THE

MOTIONS OF FLUIDS AND SOLIDS

RELATIVE TO

THE EARTH'S SURFACE.

INTRODUCTION.

Some of the results contained in the following pages were published about two years ago, in an essay in the Nashville Journal of Medicine and Surgery, edited by Professor W. K. Bowling, M. D., of Nashville, Tennessee. A small edition was also published in pamphlet form, and distributed by the Smithsonian Institution and myself amongst various scientific men, libraries, and scientific associations, both in this country and in Europe. In that essay it was attempted to show that the depression of the atmosphere at the poles and the equator, and the accumulation or bulging at the tropics, as indicated by barometric pressure, the gyratory motion of storms from right to left in the northern hemisphere, and the contrary way in the southern, and certain motions of oceanic currents, are necessary consequences of the modifying forces arising from the earth's rotation on its axis, and also that the observed flowing of the lower strata of the atmosphere in the middle latitudes towards the poles, contrary to the ordinary theory of the trade winds, is caused by the greater pressure of this accumulation of atmosphere at the tropics. It is believed that that essay was the first attempt to account for those remarkable phenomena by means of the modifying influence of the earth's rotation, and that it furnishes the only satisfactory explanation of them which has yet been given.

In that essay it was inconvenient to use any mathematical formulæ, and consequently the results merely of only a partial and imperfect investigation of the subject were given; but it is thought that on account of the importance of the subject, it deserves a more thorough investigation. It is proposed, therefore, in the following pages, to go into a complete analytical investigation of the general motions of fluids surrounding the earth, and of projectiles at its surface, arising from disturbing forces and the earth's attraction, combined with the modifying forces arising from its rotation on its axis. We shall accordingly, in the first section, investigate the general equations of motion relative to the earth's surface, applicable to both fluids and solids, and in the subsequent sections treat, first of the motions and figure of the whole or a part of a fluid surrounding the earth, upon the hypothesis that its motions are not resisted by the earth's surface, and then apply the results thus obtained to the explanation of the general motions of the atmosphere, the motions of storms or hurricanes, and the currents of the ocean. We shall also give a complete but concise treatise on projectiles, taking into account the effect of the earth's rotation.

We hope to be able in this investigation to give a satisfactory explanation of all the general motions of the atmosphere and of the ocean; the cause of the greater pressure of the atmosphere near the tropics than at the equator and the poles, and of the greater pressure generally in the northern hemisphere than in the southern; to account for the motion of all great storms in both hemispheres from the equator towards the poles in parabolic paths, and to completely establish their gyratory character; none of which phenomena have ever been satisfactorily accounted for by any of the usual theories, which do not take into account the influence of the earth's rotation.

SECTION I.

OF THE GENERAL EQUATIONS OF MOTION RELATIVE TO THE EARTH'S SURFACE.

1. Let x, y, and z be three rectangular coördinates, having their origin at the centre of the earth, x corresponding with the axis of rotation. Also let

Ω be the potential of all the attractive forces of the earth,

P the pressure of the fluid, and

k its density.

Then $k\,D_x\,\Omega$, $k\,D_y\,\Omega$, and $k\,D_z\,\Omega$ are the forces for a unit of volume, arising from the earth's attraction, and $D_x P$, $D_y P$, and $D_z P$, those arising from the pressure of the fluid, in the directions respectively of x, y, and z; and we have for the equations of the absolute motions of the fluid, regarding the centre of the earth at rest,

$$(1)\qquad \begin{aligned} D_t^2 x + D_x \Omega + \frac{1}{k} D_x P &= 0, \\ D_t^2 y + D_y \Omega + \frac{1}{k} D_y P &= 0, \\ D_t^2 z + D_z \Omega + \frac{1}{k} D_z P &= 0. \end{aligned}$$

Putting $P = 0$, they are the equations of a projectile.

2. Let r be the distance from the earth's centre,

θ, the polar distance,

φ, the longitude, and

n, the angular velocity of the earth's rotation.

Then we have

$$(2)\qquad \begin{aligned} x &= r \cos \theta, \\ y &= r \sin \theta \cos (n t + \varphi) = r \sin \theta \cos \omega, \\ z &= r \sin \theta \sin (n t + \varphi) = r \sin \theta \sin \omega, \end{aligned}$$

by putting for brevity $n t + \varphi = \omega$.

The position of the ordinates y and z, and also the origin of the time t, being entirely arbitrary, they must be so taken as to make $\sin(nt+\varphi)$ vanish in the plane of x, y.

Using these values of x, y, and z in equations (1), we obtain equations in which the first derivatives of r, θ, and φ represent the motions of the fluid or projectile relative to the earth's surface.

3. Taking the first derivatives of (2) with regard to t, we get

$$D_t x = \cos\theta\, D_t r - r\sin\theta\, D_t\theta,$$
$$D_t y = \sin\theta\cos\omega\, D_t r + r\cos\theta\cos\omega\, D_t\theta - r\sin\theta\sin\omega\, D_t\omega,$$
$$D_t z = \sin\theta\sin\omega\, D_t r + r\cos\theta\sin\omega\, D_t\theta + r\sin\theta\cos\omega\, D_t\omega,$$

Taking the second derivatives, we get

$$D_t^2 x = \cos\theta\, D_t^2 r - 2\sin\theta\, D_t r\, D_t\theta - r\cos\theta\,(D_t\theta)^2 - r\sin\theta\, D_t^2\theta,$$

$$(3)\quad \begin{aligned} D_t^2 y = {} & \sin\theta\cos\omega\, D_t^2 r + 2\cos\theta\cos\omega\, D_t r\, D_t\theta - 2\sin\theta\sin\omega\, D_t\omega\, D_t r \\ & + r\cos\theta\cos\omega\, D_t^2\theta - r\sin\theta\cos\omega\,(D_t\theta)^2 - 2r\cos\theta\sin\omega\, D_t\omega\, D_t\theta \\ & - r\sin\theta\sin\omega\, D_t^2\varphi - r\sin\theta\cos\omega\,(D_t\omega)^2. \end{aligned}$$

$$\begin{aligned} D_t^2 z = {} & \sin\theta\sin\omega\, D_t^2 r + 2\cos\theta\sin\omega\, D_t r\, D_t\theta + 2\sin\theta\cos\omega\, D_t\omega\, D_t r \\ & + r\cos\theta\sin\omega\, D_t^2\theta - r\sin\theta\sin\omega\,(D_t\theta)^2 + 2r\cos\theta\cos\omega\, D_t\omega\, D_t\theta \\ & + r\sin\theta\cos\omega\, D_t^2\varphi - r\sin\theta\sin\omega\,(D_t\omega)^2. \end{aligned}$$

Since x, y, and z are functions of r, θ, and φ, we must put

$$(4)\quad \begin{aligned} D_x\Omega &= D_r\Omega\,.\,D_x r + D_\theta\Omega\,.\,D_x\theta + D_\phi\Omega\,.\,D_x\varphi, \\ D_y\Omega &= D_r\Omega\,.\,D_y r + D_\theta\Omega\,.\,D_y\theta + D_\phi\Omega\,.\,D_y\varphi, \\ D_z\Omega &= D_r\Omega\,.\,D_z r + D_\theta\Omega\,.\,D_z\theta + D_\phi\Omega\,.\,D_z\varphi. \end{aligned}$$

Now we have

$$r^2 = x^2 + y^2 + z^2,$$
$$\tan\theta = \frac{\sqrt{y^2+z^2}}{x},$$
$$\tan\omega = \frac{z}{y}.$$

Hence,

$$D_x r = \frac{x}{r} = \cos\theta,$$
$$D_y r = \frac{y}{r} = \sin\theta\cos\omega,$$
$$D_z r = \frac{z}{r} = \sin\theta\sin\omega,$$

$$D_x\theta = -\frac{\sqrt{y^2+z^2}}{r^2} = -\frac{\sin\theta}{r},$$

$$D_y\theta = \frac{xy}{r^2\sqrt{y^2+z^2}} = \frac{\cos\theta\cos\omega}{r},$$

$$D_z\theta = \frac{xz}{r^2\sqrt{y^2+z^2}} = \frac{\cos\theta\sin\omega}{r},$$

$$D_x\varphi = 0,$$

$$D_y\varphi = \frac{-z}{y^2+z^2} = \frac{\sin\omega}{r\sin\theta},$$

$$D_z\varphi = \frac{y}{y^2+z^2} = \frac{\cos\omega}{r\sin\theta}.$$

By means of these equations, equations (4) become

$$D_x\Omega = D_r\Omega\cos\theta - D_\theta\Omega\frac{\sin\theta}{r},$$

$$(5)\quad D_y\Omega = D_r\Omega\sin\theta\cos\omega + D_\theta\Omega\frac{\cos\theta\cos\omega}{r} - D_\phi\Omega\frac{\sin\omega}{r\sin\theta}.$$

$$D_z\Omega = D_r\Omega\sin\theta\sin\omega + D_\theta\Omega\frac{\cos\theta\sin\omega}{r} + D_\phi\Omega\frac{\cos\omega}{r\sin\theta}.$$

In the same manner we obtain

$$D_xP = D_rP\cos\theta - D_\theta P\frac{\sin\theta}{r},$$

$$(6)\quad D_yP = D_rP\sin\theta\cos\omega + D_\theta P\frac{\cos\theta\cos\omega}{r} - D_\phi P\frac{\sin\omega}{r\sin\theta},$$

$$D_zP = D_rP\sin\theta\sin\omega + D_\theta P\frac{\cos\theta\sin\omega}{r} + D_\phi P\frac{\cos\omega}{r\sin\theta}.$$

Substituting the values of the first members of equations (3), (5), and (6) in equations (1), and multiplying them respectively by $\cos\theta$, $\sin\theta\cos\omega$, $\sin\theta\sin\omega$, and adding, we obtain the first of the following equations. Again, multiplying them respectively by $r\sin\theta$,—$r\cos\theta\cos\omega$, and $-r\cos\theta\sin\omega$, and adding, we obtain the second of those equations. Finally, multiplying the last two respectively by $r\sin\theta\sin\omega$, and $-r\sin\theta\cos\omega$, and adding, we get the last of the following equations:—

$$\frac{1}{k}D_rP = -D_t^2r + r(D_t\theta)^2 + r\sin^2\theta\,(n + D_t\omega)\,D_t\varphi$$
$$+ rn^2\sin^2\theta - D_r\Omega,$$

$$(7)\quad \frac{1}{k} D_\theta P = - r^2 D_t^2 \theta - 2r D_t r\, D_t \theta + r^2 \sin\theta \cos\theta\, (n + D_t \omega)\, D_t \varphi$$
$$+ r^2 n^2 \sin\theta \cos\theta - D_\theta \Omega,$$
$$\frac{1}{k} D_\varphi P = - r^2 \sin^2\theta\, D_t^2 \varphi - 2r \sin^2\theta\, D_t \omega\, D_t r$$
$$- 2 r^2 \sin\theta \cos\theta\, D_t \omega\, D_t \theta - D_\varphi \Omega.$$

In these equations $D_r P$, $\frac{1}{r} D_\theta P$, and $\frac{1}{r \sin\theta} D_\varphi P$ represent the forces arising from the pressure in the directions respectively of r, θ, and φ.

4. If N be the normal distance to the surface of the earth, or to any level surface, and the forces in the first two of the preceding equations be resolved in the directions of the normal and a perpendicular to the normal in the plane of the meridian, putting $\cos \frac{r}{N} = 1$, and neglecting the small terms multiplied by $\sin \frac{r}{N}$, which are of the second order of the earth's ellipticity, and letting $\frac{1}{r} D_{\theta'} P$ represent the force arising from the pressure, resolved in the direction of the perpendicular to the normal in the plane of the meridian, the preceding equations give, when the fluid is at rest,

$$\frac{1}{k} D_N P = r n^2 \sin^2\theta - D_N \Omega = - g,$$
$$(8)\quad \frac{1}{k} D_{\theta'} P = r^2 n^2 \sin\theta \cos\theta - D_{\theta'} \Omega = 0,$$
$$\frac{1}{k} D_\varphi P = - D_\varphi \Omega = 0,$$

and hence, neglecting the very small terms multiplied by $\sin \frac{r}{N}$, depending upon the motions of the fluid relative to the earth's surface, they give for the fluid in motion,

$$\frac{1}{k} D_N P = - D_t^2 r + r\, (D_t \theta)^2 + r \sin^2\theta\, (n + D_t \omega)\, D_t \varphi - g,$$
$$(9)\quad \frac{1}{k} D_{\theta'} P = - r^2 D_t^2 \theta - 2r D_t r\, D_t \theta + r^2 \sin\theta \cos\theta\, (n + D_t \omega)\, D_t \varphi,$$
$$\frac{1}{k} D_\varphi P = - r^2 \sin^2\theta\, D_t^2 \varphi - 2 r \sin^2\theta\, D_t \omega\, D_t r$$
$$- 2 r^2 \sin\theta \cos\theta\, D_t \omega\, D_t \theta.$$

Integrating, we obtain

$$(10)\quad \begin{aligned} P &= \Pi - \int_N g\,k = \Pi - g\,k\,N + \int_N N\,D_N(g\,k), \\ &= \Pi - g\,k\,N + \int_N g\,N\,D_N k + \int_N N k\,D_N g, \\ &= \Pi + K, \end{aligned}$$

in which Π must satisfy the following equations of partial differentials:—

$$(11)\quad \begin{aligned} \frac{1}{k} D_N \Pi &= -D_t^2 r + r\,(D_t\,\theta)^2 + r\,\sin^2\theta\,(n + D_t\,\omega)\,D_t\,\varphi, \\ \frac{1}{k} D_{\theta'} \Pi &= -r^2 D_t^2\,\theta - 2\,r\,D_t r\,D_t\,\theta + r^2 \sin\theta\,\cos\theta\,(n + D_t\,\omega)\,D_t\,\varphi, \\ \frac{1}{k} D_\phi\,\Pi &= -r^2 \sin^2\theta\,D_t^2\,\varphi - 2\,r\,\sin^2\theta\,D_t\,\omega\,D_t\,r \\ &\quad - 2\,r^2 \sin\theta\,\cos\theta\,D_t\,\omega\,D_t\,\theta, \end{aligned}$$

and in which

$$(12)\quad K = -g\,k\,N + \int_N g\,N\,D_N k + \int_N N\,k\,D_N g.$$

Hence Π is the pressure arising from the motions of the fluid, and K that arising from its gravity. If g and k are functions of N, θ', and φ, K is a function of the same.

5. For a stratum of equal pressure, P is constant, and hence

$$D_{\theta'} P = 0, \qquad D_\phi\,P = 0.$$

If we therefore put K' and h for the special values respectively of K and N, belonging to a stratum of equal pressure, and take the derivatives of (10) with regard to θ' and φ, and neglect the very small terms containing $D_t\,r$ as a factor, which, in all ordinary motions of the fluid, will be shown to be insensible, we obtain for the general equations of horizontal motions, by restoring the value of ω in § (2).

$$(13)\quad \begin{aligned} 0 &= \frac{1}{k} D_{\theta'} K' - r^2 D_t^2\,\theta + r^2 \sin\theta\,\cos\theta\,(2\,n + D_t\,\varphi)\,D_t\,\varphi, \\ 0 &= \frac{1}{k} D_\phi\,K' - r^2 \sin^2\theta\,D_t^2\,\varphi - 2\,r^2 \sin\theta\,\cos\theta\,(n + D_t\,\varphi)\,D_t\,\theta, \end{aligned}$$

in which

$$(14)\quad K' = -g\,k\,h + \int_h g\,h\,D_h k + \int_h h\,k\,D_h g.$$

6. If we suppose the fluid to be elastic, and the ratio of the density to the elastic force or pressure to depend upon the temperature, we may put

$$k = \alpha\, P, \tag{15}$$

in which α may be a function of h, θ', and φ. Substituting this value of k in (14), we get, when g may be regarded as constant, since in that case the last term of (14) vanishes.

$$\begin{aligned} D_{\theta'} K' &= - D_{\theta'} (g\, \alpha\, P\, h) + D_{\theta'} \textstyle\int_h g\, h\, D_h (\alpha\, P), \\ &= - g\, \alpha\, P\, D_{\theta'} h - g\, h\, P\, D_{\theta'}\, \alpha + g\, P\, D_{\theta'} \textstyle\int_h h\, D_h\, \alpha, \end{aligned} \tag{16}$$

Hence, dividing by $k = \alpha\, P$, we get

$$\begin{aligned} \frac{1}{k} D_{\theta'} K' &= - g\, D_{\theta'} h - g\, h\, D_{\theta'} \log \alpha + A_{\theta'}, \\ \frac{1}{k} D_{\phi} K' &= - g\, D_{\phi} h - g\, h\, D_{\phi} \log \alpha + A_{\phi}, \end{aligned} \tag{17}$$

by changing θ' to φ for the last equation, and putting

$$\begin{aligned} A_{\theta'} &= \frac{g}{\alpha} D_{\theta'} \textstyle\int_h h\, D_h\, \alpha, \\ A_{\phi} &= \frac{g}{\alpha} D_{\phi} \textstyle\int_h h\, D_h\, \alpha. \end{aligned} \tag{18}$$

7. When $D_h\, \alpha$ is constant, that is, when α varies as the altitude,

$$\begin{aligned} A_{\theta'} &= \frac{g e}{2 \alpha} D_{\theta'} h^2, \\ A_{\phi} &= \frac{g e}{2 \alpha} D_{\phi} h^2, \end{aligned} \tag{19}$$

in which

$$e = D_h\, \alpha,$$

depending upon the constant ratio of increase or decrease of α with h.

8. When α is constant, or when the fluid is homogeneous, the last two terms of (17) vanish, and in the latter case, h may represent the height of the surface of the fluid above any level surface.

9. In the preceding investigation, the effect upon g arising from a change of the figure of the fluid has been neglected. It is small

in the case of water, and in the case of the atmosphere, entirely insensible.

10. Equations (13), together with the condition that the same amount of fluid must always occupy a space which is inversely as its density, the analytical expression of which is called the equation of continuity, are the conditions which must be satisfied by the motions of a fluid surrounding the earth, and are sufficient to determine its horizontal motions, and also the value of h, which gives the figure of the fluid. When h is determined, equation (10) gives the pressure of the fluid. As only a very special form of the general equation of continuity will be needed in this investigation, it is unnecessary to give it here.

[TO BE CONTINUED.]

PROBLEM.

BY J. FOSTER FLAGG, C. E.

REQUIRED a formula for the strength of a circular flat iron plate of uniform thickness, supported throughout its circumference and uniformly loaded; likewise a formula, supposing the plate to be bolted down. Also find the equation of the curve, for each case, which should be given to a section of the plate, in order that it may have the greatest strength with the least material. The problem may be further varied by supposing the plate *square*, instead of round.

We have recently had great need of such formulas on the Washington Aqueduct, and if they were obtained in a moderately simple shape, they would, I think, be quite valuable. I know of no place where the discussion of these questions may be found, although it is quite possible that it may have been made.

In a German work on bridges, the following formula is given, but without investigation; namely,

$$x = \frac{r}{\sin\theta} \log \cot \tfrac{1}{2} \varphi,$$

which is evidently incomplete, since φ can have but one value for any assumed value of x; whereas it should have several values, one for each coursing joint which depends upon the initial value φ_1 of φ for $x = 0$.

THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from page 148.]

SECTION II.

ON THE MOTIONS AND FIGURE OF A FLUID SURROUNDING THE EARTH.

11. THE results obtained in this and the following section, will be upon the hypothesis that the motions of the fluid are not resisted by the earth's surface. The motions and figure of the fluid must be such as to satisfy equations (13), and also the condition of continuity. To determine them in the general case in which both a and g are functions of h, θ' and φ, would be very difficult. We shall take here the special case only in which a is a function of θ' which increases or decreases, from the equator to the pole, and in which g is regarded as constant at all parts of the earth's surface, and throughout the whole range of altitude. Equations (13) become in this case, when the fluid is elastic,

$$(20) \quad \begin{aligned} g\, D'_\theta h &= r^2 \sin\theta \cos\theta\,(2\,n + D_t \varphi)\, D_t \varphi - r^2 D_t^2 \theta - g\, h\, D'_\theta \log a \\ g\, D_\phi h &= -\, 2\, r^2 \sin\theta \cos\theta\,(n + D_t \varphi)\, D_t \theta - r^2 \sin^2\theta\, D_t^2 \varphi. \end{aligned}$$

In inelastic fluids we have $\log k$ instead of $\log a$.

12. To determine completely the motions and figure of the fluid which, would satisfy the conditions for any initial state of the fluid, upon the hypothesis that the motions are entirely free from resistances arising from the motions of the particles amongst themselves, would be impossible. But since in all fluids there are slight resistances to the motions of the particles amongst themselves, which, however small, eventually destroy all oscillatory or wave motions depending upon the initial state of the fluid, and reduce the motions of the particles amongst themselves to the minimum which satisfies the conditions, it is not necessary to integrate the equations generally, but merely to satisfy them with the least possible motion of the particles amongst themselves. Hence both the motions and density of the fluid at any place, and likewise its figure, must be independent of the time, and therefore constant.

13. Since $D_\varphi h$ in the last of equations (20) can only have a value arising from an oscillatory or wave motion of the fluid, which would soon be destroyed, it must be put equal 0, and then the equation gives by integration for each particle supposed to be entirely free from the resistances arising from the motions of the particles amongst themselves,

$$r^2 \sin^2 \vartheta \, (n + D_t \varphi) = c, \tag{21}$$

in which c is a constant depending upon the initial motion of the particle. Let

R be the radius of the earth regarded as constant,

m the mass of the fluid, and

l its uniform depth when at rest relative to the earth.

As the quantity of motion in the whole mass cannot be affected by the mutual actions of the particles upon each other, we have, even in the case in which the particles are not free from mutual resistances,

$$\int_m r^2 \sin^2 \vartheta \, (n + D_t \varphi) = \int_m c = Cm, \tag{22}$$

in which C is a constant depending upon the initial motions of all the particles.

The first member of this equation expresses the sum of the areas projected upon the plane of the equator, arising from the absolute motions of all the particles for a unit of time, and hence this sum is constant.

14. If we put v for $D_t\varphi$ belonging to the initial state of the fluid, the last equation gives, neglecting quantities of the order of the range of altitude compared with the earth's radius,

$$Cm = R^2 \int_m \sin^2 \vartheta\,(n + v),$$

$$= R^4 \int_0^l {}_N \int_0^{2\pi} {}_\phi \int_0^{\pi} {}_\theta\, k \sin^3 \vartheta\,(n + v),$$

$$= \tfrac{2}{3} R^2 m\,(n + v')$$

in which

$$v' = \frac{3R^2}{2m} \int_0^l {}_N \int_0^{2\pi} {}_\phi \int_0^{\pi} {}_\theta\, k\, v \sin^3 \vartheta.$$

Hence,

$$C = \tfrac{2}{3} R^2 (n + v').$$

In the preceding integration k is supposed to be independent of ϑ. If the density should vary considerably with the latitude, it would affect the preceding result slightly.

When, by the mutual actions of the different strata upon one another, $D_t\varphi$ becomes the same at all altitudes, upon the same parallel of latitude, c then becomes equal to C, and equation (21) gives

$$D_t\varphi = \frac{C}{R^2 \sin^2\theta} - n = \frac{2\,(n + v')}{3 \sin^2\theta} - n. \tag{23}$$

This value of $D_t\varphi$ satisfies the last of equations (20), and since it gives a uniform motion of all the particles of the fluid upon the same parallels of latitude, as much fluid flows from any place as flows into it, while the density, from what has been stated, remains the same, and hence it also satisfies the condition of continuity.

15. If we suppose the initial state of the fluid to be that of rest relative to the earth's surface, equation (23) becomes

$$(24)\qquad D_t\varphi = \left(\frac{2}{3\sin^2\theta} - 1\right) n,$$

substituting this value of $D_t\varphi$ in the first of equations (20) we get, by putting R for r,

$$(25)\quad g\,D_\theta' h = R^2 n^2 \sin\theta \cos\theta \left(\frac{4}{9\sin^4\theta} - 1\right) - R^2 D_t^2\theta - g\,h\,D_\theta' \log\alpha.$$

Since the last term of this equation is a function of h, the height of the strata, the equation can only be satisfied by counter currents of the strata between the equator and the poles; and as we have seen that the figure of the fluid, and its density at the same place, are constant, in order to satisfy the condition of continuity, these currents must be such as to satisfy, for every vertical column of the fluid, the following equation,

$$(26)\qquad \int_m D_t\theta' = 0.$$

In order to satisfy this condition, h, which is a general integral, must have a negative constant added to it. Hence at a certain altitude the last term of (20) vanishes, and the fluid there has no motion towards or from the equator.

If the density increases towards the poles, this term is positive for the lower strata, but negative for the upper ones, and hence the motion is toward the equator below, and from it above. If the density decreases towards the poles, the motions are the reverse.

If there were no resistances of any kind, the motions would be continually accelerated so long as the density is different between the equator and the poles; but where there are slight resistances, the motions are only accelerated until the resistances become equal to the accelerating force.

16. Since the last two terms of (25) have a very little effect upon the value of h, and consequently upon the figure of the fluid,

in comparison with the remaining term of that member, unless the difference of density, and the motion of the fluid between the equator and the poles, are very great, we shall neglect them, and determine the figure depending upon the remaining term arising from the earth's rotation. Equation (25) gives by integration in this case, since $D_\theta' h$ does not differ sensibly from $D_\theta h$,

$$2gh = -R^2 n^2 \left(\frac{4}{9 \sin^2 \theta} + \sin^2 \theta\right) + C.$$

If h' be put for the value of h at the equator, putting $\sin \theta = 1$, we get

$$2gh' = -\tfrac{13}{9} R^2 n^2 + C.$$

Hence,

(27) $$h = h' + \frac{R^2 n^2}{2g}\left(\tfrac{13}{9} - \frac{4}{9 \sin^2 \theta} - \sin^2 \theta\right).$$

Since one of the terms in this value of h has $\sin \theta$ in the denominator, whatever be the value of h at the equator, it must become 0 towards the poles, and the surface of the fluid meet the surface of the earth; and this must be the case, however large the terms which have been neglected. Hence *the fluid, however deep it may be at the equator, cannot exist near the poles.*

17. If θ_0 be the value of θ where $h = 0$, the last equation gives

(28) $$\sin^4 \theta_0 - \left(\frac{2g}{R^2 n^2} h' + \tfrac{13}{9}\right) \sin^2 \theta_0 = -\tfrac{4}{9},$$

which determines θ_0 when h' is given.

If we put θ_1 for the value of θ where h is a maximum, equation (25), putting $D_\theta' h = 0$, and neglecting the last two terms, gives

(29) $$\sin^2 \theta_1 = \tfrac{2}{3},$$

This gives $\theta_1 = 55°$ nearly, answering to the parallel of 35°, where h is a maximum.

If, therefore, we assume h', equation (27) gives the figure which the fluid assumes, *which must be somewhat as represented in the external* *part of* Fig. (1), *the surface of the fluid being slightly depressed at the equator, having its maximum height about the parallel of* 35°, *and meeting the surface of the earth towards the poles.*

18. In the applications of the preceding equations we must put

$$R = 3956 \text{ miles} = 20887680 \text{ feet},$$

$$n = \frac{2\pi}{(23 \times 60 + 56)\, 60} = .000072924$$

$$g = 32.2 \text{ feet}.$$

Hence $$R n = 1523.2 \text{ feet, and } \frac{R^2 n^2}{2g} = 36017 \text{ feet}.$$

With these values, if we assume $h' = 5$ miles, (28) gives $\theta_0 = 28° \, 30'$ for the polar distance of the parallel where the surface of the fluid, or the stratum of equal pressure, meets the surface of the earth.

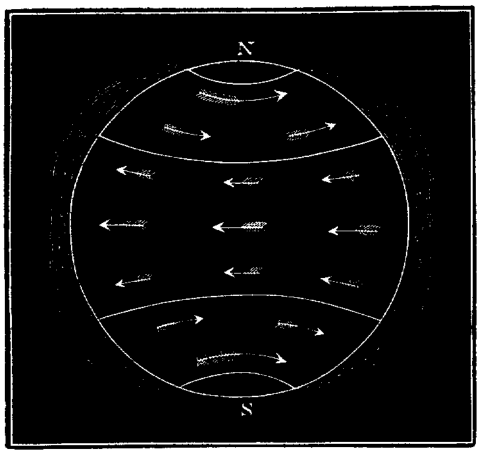


Fig. 1.

If in (27) we substitute for $\sin\theta$ its special value in (29), we obtain $h - h' = 4002$ feet for the excess of the height of the fluid at its maximum, above its height at the equator; which is a constant independent of the amount or depth of the fluid.

19. If we put $D_t \varphi = 0$ in (24), it gives

(30) $$\sin^2\theta = \sin^2\theta_1 = \tfrac{2}{3}.$$

Hence, *the latitude of no motion of the fluid east or west, is the latitude of the maximum of* h.

20. If $\sin^2\theta < \frac{2}{3}$ in (24), $D_t \varphi$ is positive, but if $\sin^2\theta > \frac{2}{3}$, it is negative. Hence, *between the parallels of* 35° *and the poles, the motion of the fluid is eastward, but between those parallels and the equator it is toward the west.*

21. The lineal velocity of the fluid east or west relative to the earth's surface is $R \sin \theta \, D_t \varphi$. Representing it by v'', equation (24) gives

$$v'' = R n \left(\frac{2}{3 \sin \theta} - \sin \theta\right). \tag{31}$$

Putting $\sin \theta = 1$, this equation gives $v'' = -\frac{1}{3} R n = 508$ feet for the velocity of the fluid westward per second at the equator. Towards the poles it is evident, from an inspection of the preceding equation, that the velocity must become very great.

The east and west motions of the fluid, as well as its figure, are represented by Fig. (1), the different lengths of arrows representing, in some measure, the different velocities of the fluid.

22. The whole lineal velocity of the fluid east, arising from both the earth's rotation and the velocity of the fluid relative to the earth's surface, is $R \sin \theta \, (n + D_t \varphi)$. Representing this by v''', equation (24) gives

$$v''' = \frac{2 R n}{3 \sin \theta}. \tag{32}$$

Hence this velocity is inversely as the distance from the axis of rotation, which is a necessary consequence of the preservation of areas, as shown in § 13. For as the fluid in moving from the equator towards the poles approaches the axis of rotation, it must have its velocity increased, and in receding from the axis it must be decreased, just as a planet is accelerated in its perihelion but retarded in its aphelion. The reasoning, therefore, of those who, in attempting to explain the trade winds, assume that the fluid, in moving towards or from the equator, has a tendency to retain the same lineal velocity, is erroneous.

23. If, instead of a state of rest relative to the earth's surface, we suppose that the fluid has an initial angular velocity, we must put $n + v'$ instead of n in the preceding equations.

ON PRACTICAL GEOMETRICAL METHODS OF LOCI.

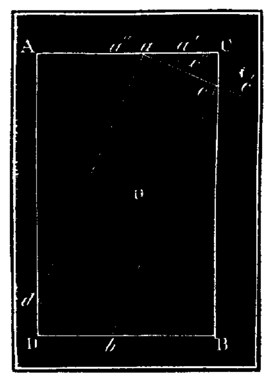
BY D. H. MAHAN,
Professor of Engineering in the U. S. Military Academy, West Point, N. Y.

EDITOR OF THE MATHEMATICAL MONTHLY:

SIR, — Permit me to call the attention of your younger mathematical readers to the subject of the practical geometrical methods of *loci*, as affording both an interesting and profitable exercise in the use of mathematical instruments, particularly for those who are pursuing scientific studies with the view of becoming civil engineers or architects. I have selected two cases out of the many that have fallen under my notice in their applications to practice.

Prob. To inscribe within a given rectangle another, one side of which is given.

Let AB be the given rectangle; ab the one to be found, the side ac of which is given.



An examination of the Fig. will show that the diagonals of the two rectangles pass through the same point O. From O then, with any assumed radius Oa' describe an arc, and from a' set off the chord $a'c' = ac$. If the point c' falls without BC, take any radius $Oa'' < Oa'$ and repeat the same construction, thus determining the point c'' within BC. Having, in this way, found as many points c', c'' &c., as desired, on either side of BC draw through them the line $c'cc''$, &c. This line will be the *locus* of the required conditions, and the point c, where it intersects BC, will be one angle of the inscribed rectangle to be found, from which the others are readily determined by the inverse order of the construction.

Remark. This problem finds an application in HOWE's Truss for finally $\frac{PS}{PQ\sqrt{PF}} = \frac{QT}{QR\sqrt{QG}}$, which is precisely the property of the cycloid proved in Proposition 3. So that an arc of a cycloid is the brachistochrone, or curve of quickest descent, under the supposed conditions.

The points F and G in Fig. 1 are in the base of the cycloid, so that, in Fig. 2, *the base of the cycloid must pass through the point from which the descent begins;* this condition is often not distinctly mentioned. If the two points H and I be nearly on a level, the arc may be almost the whole cycloid.

THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from page 216.]

SECTION III.

ON THE MOTIONS AND FIGURE OF A SMALL CIRCULAR PORTION OF FLUID ON THE EARTH'S SURFACE.

24. We shall, in this case, suppose that α is a function of the distance from the centre of the fluid. It will be more convenient, therefore, to express our general equations (20) in terms of other polar coördinates, of which the pole P, Fig. 2, does not correspond with the pole of the earth. Regarding the earth as a perfect sphere, let

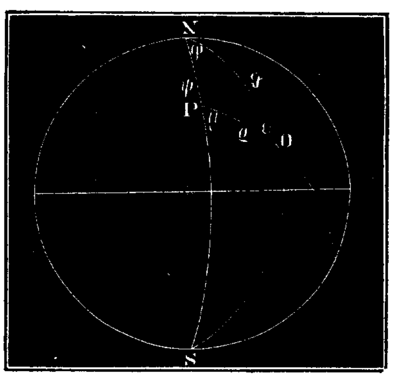


Fig. 2

ψ be the distance in arc of the new pole P from the pole of the earth; also let

ϱ be the distance in arc from the pole P,

μ the angle SPO between ϱ and the meridian,

ε the alternate angle NOP.

If in equations (20) we put $n = 0$, they become the equations of horizontal motions in the case in which the earth has no rotary motion, and the pole of the coördinates can in this case be assumed at pleasure. Hence, when the earth has no rotation, by putting ϱ for θ, and μ for φ, we have

$$(33)\quad \begin{aligned} g\, D_\rho h &= r^2 \sin\varrho \cos\varrho\, (D_t \varphi)^2 - r^2 D_t^2 \varrho - g\, h\, D_\rho \log\alpha, \\ g\, D_\mu h &= -2\, r^2 \sin\varrho \cos\varrho\, D_t \varrho\, D_t \mu - r^2 \sin^2\varrho\, D_t^2 \mu. \end{aligned}$$

When the earth has a rotation, we must add to the second members of these equations respectively the terms $D_\rho F$, and $D_\mu F$, in which F is the part of P, equation (10) depending upon the earth's rotation, and must satisfy the following equations,

$$\begin{aligned} D_\theta F &= 2\, r^2 n \sin\theta \cos\theta\, D_t \varphi, \\ D_\phi F &= -2\, r^2 n \sin\theta \cos\theta\, D_t \theta. \end{aligned}$$

Since θ and φ are functions of ϱ and μ, we must put

$$\begin{aligned} D_\rho F &= D_\theta F . D_\rho \theta + D_\phi F . D_\rho \varphi, \\ D_\mu F &= D_\theta F . D_\mu \theta + D_\phi F . D_\mu \varphi. \end{aligned}$$

Hence, substituting the preceding values of $D_\theta F$ and $D_\phi F$, we get

$$(34)\quad \begin{aligned} D_\rho F &= 2\, r^2 n \sin\theta \cos\theta\, (D_t \varphi\, D_\rho \theta - D_t \theta . D_\rho \varphi), \\ D_\mu F &= 2\, r^2 n \sin\theta \cos\theta\, (D_t \varphi\, D_\mu \theta - D_t \theta . D_\mu \varphi). \end{aligned}$$

Now, from the relations of the different parts of a spherical triangle, we have

$$(35)\quad \begin{aligned} \cos\theta &= \cos\psi \cos\varrho - \sin\psi \sin\varrho \cos\mu, \\ \cot\varphi &= \frac{\sin\psi \cos\varrho + \cos\psi \sin\varrho \cos\mu}{\sin\psi \sin\mu}. \end{aligned}$$

Hence, taking the derivatives and reducing, we get

$$D_\rho \theta = \frac{\cos\psi \sin\varrho + \sin\psi \cos\varrho \cos\mu}{\sin\theta} = \cos\varepsilon,$$

$$D_\mu \theta = -\frac{\sin\psi \sin\varrho \sin\mu}{\sin\theta} = -\sin\varrho \sin\varepsilon,$$

$$D_\rho \varphi = \frac{\sin^2\varphi}{\sin\psi \sin\mu} = \frac{\sin\varepsilon}{\sin\theta},$$

$$D_\mu \varphi = \frac{\cos\psi \sin\varrho + \sin\psi \cos\varrho \cos\mu}{\sin^2\mu \sin\varrho} \sin^2\varphi = \frac{\sin\varphi \cos\varepsilon}{\sin\theta},$$

$$D_t \theta = D_\rho \theta \,.\, D_t \varrho + D_\mu \theta \,.\, D_t \mu = \cos\varepsilon\, D_t \varrho - \sin\varrho \sin\varepsilon\, D_t \mu,$$

$$D_t \varphi = D_\rho \varphi \,.\, D_t \varrho + D_\mu \varphi \,.\, D_t \mu = \frac{\sin\varepsilon}{\sin\theta} D_t \varrho + \frac{\sin\varrho \cos\varepsilon}{\sin\theta} D_t \mu.$$

These values being substituted in (34), we get

$$(36) \qquad \begin{aligned} D_\rho F &= 2\, r^2 n \sin\varrho \cos\theta\, D_t \mu, \\ D_\mu F &= -2\, r^2 n \sin\varrho \cos\theta\, D_t \varrho. \end{aligned}$$

If we add these values of $D_\rho F$ and $D_\mu F$ respectively to the second members of (33), we get for the equations of motion, in terms of ϱ and μ, when the earth has a rotation,

$$(37) \qquad \begin{aligned} g D_\rho h &= r^2 \sin\varrho\, (2n\cos\theta + D_t\mu \cos\varrho)\, D_t\mu - r^2 D_t^2 \varrho - g h D_\rho \log\alpha, \\ g D_\mu h &= -2\, r^2 \sin\varrho\, (n\cos\theta + D_t\mu \cos\varrho)\, D_t\varrho - r^2 \sin^2\varrho\, D_t^2 \mu, \end{aligned}$$

in which $\cos\theta$ has the value in terms of ϱ and μ, in the first of (35).

25. When $\sin\varrho$ is so small that the last term of the value of $\cos\theta$ may be neglected in comparison with the first, we have $\cos\theta = \cos\psi \cos\varrho$, which being substituted in the last equations, they become

$$(38) \qquad \begin{aligned} g D_\rho h &= r^2 \sin\varrho \cos\varrho\, (2n\cos\psi + D_t\mu)\, D_t\mu - r^2 D_t^2 \varrho - g h D_\rho \log\alpha, \\ g D_\mu h &= -2\, r^2 \sin\varrho \cos\varrho\, (n\cos\psi + D_t\mu)\, D_t\varrho - r^2 \sin^2\varrho\, D_t^2 \mu. \end{aligned}$$

These equations are similar to equations (20), having ϱ and μ instead of θ and φ, and, instead of n, having $n\cos\psi$, which is the earth's angular velocity of rotation around the axis, corresponding with the pole P (Peirce's *Analytical Mechanics*, § 25). Hence we can treat these equations precisely as equations (20) in the last section, and, instead of (21), we get

(39) $$r^2 \sin^2 \varrho \, (n \cos \psi + D_t \mu) = c,$$

and, instead of (22), we get

(40) $$\textstyle\int_m r^2 \sin^2 \varrho \, (n \cos \psi + D_t \mu) = \int_m c = Cm.$$

On account of the term which has been neglected in the value of $\cos \theta$, these equations cannot be used for large values of ϱ, and hence we may put $\sin \varrho = \varrho$. Let

$s = R\varrho$ be the lineal distance from the centre,

s' be the value of s at the external part of the fluid,

u be the initial value of $D_t \mu$.

The last equation then gives, putting R for r,

$$\begin{aligned} Cm &= \textstyle\int_m s^2 (n \cos \psi + u), \\ &= \int_0^l {}_N \int_0^{2\pi} {}_\mu \int_0^{s'} {}_s \, k s^3 (n \cos \psi + u), \\ &= \tfrac{1}{2} s'^2 m \, (n \cos \psi + u'), \end{aligned}$$

in which

$$u' = \frac{2}{s'^2 m} \int_0^l {}_N \int_0^{2\pi} {}_\mu \int_0^{s'} {}_s \, k s^3 u.$$

Hence,

(41) $$C = \tfrac{1}{2} s'^2 (n \cos \psi + u').$$

In the preceding integration k is regarded as a constant. When, by the mutual action of the different strata upon each another, $D_t \mu$ becomes the same at all altitudes at the same distance from the centre P, c becomes equal to C, and equation (39) then gives

(42) $$D_t \mu = \frac{C}{R^2 \sin^2 \varrho} - n \cos \psi = \frac{s'^2 (n \cos \psi + u')}{2 s^2} - n \cos \psi.$$

26. If we suppose the initial state of the fluid to be that of rest relative to the earth's surface, the last equation becomes

(43) $$D_t \mu = \left(\frac{s'^2}{2 s^2} - 1 \right) n \cos \psi.$$

Substituting this value of $D_t\,\mu$ in the first of equations (38), it becomes, by putting R for r and $\cos\varrho = 1$,

$$(44)\qquad g\,D_s\,h = n^2 \cos^2\psi \left(\frac{s'^4}{4\,s^3} - s\right) - D_t^2\,s - g\,h\,D_s \log\alpha.$$

This equation is similar to (27), and, like it, can only be satisfied by means of an interchanging motion between the internal and external part of the fluid; and the remarks following that equation in § 15 are also applicable to this.

27. By omitting the last two terms in the preceding equation, as was done in equation (25), (§ 16), we get by integration,

$$2\,g\,h = -\,n^2 \cos^2\psi \left(\frac{s'^4}{4\,s^2} + s^2\right) + C.$$

Hence, eliminating C,

$$(45)\qquad h = h' + \frac{n^2 \cos^2\psi}{2\,g}\left(\tfrac{5}{4}\,s'^2 - \frac{s'^4}{4\,s^2} - s^2\right).$$

Since one of the negative terms in this value of h has s in the denominator, it must become equal 0 towards the centre where s vanishes. Hence *the fluid, however deep it may be at the external part, cannot exist at the centre.*

28. If we put s_0 for the value of s where $h = 0$, the last equation gives

$$(46)\qquad 0 = h' + \frac{n^2 \cos^2\psi}{2\,g}\left(\tfrac{5}{4}\,s'^2 - \frac{s'^4}{s_0^2} - s_0^2\right),$$

from which we obtain s_0 for any assumed value of h'.

Since s_0 is very small, the terms $\frac{5}{4}\,s'^2$ and $-\,s_0^2$ may generally be omitted in the last equation, and it then becomes

$$(47)\qquad s_0 = \frac{n \cos\psi\, s'^2}{\sqrt{2\,g\,h'}}.$$

If we put s_1 for s where h is a maximum, equation (44) gives, by putting $D_s\,h = 0$, and neglecting the last two terms,

$$(48)\qquad s_1 = \frac{s'}{\sqrt{2}}.$$

Equation (45) determines the figure of the surface of the fluid, *which is very slightly convex towards the external part, and meets the surface of the earth near the centre* c, as represented in Fig. 3.

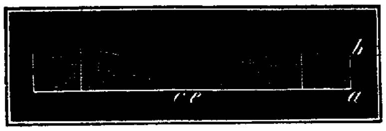


Fig. 3.

If we assume h', or $a\,b$, Fig. 3, equal 5 miles, and $a\,c = 100$ miles, equation (49) gives $c\,e = 2$ miles nearly.

29. Equation (43) gives the angular velocity of gyration, which must be very great near the centre, where s is small.

Putting $D_t\,\mu = 0$, it gives

$$(49) \qquad s = \frac{s'}{\sqrt{2}} = s_1.$$

Hence, *at the distance of* s_1, *which is the distance of the maximum of* h, *there is no gyratory motion.*

In the northern hemisphere, where $\cos\psi$ is positive, if $s < s'$, $D_t\,\mu$ is positive, but if $s > s'$, it is negative. Hence *the inner part of the fluid gyrates from right to left, but the external part from left to right,* as represented in Fig. 4. In the southern hemisphere, where $\cos\psi$ is negative, the gyrations are the reverse.

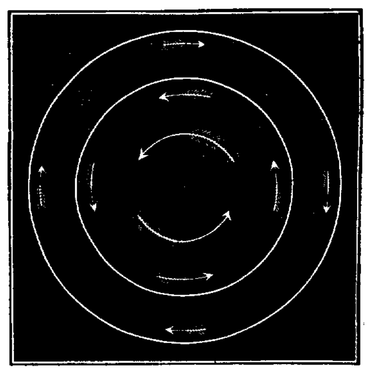


Fig. 4.

30. If the fluid is of uniform density, and every part gyrates with the same angular velocity u, it satisfies equations (38) by satisfying the following equation:

$$g\,D_s\,h = 2\,s\,u\,n\cos\psi + s\,u^2,$$

since all the other terms vanish; and this motion also satisfies the condition of continuity. By integrating, we get

$$(50) \qquad g\,h = \tfrac{1}{2}\,s^2\,u\,(2\,n\cos\psi + u) + C.$$

This is the equation of a parabola. Hence *the surface of the fluid,* *relative to the earth's surface, is the surface of a paraboloid.* If the portion of the fluid is so small that the earth's surface may be regarded as a plane, it becomes absolutely the surface of a paraboloid; and when the angular velocity of gyration is great in comparison with that of the earth's rotation, $2n \cos\psi$ may be omitted, in the preceding equation, in connection with u.

If $u = -2n\cos\psi$, or $u = 0$, h is constant, and then the surface of the fluid is a level surface. If u is negative and less than $2n\cos\psi$, the surface is convex; in all other cases it is concave.

31. If the whole of a gyrating mass of fluid has a tendency to move in the direction of the meridian with a force V, if we regard the forces which act upon each part of the fluid in the directions of the meridians as parallel, we have, using R for r,

$$V = m\, D_t^2\, \psi = \frac{1}{R} \int_m D_\theta' P.$$

The error arising from regarding the forces in the directions of the meridians parallel is of the second order of their deviation from parallelism, and consequently very small, unless the lateral extent of the fluid is very great.

From the last equation and the second of equations (9), omitting the term containing $D_t r$ as a factor, since it can produce no sensible effect, we get

$$V = \int_m [-R\, D_t^2\, \theta + R \sin\theta \cos\theta\, (2n + D_t \varphi)\, D_t \varphi].$$

If in this equation we substitute for $D_t \varphi$ its value in § 24, and for $D_t^2 \theta$ its value derived from that of $D_t \theta$ in the same section, and also for $\cos\theta$ its value in the first of equations (35), putting $\varepsilon = \mu$, since the meridians are regarded as parallel, and omitting all terms which give 0 by integration, we get

$$(51) \qquad \begin{aligned} R\, V &= -2n \sin\psi \int_m s^2 \cos^2\mu\, D_t \mu, \\ &= -n \sin\psi \int_m s^2\, D_t \mu. \end{aligned}$$

If $D_t \mu$, the angular velocity of gyration, is positive, V is negative; but positive, if $D_t \mu$ is negative. Hence *if the fluid gyrates from right to left, the whole mass has a tendency to move towards the north; but if from left to right, towards the south.*

If every part of a cylindrical mass having its axis of revolution vertical has the same angular velocity of gyration as in the case of solids, calling this velocity u, the preceding equation gives for the accelerating force in the direction of the meridian,

$$(52) \quad \begin{aligned} \frac{V}{m} &= -\frac{s'^2 u n \sin \psi}{2 R} = -\frac{s'^2 u \sin \psi}{2 R^2 n} \times R n^2, \\ &= -\frac{s'^2 u \sin \psi}{2 R^2 n} \times \frac{g}{289} = -\frac{g}{578} \cdot \frac{u \sin \psi}{n} \cdot \frac{s'^2}{R^2}. \end{aligned}$$

32. If a body move in the direction of ϱ or s with a velocity $v = D_t s$, and p be the direction of a perpendicular to it on the left, we obtain from the last of equations (36) for the deflecting force in the direction of p, arising from the earth's rotation,

$$(53) \quad \begin{aligned} D_p F &= \frac{D_\mu F}{R \sin \varrho} = -2 R n \cos \theta \, D_t \varrho, \\ &= -2 n \cos \theta \, D_t s = -\frac{2 \cos \theta \, D_t s}{R n} \times R n^2, \\ &= -\frac{2 \cos \theta \, D_t s}{R n} \times \frac{g}{289} = -\frac{2 g v \cos \theta}{289 R n}. \end{aligned}$$

This force is negative in the northern hemisphere, and positive in the southern. Hence *in whatever direction a body moves on the surface of the earth, there is a force arising from the earth's rotation, which deflects it to the right in the northern hemisphere, but to the left in the southern.* This is an extension of the principle upon which the theory of the trade winds is based, and which has been heretofore supposed to be true only of bodies moving in the direction of the meridian.

points by $x'\,y'$ and $x''\,y''$, and the intercepts of the axes of reference by α and β; then, by known properties* of great circles arcs, we have

$$(1)\qquad x = -\frac{1}{\alpha},\quad y = -\frac{1}{\beta},$$

$$(2)\qquad y' = -\frac{\beta}{\alpha}x' + \beta,$$

$$(3)\qquad y'' = -\frac{\beta}{\alpha}x'' + \beta.$$

The elimination of α and β from (1), by means of (2) and (3), will obviously satisfy the required conditions of the problem. But the elegant solution of the problem by Mr. Osborne in the last Monthly, page 292, by an entirely different method, would seem to render further remarks unnecessary.

THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from page 307.]

SECTION IV.

ON THE GENERAL MOTIONS AND PRESSURE OF THE ATMOSPHERE.

33. By the general motions of the atmosphere are meant all those motions produced by a difference of density between the equatorial and polar regions arising principally from a difference of temperature. If the motions of the atmosphere were not resisted by the earth's surface, the results of the preceding sections could be at once applied to them without any modifications, and hence towards the poles there would be a very rapid motion eastward, and in the equatorial

* These properties, and many other analogous ones, of great circle arcs, it is proposed to investigate in subsequent numbers of the Monthly.

regions towards the west, and the atmosphere would entirely recede from the poles, and be also depressed about 4,000 feet at the equator, as has been shown in section (2). Although the preceding results, when applied to the atmosphere, are very much modified by the resistances of the earth's surface, yet they will be of great advantage in explaining its general motions; for as there can be no resistance until there is motion, the atmosphere must have a tendency to assume, in some measure, the same motions and figure as in the case of no resistances. Hence, towards the poles the general motions of the atmosphere must be towards the east, and in the torrid zone towards the west; but as these motions, in consequence of the resistances, are small in comparison with those in the case of no resistances, instead of the atmosphere's receding entirely from the poles, as represented in Fig. 1, page 215, there must be only a comparatively small depression there, as represented in Fig. 5, and instead of its being about 4,000 feet lower at the equator than at the place of its maximum height near the tropics (§ 18), there must be only a very slight depression there.

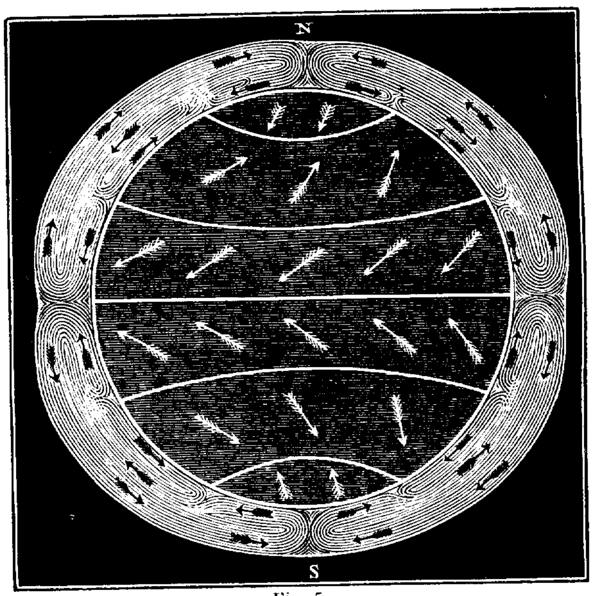


Fig. 5.

34. The force which overcomes the resistance of the earth's surface to the east and the west motions of the atmosphere depends upon the term in the least of our general equations (13) containing

$D_t\theta$ as a factor, which depends upon the interchanging motion of the fluid between the equatorial and the polar regions, and hence the term must vanish at the equator and the poles. All the east or west motion of the atmosphere is consequently destroyed by the resistances at these places, and hence as $D_t\theta$ vanishes there also, there is a belt of calms at the equator, called the equatorial calm belt, and there must be also a region of calms about the poles.

35. As the motion of the atmosphere is east towards the poles and west near the equator, somewhere between the equator and the poles there must be a parallel of no motion east or west, which, in the case of no resistance, was determined upon the hypothesis of an initial state of rest, and found to be at the parallel of 35°, § (18). In the case of the atmosphere this parallel is entirely independent of the initial state of the atmosphere, and depends in a great measure upon the law of resistance, and hence it cannot be accurately determined. It is evident, however, that the east and west motions of the atmosphere at the earth's surface must be such that the sum of the resistances of each part of the earth's surface multiplied into its distance from the axis of rotation, must be equal 0, else the velocity of the earth's rotation would be continually accelerated or retarded, which cannot arise from any mutual action between the surface of the earth and the surrounding atmosphere. Now, as the part of the earth's surface where the motion of the atmosphere is west is much farther from the axis than the part where it is east, the latter part must comprise more than half of the earth's surface, unless the velocity of the eastern motion towards the poles is much greater than that of the western motion near the equator. Therefore, since one-half of the earth's surface is contained between the parallels of 30°, the parallels of no east or west motion at the earth's surface must fall within these parallels, and they are accordingly found to be near the tropics, on the ocean. Hence the maximum height

of the atmosphere, as represented in Fig. (5), must also be near the same parallels.

36. The increase of pressure arising from the accumulation of atmosphere near the tropics, caused principally by the deflecting forces (§ 32) arising from the more rapid east and west motions of the atmosphere in the upper regions, where there is least resistance, gives the atmosphere a tendency to flow from beneath this accumulation both towards the equator and the poles, since the motions, and consequently the forces, which cause this accumulation, are much less near the surface. But on account of the greater density of the atmosphere towards the poles, it has a tendency also to flow, at the earth's surface, from the poles towards the equator. Between the parallels of greatest pressure and the equator, these tendencies combine, and produce a strong surface current, which, combining with the westward motion there, gives rise to the well-known north-east wind in the northern hemisphere, and the south-east wind in the southern hemisphere, called the trade winds. But between the parallels of greatest pressure and the poles, these tendencies are opposed to each other, and the one arising from the accumulation of atmosphere near the tropics being the greater in the middle latitudes, causes the atmosphere to flow at the earth's surface towards the poles; and this motion, combining with the general eastward motion of the atmosphere in those latitudes, gives rise to the south-west wind in the northern hemisphere and the north-west wind in the southern hemisphere, called the passage winds.

37. Near the poles, the tendency to flow towards the equator seems to be the greater, and causes a current there *from* the poles, which, being deflected westward (§ 32), causes a slight north-east wind in the north frigid zone, and a south-east wind in the south frigid zone. But this is only near the earth's surface; and the gener-

al tendency of the atmosphere in the upper regions must be towards the east, as will be seen.

38. Since the atmosphere near the tropics can have no motion in any direction at the earth's surface, there are calm belts there, called the tropical calm belts. Near the polar circles, where the polar and passage winds meet, there must also be calm belts, which may be called polar calm belts. The motions of the atmosphere, therefore, at the earth's surface, if they were not modified by the influence of continents, would be as represented in the interior of Fig. (5), in which the heavy lines represent the calm belts. On account of the influence of the continents, these belts are somewhat displaced and irregular, and on account of the varying position of the Sun, they change their positions a little in different seasons of the year.

The southern limit of the polar winds in the northern hemisphere, and also the limit between the trade and passage winds, has been determined by Prof. J. H. Coffin, from the discussion of a great number of observations at different points, and given in a chart, in his treatise on the winds, published in the seventh volume of the Smithsonian Contributions.

39. That the atmosphere is depressed at the equator and the poles, and has its maximum height near the tropics, as has been represented, is indicated by barometrical pressure. It was formerly thought that this pressure, at the level of the ocean, was very nearly 30 inches in all latitudes; but it is now well established that it is much less towards the poles than near the tropics, and also a little less at the equator. Says Captain Wilkes: "The most remarkable phenomenon which our observations have shown is the irregular outline of the atmosphere surrounding the earth as indicated by the pressure upon the measured column at different parts of the surface. Our barometrical observations show a depression

within the tropics, a bulging in the temperate zone, again undergoing a depression on advancing towards the arctic and antarctic circles." The mean of all the observations, as given in the Report of the Exploring Expedition, from Cape Henry to Madeira, taken between the parallels of 28° and 32°, was 31.215 inches; at Maderia, latitude 32° 53′, 30.176 inches; and in the rainy belt between the parallels of 8° and 12°, 29.987 inches. After passing the equator there was a slight elevation, again reaching its maximum near the tropic of Capricorn. Beyond this there was a gradual depression until about the parallel of 55°, where the barometer was rapidly depressed below 29 inches. After doubling Cape Horn and proceeding towards the equator, the height of the barometer gradually increased again to its usual height in the middle and equatorial latitudes. On sailing south again, in the Pacific Ocean, a depression of the barometer was again observed. The mean of all the observations taken on 22 days, in sailing from Callao to Tahiti, between the parallels of 10° and 15°, was 30.109 inches; and of those made on 32 days, between the parallels of 15° and 20°, was 30.147 inches. The mean of the observations made on 5 days, after leaving Sydney, between the parallels of 35° and 45°, was 30.305 inches; of those made on 7 days, between the parallels of 45° and 55°, was 29.790 inches; of those taken on 8 days, between the parallels of 55° and 65°, was 29.378 inches. The mean also of all those taken along the antarctic continent was 29.040 inches.

40. Says Sir James Ross (*Voyage to the Southern Seas,* Vol. 2, p. 383): "Our barometrical experiments appear to prove that the atmospheric pressure is considerably less at the equator than near the tropics; and to the south of the tropic of Capricorn, where it is greatest, a gradual diminution occurs as the latitude is increased, as will be shown from the following Table, derived from hourly observations of the height of the column of mer-

cury between the 20th of November, 1839, and the 31st of July, 1843."

EXTRACT FROM ROSS'S TABLE.

LATITUDE.	PRESSURE.	LATITUDE.	PRESSURE.	LATITUDE.	PRESSURE.
	inches.		inches.		inches.
Equator,	29.974	42° 53′	29.950	55° 52′	29.360
13° 0′ S.	30.016	45 0	29.664	60 0	29.114
22 17	30.085	49 8	29.467	66 0	29.078
34 48	30.023	51 33	29.497	74 0	28.928
		54 26	29.347		

41. The following table, first published by M. Schouw, and reduced here from millimetres to English inches, shows that there is a similar bulging of the atmosphere in the middle latitudes, and depression at the pole in the northern hemisphere, as has been observed in the southern hemisphere.

PLACE.	LATITUDE.	PRESSURE.	PLACE.	LATITUDE.	PRESSURE.
		inches.			inches.
Cape,	33° 0′ S.	30.040	London,	51° 30′	29.961
Rio Janeiro,	23 S.	30.073	Altona,	53 30	29.937
Christianburg,	5 30 N.	29.925	Dantzic,	54 30	29.925
La Guayra,	10	29.928	Konigsberg,	54 30	29.941
St. Thomas,	19	29.941	Apenrade,	55	29.905
Macao,	23	30.039	Edinburgh,	56	29.851
Teneriffe,	28	30.087	Christiana,	60	29.866
Madeira,	32 30	30.126	Bergen,	60	29.703
Tripoli,	33	30.213	Hardanger,	60	29.700
Palermo,	38	30.036	Reikiavig,	64	29.607
Naples,	41	30.012	Godthaab,	64	29.603
Florence,	43 30	29.996	Eyafiord,	66	29.669
Avignon,	44	30.000	Godhaven,	69	29.674
Bologna,	44 30	30.008	Upernavik,	73	29.732
Padua,	45	30.008	Mellville Isle,	74 30	29.807
Paris,	49	29.976	Spitzbergen,	75 30	29.795

By the above method the function a^x may easily be differentiated. We have $D_x a^x = m a^x$; where $a^{\frac{1}{m}} = 2.71828\dots$ $\therefore (2.718\,..)^m = a$, or m is $\log a$ in the system of which the base is $2.7182818\dots$

THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from page 373.]

43. The pressure of the atmosphere may be obtained from the first of equations (9). The terms in the equation depending upon the motions of the atmosphere are insensible, and consequently may be omitted. The term $D_t^2 r$ depends upon the acceleration or retardation of the vertical motion of the atmosphere, and is of the same order in comparison with g as the rate of its acceleration or retardation in comparison with that of a descending or ascending free body, and hence in all ordinary motions of the atmosphere it is insensible. Restoring the value of ω in (§ 2), the largest of the remaining terms is $2\,r\sin^2\theta\, n\, D_t\varphi$, which is of the same order in comparison with $r\,n^2$, as the east or west motion of the atmosphere in comparison with the motion of the rotation of the earth on its axis. But $r\,n^2 = \frac{1}{289}\,g$ only, hence the term is entirely insensible. We may therefore put

$$\frac{D_N\,P}{P} = -\,\alpha g.$$

Using the common system of logarithms and putting M for its modulus, we get by integration

$$(54)\qquad \log P' - \log P = M\alpha g\,N,$$

in which P' is the pressure at the earth's surface.

Hence

$$(55)\qquad D_{\theta'}\log P = D_{\theta'}\log P' - M g\,N\,D_{\theta'}\,\alpha.$$

By means of this equation the second of equations (9) becomes, by putting αP for k, (§ 6), and omitting the very small term $2\,r\,D_t\,r\,D_t\,\theta$,

$$(56)\quad D_{\theta'}\log P' - Mg\,N\,D_{\theta'}\,\alpha = M\alpha\,[r^2\sin\theta\cos\theta\,(2\,n + D_t\,\varphi)\,D_t\,\varphi - r^2\,D_t^2\,\theta].$$

44. In the case of the atmosphere, there must be a term in equations (9) to represent the resistances to the motions; and this term in the second of these equations may be denoted by $(\varphi)\,D_t\,\theta$ in the second member. Putting

$$(57)\qquad W = (\varphi)\,D_t\,\theta + r\,D_t^2\,\theta,$$

W will represent the force which overcomes the resistances to the motions of the atmosphere between the equator and the poles, and also its inertia.

Since $D_t\,\varphi$ is generally very small in comparison with $2\,n$, it may be omitted in equation (56), which then becomes, by means of equation (57),

$$(58)\quad D_{\theta'}\log P' - Mg\,N\,D_{\theta'}\,\alpha = M\alpha(2\,r^2\,n\sin\theta\cos\theta\,D_t\,\varphi - r\,W).$$

45. It will be shown, that $r\,W$ is very small in comparison with $2\,r^2\,n\sin\theta\cos\theta\,D_t\,\varphi$; and hence, if $D_t\,\varphi$ were known, the preceding equation, neglecting the term $r\,W$, would give approximately the pressure of the atmosphere at the earth's surface. But since we do not know the value of the term denoting the resistances in the last of equations (9), we cannot determine the value of $D_t\,\varphi$; therefore, since $D_{\theta'}\log P'$ can be determined from observations of the barometric pressure, we shall use the equation to determine $D_t\,\varphi$, from which we easily obtain the east or west motion of the atmosphere. Denoting the velocity of this motion per hour by v, we shall have

$$(59)\qquad v = 3600\,r\sin\theta\,D_t\,\varphi.$$

46. The ratio of the density to the elastic force decreases $\frac{1}{491}$ for

every degree of Fahrenheit. But as a higher temperature is always accompanied by a greater amount of aqueous vapor, the density of which is less than that of the atmosphere, the rate of decrease has been found to be $\frac{1}{449}$ for every degree. Let

α' be the value of α at the equator, and

i the difference of temperature between the equator and the poles.

If we suppose the temperature to decrease from the equator to the poles as the square of the sine of the latitudes, we shall have

$$\alpha = \alpha' (1 + \tfrac{1}{449}\, i \cos^2 \theta).$$

Hence

$$D'_\theta\, \alpha = -\tfrac{2}{449}\, \alpha'\, i \sin \theta \cos \theta.$$

By means of the last three equations, equation (58), putting R for r and e for $\frac{1}{M\alpha' g}$, is reduced to

$$(60)\quad v = \frac{1800}{R\, n \cos\theta\, (1 + \frac{1}{449}\, i \cos^2 \theta)} \Big(e\, g\, D_{\theta'} \log P' + \tfrac{2}{449}\, i\, g \sin \theta\, N + W \Big).$$

Since the variation of α with the altitude can produce no sensible effect in the results, α has been regarded as a function of the latitude only. We must, therefore, take the mean value of α' belonging to the atmosphere at the equator at all altitudes, which we will assume to be that belonging to the temperature of 32°.

47. By means of observations of P at different altitudes, equation (54) gives the value of $\frac{1}{M\alpha g}$, which, at the temperature of 32°, has been determined to be 60156 feet; which, consequently, is the value of e. The difference between the mean temperatures of the equator and the poles is about 60°; we shall, therefore, in the following applications, put $i = 60$.

48. The value of $D_{\theta'} P'$ in the preceding equation can be determined approximately for any latitude from the preceding tables of barometric pressure. Since the coördinates of pressure given there

have been deduced from observations made in different longitudes and at all seasons, they are somewhat irregular; but coördinates can be assumed with regular differences, and such that the interpolated values of the coördinates of pressure for the latitudes given in the tables will very nearly correspond with the pressures given there; and then, from these coördinates, the approximate value of $D_{\theta'} \log P'$ can be determined. In this manner the values of $D_{\theta'} P'$ in the following table have been determined, except the first, which has been assumed. The third column of the table contains the values of v at the earth's surface, neglecting the term W, which will be shown to have, in general, a very little effect. The fourth contains the coefficient of N, and the fifth the value of v at the height of 3 miles.

TABLE.

Latitude.	$D_{\theta'} \log P'$.	v, ($N=0$).	Coeff. of N.	v, ($N=3$ miles).
75 N.	—.0060	— 2.7 miles.	2.33	+ 4.3 miles.
65	.0000	0.0	3.87	11.6
55	+.0188	+ 9.9	5.34	25.9
45	+.0080	+ 4.5	6.71	24.6
30	.0000	0.0	8.49	25.5
15	—.0060	—10.0	9.70	19.1
15 S.	+.0060	—10.0	9.70	19.1
30	—.0147	+11.4	8.49	36.9
40	—.0372	+23.4	7.36	45.5
50	—.0295	+15.3	6.07	33.5
60	—.0133	+ 6.0	4.61	19.8

49. The term W, and its effect upon the value of v, cannot be determined, but they can be shown from observation to be, in general, very small; and, since W is positive, as may be seen from equation (75), when the motion is from the north towards the south, and negative when the contrary, except when the motion is retarded, and the term $r\, D_t^2\, \theta$ arising from the inertia of the atmosphere is greater

than the resistances, its effect for the most part is to increase the value of v algebraically where the motion is towards the south, and decrease it where it is towards the north. In the regions of the trade winds about the parallels of 15°, the current at the surface of the earth is stronger than at any other parallel, and as the resistances at the surface must be much greater than in the upper regions, the term W must be greater there than in any other part of the atmosphere. If $v = 0$, equation (60) gives, since $N = 0$ at the surface, $W = e\,g\,D_{\theta'} \log P'$; and from the preceding table, when $v = -10$ miles, $W = 0$. Now, we know from observation that the velocity of the atmosphere westward at the parallels of 15°, cannot be much less than 10 miles per hour, and hence W is small in comparison with $e\,g\,D_{\theta'} \log P'$, which at that parallel is itself small; and hence the effect of W upon the value of v in the higher latitudes, where the value of $\cos \theta$ in the denominator is much greater, must be very small. Very near the equator the formula for the value of v, equation (60), fails practically, since, on account of the small value of $\cos \theta$ there, the effect of W may be very great.

50. If the motion of the atmosphere east in the higher latitudes and west near the equator, be that given in the preceding table, or by equation (60), it must cause the observed difference of barometric pressure in the different latitudes; and hence, from what we know of those motions of the atmosphere from observations, there can be no doubt that they are adequate to account for this observed difference of pressure.

51. It is evident, where the motions of the atmosphere are resisted by the earth's surface, that all the conditions cannot be satisfied by a motion at the surface from the poles towards the equator, and by a counter motion in the upper regions. For we have seen (§ 35), that the atmosphere at the surface of the earth must have an eastern motion in the middle latitudes; but it cannot have such a motion,

unless it also have a motion towards the poles, in order that the deflecting force (§ 18) arising from this motion may overcome the resistances to the eastern motion. But it is evident that there cannot be a complete reversal of the motions in the middle latitudes, but some portion of it must flow towards the poles in the upper regions, else the eastern motion there could not be greater than at the surface, which the conditions require. The motions, therefore, must be somewhat as represented in Fig. (5). The part of the atmosphere next the earth's surface in the middle latitudes having a motion towards the poles, extends to a considerable height, since it generally embraces the region of fair weather clouds, as may be seen by observation.

52. It is seen, from the results given in the preceding table, that the eastward motion of the atmosphere in the middle and higher latitudes must be greatest in the upper strata, and that in the region of the trade winds, where the motion is westward at the surface, it must be towards the east above. This is also evident from the general consideration, that the whole amount of deflecting force eastward arising from the motion of the atmosphere towards the poles is equal to the deflecting force westward arising from its motion back towards the equator, and that the deflecting force eastward is principally above where there is less resistance than near the surface. Hence at the top of Mauna Loa in the Sandwich Islands, and on the peak of Teneriffe, both of which places are near the tropical calm belt at the surface, a strong south-west wind prevails. Hence, also, "on the eruption of St. Vincent, in 1812, ashes were deposited at Barbadoes, sixty or seventy miles eastward, and also on the decks of vessels one hundred miles still further east, whilst the trade wind at the surface was blowing in its usual direction." The eastward motion of the atmosphere above, in the latitudes of the trade winds, is also confirmed by observations made on the directions of the

clouds at Colonia Tovar, Venezuela, latitude 10° 26′, as given in the Report of the Smithsonian Institution for 1857 (p. 254). While the motion of the lower clouds was in general from some point towards the east, the observed motion of nearly all the higher clouds was from some point towards the west.

53. From what precedes, the limit between the atmosphere which moves eastward in the middle latitudes and westward nearer the equator, which at the earth's surface is at the tropical calm belt, must be a plane inclining towards the equator above. And since, according to (§ 51), the atmosphere near the earth's surface cannot have an eastward motion, unless it also has a motion toward the poles; this plane near the earth's surface must nearly coincide with the one which separates the atmosphere moving towards the poles from that moving towards the equator, in the trade wind regions, and hence the latter must also incline above towards the equator. This explains the winds at the peak of Teneriffe, which at the top always blow from the south-west, while at the base they blow alternately from the south-west and north-east, changing with the seasons. As the tropical calm belt together with this dividing plane changes its position with the seasons, as will be explained, in the latter part of summer when this plane is farthest north, it still leaves the top of the peak north of it while the base is south of it; and hence the wind at the top always blows from the south-west, even when at the base it blows from the north-east. As this plane moves south in the fall, more of the peak gradually becomes north of it; and hence the south-west wind, which always prevails at the top, gradually descends lower on the sides of the peak until it reaches the base. Hence, when this plane reaches its most southern position, in the latter part of winter, the south-west wind prevails at both the base and the top.

54. It is seen, from the first of the results given in the last table,

that if the barometric pressure increases near the poles, as it seems to do, at least in the northern hemisphere, the atmosphere at the earth's surface must have a westward motion there; and as it cannot have this motion unless it also have a motion toward the equator, so that the deflecting force arising from this motion may overcome the resistance to the westward motion, the wind there must blow slightly from the north-east, as has been shown in (§ 37). This, according to Professor Coffin's chart of the winds, already alluded to, seems to accord with observation.

55. The depression of the atmosphere at the poles and at the equator, and the accumulation near the tropics, may be explained in a general manner by means of the principle in (§ 32), that when a body moves in any direction in the northern hemisphere, it is deflected to the right, and the contrary in the southern. The atmosphere towards the poles having an eastward motion, the deflecting force arising from it causes a pressure towards the equator, and the motion near the equator being westward, the pressure is towards the poles; and hence there must be a depression at the poles and at the equator, and an accumulation near the tropics. Since this deflecting force is as $\cos \theta$, it is small near the equator; and, consequently the depression there is small.

56. According to the preceding tables of barometric pressure, there is more atmosphere in the northern than in the southern hemisphere. Says Sir James Ross, "the cause of the atmosphere being so very much less in the southern than in the northern hemisphere remains to be determined." This is very satisfactorily accounted for by the preceding principle; for as there is much more land, with high mountain ranges, in the northern hemisphere, than in the southern, the resistances are greater, and consequently the eastward motions, upon which the deflecting force depends, is much less; and the consequence is, that the more rapid motions of the

southern hemisphere cause a greater depression there, and a greater part of the atmosphere to be thrown into the northern hemisphere.

This also accounts for the mean position of the equatorial calm belt being, in general, a little north of the equator. But in the Pacific Ocean, where there is nearly as much water north of the equator as south, its position nearly coincides with the equator.

For the same reason the tropical calm belt of the northern hemisphere is farther from the equator than that of the southern hemisphere; and, on account of the irregular distribution of the land and water of the two hemispheres in different longitudes, it does not coincide with any parallel of latitude. In the longitude of Asia, where there is all land in the northern hemisphere and the Indian Ocean in the southern, this belt, which is also the dividing line which separates the winds which blow east from those which blow west, is farther from the equator than at any other place, as shown by Professor Coffin's chart.

57. In winter, the difference of temperature between the equator and the poles, upon which the disturbance of the atmosphere depends, is much greater than in summer; this causes the eastward motion of the atmosphere in either hemisphere during its winter to be greater, while in the other hemisphere it is less. Hence a portion of the volume of the atmosphere in winter is thrown into the other hemisphere; but, although the volume or height of the atmosphere is then less, yet, being more dense, the barometric pressure remains nearly the same. The difference at Paris, and in the middle latitudes generally, between winter and summer, is only about $\frac{1}{10}$ of an inch.

On account of this alternate change with the seasons of the velocity of the eastward motion of the atmosphere in the two hemispheres, the equatorial and tropical calm belts change their positions a little, moving north during our spring, and south in the fall. When the sun is near the tropics, the true law of the decrease of temperature from the equator to the poles varies from that which has been assumed, (§ 46), and is then different in the two hemispheres, which doubtless has some effect also upon the position of the calm belts.

[To be Continued.]

METHOD OF SOLVING NUMERICAL EQUATIONS.

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THE most direct method of resolving a numerical equation of any degree, is to transform it in such a way that its first member becomes a perfect power of the same degree; the solution is then reduced to extracting the root of its second member.

If the first member can be made a perfect nth power by the addition or subtraction of a *number*, the solution will be effected by simply extracting the nth root of the absolute term.

Thus, if the given equation be

$$x^3 - 3x^2 + 3x = 15,$$

subtract . . . . $1 = 1;$

then $(x - 1)^3 = x^3 - 3x^2 + 3x - 1 = 14,$

$\therefore$ $x - 1 = \sqrt[3]{14}$, and $x = 1 + \sqrt[3]{14}.$

If, however, it should be necessary to add terms involving x, then the quantity whose nth root is required, will be a polynomial, with numerical coefficients, and of a degree inferior to the nth. Thus, if we have

$$x^3 + 7x^2 + 5x = 18,$$

add $8 - x^2 + 7x = 8 - x^2 + 7x;$