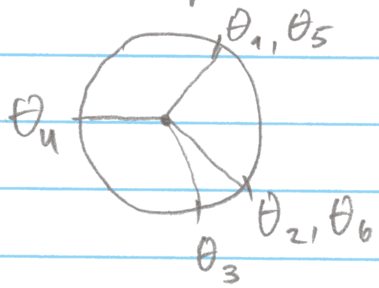
$\Rightarrow$ "quasi-periodic"

when $K>0$ (nonlinear regime), can have periodic solutions even for irrational Ω



Arnolds tongues

If we choose K, Ω values within the $\frac{1}{4}$ tongue, for example, then any initial conditions θ_0 will eventually converge to a specific 4-period solution, such as



$\Rightarrow$ unequal jumps (due to nonlinearity), but still 4-periodic.

* There are ∞ Arnold tongues, each corresponding to a nonlinear resonance between the forcing Ω & the nonlinear oscillator, each for a different rational P/Q [= "Winding #" $\equiv \frac{\theta^n - \theta_0}{n} \, (n \to \infty)$]

* [describe nonlinear resonance for an actual pendulum forced by periodic forcing, vs a linear resonance.]

<u>Devil's staircase</u>

At $K=1$, the tongues of periodic solutions cover the entire $\Omega = [0,1]$ interval, besides a fractal set of dimension <1 $\Rightarrow$ zero total length.