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A LOW-ORDER MODEL OF THE HEINRICH EVENT CYCLE

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Abstract. If Heinrich events result from free oscillations in the size and basal melting conditions of the Laurentide ice sheet, a rigorous quantitative analysis of ice sheet physics should describe their dynamics. To explore this possibility, I exploit two characteristic timescales which arise from ice sheet physics to construct a relaxation oscillator model of the North Atlantic's Heinrich events. The numerical implementation of this model confirms the notion that the periodicity of Heinrich events (approximately 7,000 years) is determined by the gross properties of a steady glacial climate (e.g., an annual average sea level temperature of -10° C and an adiabatic lapse rate).

1.0. INTRODUCTION

In my companion paper [MacAyeal, this issue], I offer a conceptual model of a Heinrich event oscillation. The periodicity implied by this model was computed using (1) the solution for heat conduction in a semi-infinite medium (an oversimplified geometry) and (2) an initial condition for the temperature field at the onset of growth that mimics the atmospheric lapse rate (thermal effects of prior history were disregarded). Here I develop a numerical model which gives quantitative rigor to the conceptual model. The mathematical treatment presented below demonstrates the validity of the assumptions and confirms the accuracy of the estimated period described previously.

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2.0. A HEINRICH EVENT OSCILLATOR MODEL

I now develop the simple conceptual model of the Laurentide ice sheet (LIS) described in the companion paper [MacAyeal, this issue]. The purpose of this effort is to check the accuracy of the predicted periodicity for Heinrich events given in the companion paper and to estimate the magnitude of the iceberg discharge flux.

The idealized view of LIS geometry and dynamics is shown in Figure 1. I consider a two-dimensional cross section of the LIS that extends from the ice divide (Hudson Bay) to the ocean at the eastern edge of the North American continent (the mouth of Hudson Strait). Normal ice sheet surface topography is disregarded, because I intend only to account for ice flow that arises from deformation of basal sediment. Ice flow associated with internal ice deformation is disregarded, because its effect is not essential in the Heinrich event mechanism. Ice thickness, $H(t)$, is thus assumed to be uniform along the two-dimensional cross section of the LIS.

The bed on which the ice slab sits is assumed to have a vertical elevation fixed at sea level, $z = 0$. At $x = 0$, the ice is contained by a frictionless impermeable wall. This represents the ice divide at the upstream end of the flow line which leads through Hudson Strait. At $x = L \approx 2000$ km, the ice is free to flow off the rigid bed into the ocean. This represents the iceberg calving margin at the mouth of Hudson Strait. Ice flow mechanics and mass balance are combined in a manner consistent with the expectation that (1) when the bed is frozen, that is, $\theta_b < 0$, horizontal ice flow is nil and the ice thickness slowly increases as dictated by the snow accumulation rate A and (2) when the bed is melted, that is, $\theta_b = 0$, deforming sediments allow rapid, z-independent horizontal flow (plug flow) with a z-independent horizontal divergence. Under these two simplifying assumptions, changes in $H(t)$ are independent of x. The rectangular

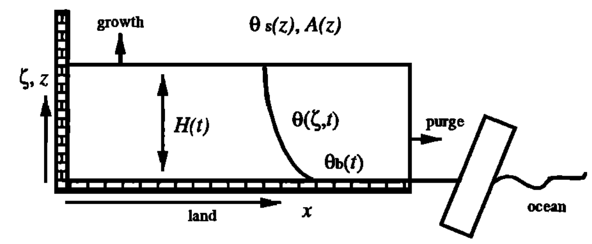

Fig.1. Idealized "ice slab" geometry of the Laurentide ice sheet.

geometry of the ice slab is thus time invariant, and the governing equations for the ice thickness can be simplified to the following approximate form:

$$
\begin{aligned}
H_t &= A \qquad \text{if} \quad \theta_b < 0 \\
H_t &= \frac{-H}{\tau_{is}} \qquad \text{if} \quad \theta_b = 0
\end{aligned}
\tag{1}
$$

here the subscript t denotes the time derivative of the subscripted variable and $\tau_{is} \approx 250$ years is a characteristic timescale for thinning associated with ice stream flow. A great deal can be said about how one would derive the value of τ_{is} from ice mechanics considerations. I will refrain from doing so, because all that will be important for the present study is that $\tau_{is} \ll 7000$ years. (Calculations using a high-order numerical model of the LIS provide additional justification of the value of τ_{is} used here [MacAyeal and Wang, 1992].)

The atmosphere is assumed to have a time-independent temperature field and snow accumulation rate. In this circumstance the temperature and accumulation rate at the surface of the ice sheet are

$$
\theta_s = \theta_{sl} - \Gamma_a H(t) \tag{2}
$$

$$
A = A_{sl} e^{\frac{-H(t)}{z_o}} \tag{3}
$$

The snow accumulation rate decays exponentially with increasing ice sheet surface elevation to reflect the reduction of precipitable water vapor at higher elevation.

The x independence of the idealized ice sheet geometry and of the atmospheric boundary conditions implies that the temperature-depth profile in the ice is also x-independent. In this circumstance, the equation governing $\theta(\zeta, t)$ is reduced to

$$
\begin{aligned}
\theta_t - \frac{\kappa}{H^2}\,\theta_{\zeta\zeta} &= \frac{\zeta A}{H}\theta_\zeta \qquad \text{if} \quad \theta_b < 0 \\
\theta_t - \frac{\kappa}{H^2}\,\theta_{\zeta\zeta} &= 0 \qquad \text{if} \quad \theta_b = 0
\end{aligned}
\tag{4}
$$

where subscripts t and ζ denote partial differentiation. The non-zero term on the right-hand side of equation (4) is generated by the fact that $\zeta = z/H(t) \in [0, 1]$ is a vertical coordinate that stretches with the growth and shrinkage of the ice column.

The boundary conditions to be applied at the surface and base of the ice slab are

$$
\theta(1, t) = \theta_s \tag{5}
$$

$$
\theta_\zeta(0, t) = \frac{-GH}{k} \tag{6}
$$

if the bed is frozen, or

$$
\theta(0, t) = \theta_b = 0 \tag{7}
$$

if the bed is melted.

Frictional heating occurs at the bed of the ice slab when gravitational potential energy is lost as the ice slab thins during the purge phase of the cycle. For some Antarctic ice streams, approximately 75% of gravitational potential energy dissipation is represented by frictional heating at the bed (e.g., the mechanical energy budget of ice stream B, West Antarctica, described by MacAyeal [1989]). The rest is dissipated by work done to deform the ice (stretching, compressing and shearing). In the present analysis, I assume that all the gravitational potential energy is dissipated at the bed and that its rate is x-independent. This is perhaps the least satisfactory of the all the simplifications I make, because it is inconsistent with the x independence of H and the horizontal ice velocity implied by equation (1). Basal friction necessarily introduces surface slopes in an ice sheet. Frictional heating is thus inconsistent with the fact that I take the ice sheet to have a flat, slablike geometry. The x-independent thinning rate of H implied by equation (1) during the purge phase of the cycle (when $\theta_b = 0$) implies that the horizontal ice velocity toward the iceberg calving margin varies linearly with x. Despite these two inconsistencies, I take frictional heating as an x-independent quantity in this study. The alternative is to abandon the simple x-independent nature of the low-order model which employs only ordinary, first-order differential equations in favor of a complicated, higher-order model of ice sheet dynamics that involves many more variables and partial differential equations (which relate spatial derivatives to time derivatives of various fields).

With the above justification, I take the local rate of frictional heating at the bed to be proportional to the rate at which gravitational potential energy stored in the ice column is dissipated by thinning. The basal heating rate is thus given by the following expression

$$
-\frac{d}{dt}\left(\frac{1}{2}\rho g H^2\right) = \frac{\rho g H^2}{\tau_{is}} \tag{8}
$$

The basal heat introduced by friction is balanced by vertical heat conduction through the ice and latent heat consumed by water production. The bed is assumed to be drained. Thus latent heat is not treated as a stored energy to be accounted for at some later time during the cycle.

The description of the model is completed by the definition of conditions which determine when the system switches between growth and purge behavior. The growth phase of the cycle ends when the basal temperature reaches the melting point. Thus if $t = T_L$ is the time when the growth phase switches to the purge phase,

$$\theta_b(T_L) = 0 \tag{9}$$

The purge phase of the cycle ends when the heat conducted through the ice at the bed exceeds the sum of the geothermal flux and the frictional dissipation. Thus if $t = T_L + T_S$ is the time when the purge phase switches to the growth phase,

$$(\theta_b)_\zeta \Big|_{t=T_L+T_S} = -\left(\frac{GH}{k} + \rho g \frac{H^3}{\tau_{is}}\right)\Big|_{t=T_L+T_S} \tag{10}$$

3.0. TWO PROBLEMS

As is customary in physical oceanography, dimensionless variables are adopted as a first step in finding the cyclic solution of equations (1)-(10). Anticipating the fact that the growth and purge phases of the Heinrich event oscillation have different characteristic timescales, I scale equations (1)-(10) differently, depending on whether the bed is melted or frozen. The scaling which applies to the frozen bed will yield what will be referred to as the long-time problem. The melted-bed scaling will yield the short-time problem. Since I expect the melted- and frozen-bed states to alternate, I will associate a time interval with each problem. For $0 < t < T_L$, the long-time scaling of the governing equations will determine the time evolution of $H(t)$ and $\theta(\zeta, t)$; for $T_L < t < T_L + T_S$, the short-time scaling will apply.

3.1. *The Long-Time Problem*

To represent the dynamics of the ice slab as it slowly grows on a frozen bed, I replace the variables appearing in equations (1)-(10) with the following nondimensional variables

$$\begin{aligned} H &\rightarrow Z_o H' \\ A &\rightarrow A_{sl} A' \\ \theta &\rightarrow \tfrac{G\, Z_o}{k}\theta' \\ t &\rightarrow \tfrac{Z_o}{A_{sl}} T' \end{aligned} \tag{11}$$

The nondimensional time T' used for the growth phase is designated by a notation that differs from what will be used below for the short-time problem. Dropping the primes, equations (1)-(7) are rewritten in dimensionless form

$$H_T = e^{-H} \tag{12}$$

$$\theta_T - \frac{\gamma}{H^2}\theta_{\zeta\zeta} = \frac{\zeta}{H}\theta_\zeta \tag{13}$$

$$\theta(1, T) = \Theta_s(H(T)) \tag{14}$$

$$\theta_\zeta(0, T) = -H \tag{15}$$

where Θ_s is the dimensionless form of the surface temperature function and $\gamma = \kappa/(A_{sl} Z_o) \approx 0.1$ is a nondimensional parameter that measures the relative effects of thermal diffusion and advection (inverse Péclét number). I take this parameter to be O(1) and take all terms in equation (13) to be of comparable importance. Equations (12)-(15) hold during the time period when the bed is frozen, which I define to be $0 < T < T_L$. The solutions to equations (12)-(15) are uniquely determined by the initial conditions

$$H(0) = H_{min} \tag{16}$$

$$\theta(\zeta, 0) = \theta_i(\zeta) \tag{17}$$

which will remain unspecified for the time being. The initial condition for H is subscripted to denote the expectation that it represents the minimum value of the ice thickness during the Heinrich event cycle.

3.2. *The Short-Time Problem*

The short-time problem, which describes ice sheet decay during a purge phase of the Heinrich event cycle, differs from the long-time problem only as a consequence of the fact that a different timescale is used to define a nondimensional time. In particular, I take

$$t \rightarrow \tau_{is}\tau \tag{18}$$

where τ_{is} is the characteristic timescale motivated by ice stream dynamics involving deformable beds. To differentiate between the short-time problem and the long-time problem, I will use τ as a new dimensionless time variable. The relationship between T, the dimensionless time in the long-time problem, and τ is given by

$$\tau = \frac{Z_o}{\tau_{is} A_o}(T - T_L) = \delta^{-1}(T - T_L) \tag{19}$$

where $\delta = \frac{Z_o}{\tau_{is} A_o} \approx 10^{-1}$ represents the ratio of an O(1) nondimensional time unit in the short-time problem to an O(1) nondimensional time unit in the long-time problem and T_L is the nondimensional time when the purge behavior begins.

The dimensionless equations governing the short-time problem are

$$H_\tau = -H \tag{20}$$

$$\theta_\tau - \frac{\epsilon}{H^2}\theta_{\zeta\zeta} = 0 \tag{21}$$

$$\theta(1, \tau) = \Theta_s(H(\tau)) \tag{22}$$

$$\theta(0, \tau) = 0 \tag{23}$$

where $\epsilon = \frac{\kappa \tau_{is}}{Z_o^2} = O(10^{-2})$ is a small parameter that determines the degree to which heat diffusion alters the temperature-depth profile over the brief time interval comprising the purge phase of the cycle. Equations (20)-(23) apply during the brief but violent time period over which the bed of the ice slab is melted, $0 < \tau < \delta^{-1}T_S$ or, equivalently, $T_L < T < T_L + T_S$. The solutions to equations (20)-(23) are uniquely determined by initial conditions which apply at $\tau = 0$ or, equivalently, at $t = T_L$. In particular,

$$H(\tau = 0) = H_{max} \tag{24}$$

$$\theta(\zeta, \tau = 0) = \Theta(\zeta) \tag{25}$$

As with the long-time problem, the above initial conditions will remain unspecified for the time being. The initial condition on H is subscripted to denote the expectation that it represents the maximum ice thickness achieved during the Hienrich event cycle.

3.3. Formal Solutions

The ice thickness solution is readily obtained by integrating equations (12) and (20):

$$H(T) = \ln(T + e^{H_{min}}) \quad \text{if} \quad 0 < T < T_L$$
$$H(T) = H_{max}e^{-\delta^{-1}(T-T_L)} \quad \text{if} \quad T_L < T < T_L + T_S \tag{26}$$

Ice thickness grows logarithmically with time (owing to the dependence of A on surface elevation; see equation (3)) during the growth phase of the cycle. During the purge phase of the cycle, ice thickness shrinks exponentially with time. At this stage, H_{min}, H_{max}, T_L, and T_S are unknown constants to be determined by additional analysis.

The temperature solution for the long-time problem is more difficult to obtain. For now, I refer to it in its formal, Green's function representation [Carslaw and Jeager, 1988]

$$\begin{aligned}\theta(\zeta, T) &= \int_0^{T_L} G_s(T, T'; \zeta)\Theta_s(\ln(T + e^{H_{min}}))dT \\ &\quad - \int_0^{T_L} G_b(T, T'; \zeta)\ln(T + e^{H_{min}})dT \\ &\quad + \int_0^1 G_i(\zeta, \zeta'; T)\theta_i(\zeta')d\zeta' \\ &= B(\zeta, T; H_{min}, T_L) + I(\zeta, T; \theta_i(\zeta))\end{aligned} \tag{27}$$

for $0 < T < T_L$. The Green's functions $G_s(T, T', \zeta)$ and $G_b(T, T'; \zeta)$ determine the influence of the surface and basal boundary conditions on the temperature evolution. The Green's function $G_i(\zeta, \zeta', T)$ determines the influence of the initial condition. The function $B(\zeta, T; H_{min}, T_L)$ and functional $I(\zeta, T; \theta_i(\zeta))$ represent a shorthand notation which captures the dependence of $\theta(\zeta, T)$ on the unknown quantities H_{min}, T_L, and $\theta_i(\zeta)$. The constants or functions which appear after the semicolons in the argument lists of $B(\ ;\ )$ and $I(\ ;\ )$ denote unknown parameters that will be determined in section 4.

To obtain the short-time evolution of the temperature field, I exploit the fact that the parameter ϵ in equation (21) is small. To lowest order, equation (21) is written

$$\theta_\tau = 0 \tag{28}$$

for $0 < \tau < \delta^{-1}T_S$. Thus to lowest order, $\theta(\zeta, \tau)$ is time independent during the purge phase of the cycle. The surface boundary condition $\Theta_s(H(\tau))$, however, is not accommodated by a time-independent profile; so, a thin boundary layer correction must be applied at the upper surface. The solution to equations (21)-(23) is thus composed of two parts:

$$\theta(\zeta, \tau) = \Theta(\zeta) + \phi(\eta, \tau) \tag{29}$$

where $\phi(\eta, \tau)$ satisfies the following boundary layer problem:

$$\phi_\tau = \left(\frac{1}{H_{max}e^{-\tau}}\right)^2 \phi_{\eta\eta} \tag{30}$$

with

$$\phi(0, \tau) = \Theta_s(H_{max}e^{-\tau}) - \Theta(1) \tag{31}$$

$$\phi(\eta \to -\infty, \tau) = 0 \tag{32}$$

The boundary layer coordinate $\eta = (\zeta - 1)/\sqrt{\epsilon}$ represents a stretched vertical coordinate that reaches unity at a depth on the order of $\epsilon^{1/2}Z_o \approx 100$ m below the surface of the ice slab. This indicates that surface warming associated with decreasing H only affects a very thin layer near the surface of the ice slab.

In what follows, I disregard the boundary layer correction $\phi(\eta, \tau)$ when determining the periodicity and amplitude of the growth/purge cycle. This disregard is justified by the fact that the penetration depth of the boundary layer (approximately 100 m) is about a factor of 10 or more smaller than the thickness of the ice sheet. In addition, the thermal dissipation timescale for nonequilibrium temperature fields with a length scale of 100 m is much shorter than the periodicity of Heinrich events. For completeness, however, I make note of the formal representation for $\phi(\eta, \tau)$ to indicate that the effects of the simplification I make can be readily determined in further study:

$$\phi(\eta, \tau) = \int_0^{\delta^{-1}T_S} \Phi(\tau, \tau'; \eta)(\Theta_s(H_{max}e^{-\tau}) - \Theta(1))d\tau' \tag{33}$$

where $\Phi(\tau, \tau'; \zeta)$ is another Green's function.

Observe that the above solutions are expressed in terms of the following unknowns: H_{min}, H_{max}, T_L, T_S, $\theta_i(\zeta)$, and $\Theta(\zeta)$. The determination of these six unknowns (four are unknown scalar constants and two are unknown scalar functions of ζ) is the crucial step in finding periodic growth/purge oscillations of the ice slab.

4.0. PERIODIC MATCHING CONDITIONS

The formal solution of the long- and short-time problems is completely described when these six unknowns are specified. Our interest is in determining the period and magnitude of the growth/purge oscillation , $T_L + T_S$ and $H_{max} - H_{min}$, respectively. To formally specify the six unknowns, I derive six additional physical constraints representing the requirements that the solution be cyclic and continuous in time and that transitions between growth and purge phases occur when specific thermodynamic conditions are met (i.e., the conditions determining basal melting or freezing).

Time continuity of H and $\theta(\zeta, T)$ require that

$$H_{min} = H_{max} e^{-\delta^{-1} T_s} \tag{34}$$

$$H_{max} = \ln(T_L + e^{H_{min}}) \tag{35}$$

$$\theta_i(\zeta) = \Theta(\zeta) \tag{36}$$

$$\Theta(\zeta) - I(\zeta, T_L; \Theta(\zeta)) = B(\zeta, T_L; H_{min}, T_L) \tag{37}$$

Equations (36) and (37) are approximate because the boundary layer correction $\phi(\eta, \tau)$ has been disregarded. Equation (37) is similar to the definition of an eigenvector: the linear operator I acts on the function $\Theta(\zeta)$ to yield the function itself plus a nonhomogeneous term, $\Theta(\zeta) - B(\zeta, T_L; H_{min}, T_L)$. This similarity is expected, because even the simplest physical systems which oscillate (e.g., a pendulum) generate eigenvector problems as part of their mathematical description.

Two more conditions relating the six unknown quantities are derived by considering the physics which determine when the basal boundary condition on the ice switches between its frozen and melted states. In particular, θ_b just reaches the melting point when $T = T_L$:

$$0 = B(0, T_L; H_{min}, T_L) + I(0, T_L; \Theta(\zeta)) \tag{38}$$

and the vertical heat flux $(\theta_b)_\zeta$ just exceeds the heat flux at the bed when $T = T_L + T_S$,

$$\Theta_\zeta(0) = -H_{min} - \frac{\rho g Z_o^2 k}{\tau_{is} G} H_{min}^3 \tag{39}$$

Equations (34)-(39) represent six equations for six unknowns. If a solution to these equations exists, it completely describes the growth/purge oscillations that I suggest are the origin of Heinrich events. Although the physical description of the idealized ice slab given in the preceeding sections has been simplified to an extreme degree, the physical problem is still too complicated to work analytically. Equations (34)-(39) are nonlinear and involve Green's functions which cannot be easily expressed analytically. I thus resort to a numerical method as a means to demonstrate that free growth/purge oscillations exist and have an amplitude and period comparable to what is observed in the geologic record.

5.0. NUMERICAL SOLUTION

The strategy I use to find the solution to the above six constraints is to examine how the ice sheet behaves after it is hit with an arbitrary initial excitation. This strategy is comparable to, for example, determining the vibrational frequency of a bell by judiciously striking it with a rubber mallet. I performed a similar exercise with a finite difference model of the ice slab.

The arbitrary initial condition used to excite free oscillations consisted of a specification of an initial thickness $H(t = 0) = 1000$ m and an initial temperature-depth profile $\theta(\zeta, t = 0) = \theta_{sl} = -10°$ C. (From now on, the use of nondimensional variables provides no advantage; thus the description of the numerical experiment is given in terms of the regular, dimensional variables.) Atmospheric forcing was specified according to expectations of atmospheric conditions over the Hudson Bay region during a glacial period: $\theta_{sl} = -10°$ C, $\Gamma_a = 9°$ C km^{-1}, $A_{sl} = 0.5$ m yr^{-1}, and $Z_o = 1000$ m. The timescale for ice stream drawdown of the ice slab during a purge phase of the cycle, τ_{is}, is taken to be 250 years. As shown in Figure 2, the ice slab oscillates readily, and settles down to a growth/purge cycle of approximately 7,260 years in duration and about 1228 m in amplitude. Transient effects resulting from the initial condition dissipate after several growth/purge cycles. This suggests that glacial periods are sufficiently long to accommodate the occurrence of many free-oscillation cycles.

Of particular interest is the sequence of events which determines a complete growth/purge cycle of the simple ice slab. As shown in Figure 3, the duration of the growth phase exceeds that of the purge phase (approximately 6,810 years versus approximately 450 years). During the growth phase, ice thickness grows logarithmically with time, as expected from equation (26). The decay of the ice thickness is exponential with time during the purge phase, again as expected from equation (26). The time histories of θ_b and $(\theta_b)_\zeta$, also shown in Figure 3, tell the story of the ice slab thermodynamics. During the growth phase of the cycle, $\theta_b < 0$ and $|(\theta_b)_\zeta| = (GH/k)$. During the purge phase of the cycle, $\theta_b = 0$ and $|(\theta_b)_\zeta| \geq (GH/k)$. The switch from growth to purge behavior occurs when $\theta_b = 0$, and the switch from purge to growth behavior occurs when $|(\theta_b)_\zeta| > ((GH/k) + \rho g H^3/\tau_{is} k)$.

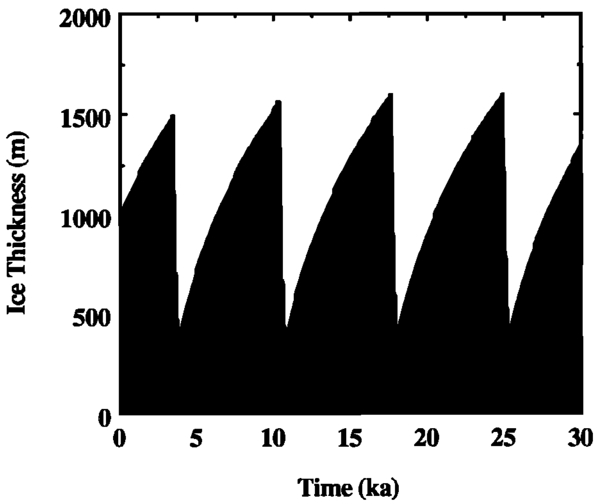

Fig. 2. Ice thickness history. The period of oscillation is approximately 7,260 years.

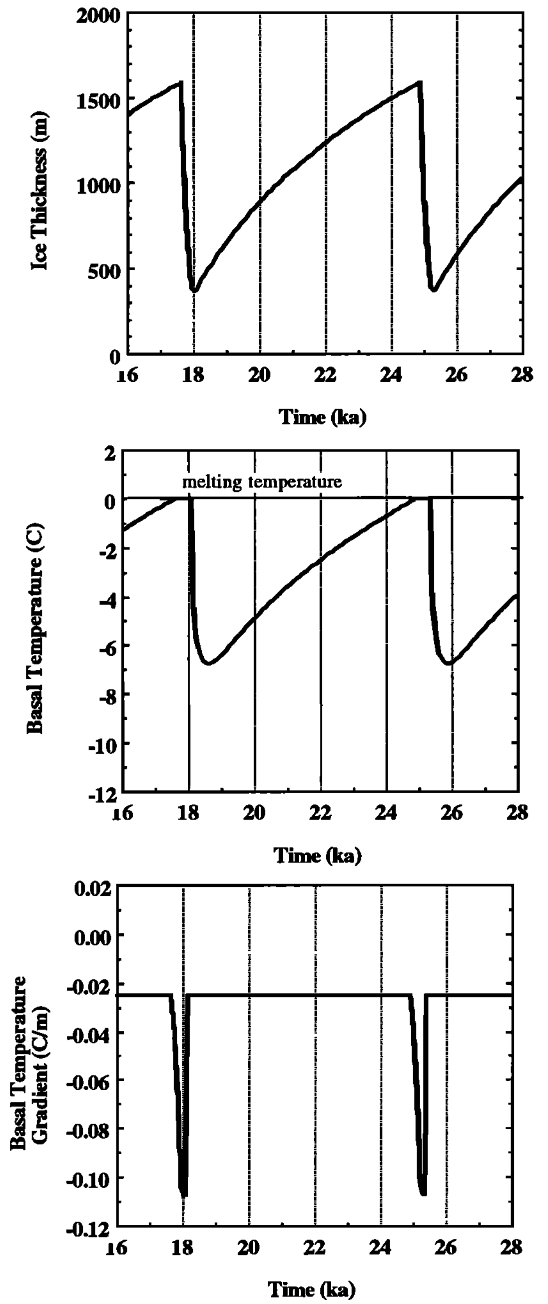

Fig. 3. (Top) Ice thickness, (middle) basal temperature, and (bottom) vertical temperature gradient at the bed.

6.0. CONCLUSION

The free-oscillation mechanism for Heinrich events has passed an important test; it can be modeled by an idealized, low-order mathematical description of the physical principles which govern ice sheet behavior. To capture more of the details implied by the geologic record (e.g., slight aperiodicity, modifications of Heinrich event timing relative to other events such as the Younger Dryas, and differences in amplitude from one cycle to the next), a high-order numerical modeling approach (involving the three dimensional, time-dependent mass, momentum, and heat continuity equations) is likely to be required. Physical features to be embraced by such a high-order model include ice shelves, horizontal advection, spatially and temporially variable accumulation rate and surface temperature, nonlinear thermal and rheological ice properties, and the hydrological, rheological, and geological properties of the bed.

The approximate formula for the periodicity of Heinrich events derived in the companion paper [MacAyeal, this issue] has been confirmed:

$$T = \frac{\pi}{\kappa}\left(\frac{-k\,\theta_{sl}}{2(G - k\Gamma)}\right)^2 \tag{40}$$

This formula predicts a periodicity of 6943 years for $\theta_{sl} = -10°$ C and $\Gamma = 0.009°$ C m^{-1}. The periodicity derived from the modeling experiment was 7260 years. This suggests that the accuracy of equation (40) should be approximately 4%. The utility of equation (40) is that it provides a simple framework for understanding the variations of Heinrich event periods observed in the geologic record. Grousset et al. [1993], for example, suggest that T is approximately 12,000 years prior to "H3"(the Heinrich event that occurred approximately 28,000 years ago) and approximately 7,000 years (or shorter) thereafter. This shift in periodicity can be explained either by a decrease in the atmospheric lapse rate or an increase in sea level climate temperature after approximately 28,000 years ago [MacAyeal, this issue]. Both shifts can perhaps be explained in terms of the background climate variation. What this explanation will entail is not immediately clear, however. The period between 14,000 and 28,000 years ago (the time when the Heinrich event periodicity was 7,000 years) was characterized by reduced summer insolation at 65° N; thus one could not appeal to direct Milankovitch variation of the sea level temperature θ_{sl} as a direct explanation of the shift in periodicity. Perhaps a more plausible explanation of the periodicity shift will come from Milankovitch forcing of the atmospheric lapse rate. A decrease in the lapse rate by about 3° C km^{-1} 28,000 years ago could easily explain the periodicity shift [see MacAyeal, this issue, Figure 4].

The results of the modeling exercise presented here suggest ways to test the free-oscillation hypothesis. The comparison between predicted and observed periodicity will probably fail to provide an unambiguous test, because the geologic record suggests somewhat more irregularity than what is implied here [Bond et al., 1992]. Undoubtedly, the effects of external climate change (e.g., Milankovitch insolation variations and short-term swings in temperature seen in the Greenland ice core records) will make a freely oscillating ice sheet appear to oscillate somewhat more irregularly than will a static external climate. The tests which may be most productive involve direct measures of the amplitude of ice sheet change. If the results here are correct, global sea level should change by approximately 3.5 m over the Heinrich event cycle (see the description in the conclusion of the companion paper [MacAyeal, this issue]). External climate forcing certainly can cause sea level changes, but not with the rapidity of a mechanism involving ice stream dynamics.

Additional tests of the mechanism may come from continued analysis of the condition and distribution of sediments both on land and at sea [e.g., Grousset et al., 1993; J. T. Andrews et al., Chronology and processes, East-central Laurentide ice sheet and NW Labrador Sea, unpublished manuscript, 1992]. The sediments flooring the Hudson Bay, Hudson Strait, and surrounding lowlands offer a means to predict the temporal phase relationship between net iceberg volume flux and iceberg debris content. This phase relationship should be reflected in the depositional sequence of materials flooring the North Atlantic and should provide important constraints on basal temperature conditions and the behavior of the deforming subglacial sediment.

As a final remark, I make note of the possibility that the mathematical methodology (in particular, the asymptotic analysis of the governing equations in two timescales) used here may have additional application in other oceanographic and glaciological problems. The key aspect which differentiates this methodology from others is that it takes advantage of the fact that there are two naturally arising timescales which govern the physical system. By developing two separate simplifications to the underlying dynamics, each motivated by the appropriate timescale, and then hooking them together with physically motivated matching conditions, a closed mathematical description of an otherwise intractible oscillation is developed. This technique may have merit for use in the study of surging valley glaciers and for understanding the underlying cause of instability in the West Antarctic ice sheet [e.g., MacAyeal, 1992; see also Hodell, 1993].

NOTATION

$A(z)$ accumulation rate (m s^{-1} ice equivalent).
A_{sl} accumulation rate at sea level (0.5 m s^{-1} ice equivalent).
δ the ratio of two timescales (nondimensional, $O(10^{-1})$).
ϵ diffusive boundary layer parameter (nondimensional, $O(10^{-2})$).
$\phi(\eta, \tau)$ boundary layer correction temperature.
G geothermal flux (0.05 W m^{-2}).
g gravity (9.8 m s^{-2}).
Γ atmospheric lapse rate (9° C km^{-1}).
γ inverse Péclét number (nondimensional, $O(10^{-1})$).
Z_o accumulation rate scale height. (10^3 m)
$H(t)$ ice column thickness.
H_{max} maximum ice thickness achieved at the end of a growth phase.
H_{min} minimum ice thickness achieved at the end of a purge phase.
η vertical coordinate scaled for boundary layer.
k thermal conductivity of ice (2 W °C^{-1} m^{-1}).
κ thermal diffusivity of ice (1.4×10^{-6} m^2 s^{-1}).
ρ ice density (917 kg m^{-3}).
t time.
τ nondimensional short-time time variable.
T nondimensional long-time time variable.
T_L duration of growth phase.
T_S duration of purge phase.
τ_{is} ice stream drawdown timescale (250 years).
$\theta(\zeta, t)$ temperature-depth profile.
θ_b basal temperature.
$\theta_s(z)$ annual average surface temperature.
θ_{sl} annual average sea level temperature of the atmosphere (-10°C).
z vertical coordinate (positive upward, zero at the ice/ground contact).
ζ stretched vertical coordinate.
Z_o scale height.

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