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THEORY OF SCALE MODELS AS APPLIED TO THE STUDY OF GEOLOGIC STRUCTURES

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ILLUSTRATIONS

INTRODUCTION

Many of the phenomena of physical science are simple enough and well enough understood that they are amenable to complete mathematical analysis without recourse to auxiliary experimentation. There are other phenomena, however, which, though being made up of well-understood simple systems, are so complicated as a whole as to render complete mathematical analysis difficult or impossible. The distribution of stress in a complicated machine part, or the flow of water in an irregularly shaped vessel, would constitute examples of the latter kind.

When something must be known about one of these more complicated problems it is usual, whenever possible, to obtain the desired information empirically by direct experimentation. Often, however, the thing

studied is too large to be experimented with. Or, as in the case of large engineering structures, the information on a bridge, dam, or building is needed in advance of designing the structure.

Under these conditions, where mathematical analysis is inadequate, and where for one reason or another direct experimentation is precluded, the best remaining alternative is to construct and study a scale model. This in fact is what is being done successfully in aerodynamical, hydraulic, and mechanical and electrical engineering. Aerodynamical studies are being made on scale models of aeroplanes in wind tunnels. In modern hydraulic laboratories such as that of the United States Army Engineering Corps at Vicksburg, Mississippi, harbors, canals, rivers, ground water flow, and other similar problems are being studied by means of models. Civil engineers are studying models of buildings, bridges, and similar structures. Mechanical engineers are making transparent models of machine parts and studying their stress distributions by the strain figures revealed by crossed nicol prisms and polarized light. Electrical engineers are studying both electrical models of electrical systems, and what may seem surprising, electrical models of mechanical systems.

The geological problems of mountain making and of diastrophism in general are peculiarly of the type that do not lend themselves readily to analysis, and the size of the elements involved places them beyond the range of direct experimentation. In this case also there remains the alternative of studying such phenomena by means of experiments performed upon properly built small scale models. For more than a hundred years attempts have been made to study the mechanics of the rock deformation in mountain making by this means. Among the earliest of such experiments were those of Sir James Hall [1] who studied the formation of folds by the use of models, employing layers of cloth in some instances, and of clay in others, to represent strata. A. Daubree [2] was a somewhat later pioneer. He performed a wide range of experiments studying both fracture and folding. For materials, he used glass, plaster, wax, and strips of metal. Later came Willis' [3] well-known experiments on the mechanics of the Appalachian type of structure. Willis used a pressure box allowing for a model about a meter in length. The materials used in his various models were composed of layers of plaster of paris, and of waxes of various consistencies, weighted from above by a heavy load of shot.

Subsequently, numerous others have performed model experiments to elucidate various geological problems. A few of the more recent ex-

[1] W. Paulcke: *Das Experiment in Geologie* (1912) Karlsruhe.

[2] A. Daubree: *Etudes synthétiques de géologie expérimentale*, pt. 1 (1879) Paris.

[3] Bailey Willis: *The mechanics of Appalachian structure*, U. S. Geol. Surv., 13th Ann. Rept., pt. 2 (1891-1892) p. 211-289.

perimenters have been Chamberlin and Miller,[4] Mead,[5] Link,[6] Hans Cloos,[7] Fujiwhara, Tsujimura and Kusamitsu,[8] Ph. Kuenen,[9] and Nettleton.[10] Most of these experimenters employed materials ranging in strength from plaster to rather soft waxes and clay. Cloos, Kuenen, and Nettleton differed from the others by using materials of extreme weakness. In his experiments on tectonic structures, Cloos used an almost liquid clay. Kuenen used china clay, vaseline, mineral oil, and paraffin at a temperature only slightly below that of melting. He tried gypsum but discarded it as being too strong. Nettleton, in experiments representing salt domes, employed two viscous liquids of different densities.

Much excellent experimentation has also been done by Adams [11] and associates, von Kármán,[12] Bridgman,[13] Griggs,[14] and others, on the elastic and plastic properties of rocks. This kind of work provides some of the necessary data for model experimentation but it is not to be classed as model experimentation itself.

Ever since the days of John Hutton [15] there have been some geologists who have maintained a legitimate skepticism regarding the significance of the results obtained by means of models of plaster and clay, when applied to mountain structures where the material is hard rock. Besides the discrepancy in the strengths of the materials in the two cases, there has also been the disparity in the time required, the model being de-

4 R. T. Chamberlin and W. Z. Miller: *Low-angle faulting*, Jour. Geol., vol. 26 (1918) p. 1-44.

5 Warren J. Mead: *Notes on the mechanics of geologic structures*, Jour. Geol., vol. 28 (1920) p. 505-523.

6 Theodore A. Link: *Experiments relating to salt dome structures*, Am. Assoc. Petrol. Geol., Bull., vol. 14 (1930) p. 483-508.

7 Hans Cloos: *Kunstliche Gebirge*, Nat. u. Mus. Senckenbergische Naturforschende Gesellschaft, Frankfort, pt. 1 (1929) p. 225-243; pt. 2 (1930) p. 258-269. *See also Einfuhrung in die Geologie* (1936) Berlin.

8 S. Fujiwhara T. Tsujimura, and S. Kusamitsu: *On the earth-vortex, échelon faults and allied phenomena*, Gerlands Beiträge zur Geophysik, Suppl. vol. 2 (1933) p. 301-360.

9 Ph. H. Kuenen: *Negative isostatic anomalies in the East Indies (with experiments)*, Leidsche Geologische Mededeelingen, Deel 8, Aflevering 2 (1936) p. 169-214.

B. G. Escher and Ph. H. Kuenen: *Experiments in connection with salt domes*, Leidsche Geologische Mededeelingen, Deel 3, Aflevering 3 (II 1929) p. 151-182.

10 L. L. Nettleton: Fluid mechanics of salt domes, Am. Assoc. Petrol. Geol., Bull. vol. 18 (1934) p. 1175-1204.

11 F. D. Adams and J. T. Nicholson: *An experimental investigation into the flow of marble*, Royal Soc. London, Philos. Tr., ser. A, vol. 195 (1901) p. 363-401.

F. D. Adams: *An experimental investigation into the action of differential pressure on certain minerals and rocks, employing the process suggested by Professor Kick*, Jour. Geol., vol. 18 (1910) p. 489-525.

12 Th. von Kármán: *Festigkeitversuche unter allseitigen druck*, Zeitschr. des Vereins deutscher Ingeniere, vol. 55 (1911) p. 1749-1757.

13 P. W. Bridgman: *The physics of high pressure* (1931) (New York); *Shearing phenomena at high pressure of possible importance to geology*, Jour. Geol., vol. 44 (1936) p. 653-669.

14 David T. Griggs: *Deformation of rocks under high confining pressures*, Jour. Geol., vol. 44 (1936) p. 541-577.

15 *See* Sir Archibald Geikie: *The founders of geology* (1897) London. Also W. Paulcke: *Das experiment in Geologie* (1912) Karlsruhe.

formed in a few hours by an amount which in the mountain may have required a million years.

Still another reason for questioning the validity of model experiments is to be found in the artifices adopted by some of the experimenters. Willis, for example, found it necessary to load his models of Appalachian structures with an overburden of 3 to 5 pounds per square inch of lead shot to make them stay down properly. This, on the scale of his models, was equivalent to an overburden of something like 100 miles of sediments. Even then the more competent plaster layers broke into rigid slabs instead of folding plastically, as even the quartzites in the Appalachians themselves are known to have done.

In view of the diversity of the materials that have been used in model experiments, and of the wide range of methods employed by various experimenters, one wonders which, if any, of the results are trustworthy. Particularly there is the need for an objective criterion to enable one to determine what the correct properties of a model should be for the best similarity, when the properties of the original are known, or whether it is even possible to build a correct model from available materials.

Dating from the time of Galileo [16] there has been accumulating a body of knowledge of how a model is related to its original, or of how the physical properties of a body change with change of size. During the last 50 years this body of knowledge has been brought to a high state of advancement in the various branches of engineering, especially in connection with the work in modern hydro- and aerodynamic laboratory practice using scale models.[17] Recently Churchill [18] has worked out the theory for problems involving the conduction of heat.

It is the purpose of the present paper to derive the general theory of the similarity between a model and its original for purely mechanical systems. This theory will then be applied to a number of illustrative geological problems and a set of model ratios between properties of the model and the corresponding properties in the original will be determined. This will enable us, if we know given properties of the original, to determine what the corresponding properties of the model should be. The knowledge thus acquired may be used either as a guide in actually performing experiments or as a criterion for evaluating experiments which have already been performed. In many cases it will allow one

16 Galileo Galilei: *Dialogues concerning two new sciences* [translation by Henry Crew and Alfonso de Salvio] (1914) p. 109-152. New York.

17 Francesco Marzola: *Some considerations regarding hydraulic models*, Hydraulic Laboratory Practice, 1926-1929, A.S.M.E. (1929) appendix 13, p. 743-758.

Alton C. Chick: *Dimensional analysis and principle of similitude*, Hydraulic Laboratory Practice 1926-1929, A.S.M.E. (1929) appendix 13, p. 782-827.

L. Prandtl and O. G. Tietjens: *Applied hydro- and aeromechanics* (1934) Chap. 2. New York.

18 R. V. Churchill: *Comparison of the temperatures in a solid and its scaled model*, Physics, vol. 6 (1935) p. 100.

to bring a problem of which he has little sensory experience down to a scale of things with which he is already familiar, and thereby enable him to have some intuitive understanding of it, even if no experiment is performed.

It is possible to extend dimensional analysis (which is what we shall be using) to include thermodynamic relations accompanying mechanical changes. To attempt to do this here, however, would complicate our problem more than the results to be obtained seem to warrant. Consequently we shall confine our attention chiefly to the mechanical aspects of the problems considered.

ACKNOWLEDGMENTS

The present paper is in a sense the integration of many of the writer's past experiences, by no means the least important of which have been conversations with friends. Among those friends who perhaps unknowingly have contributed most along these lines have been J Harlen Bretz, J. T. Stark, Waldo W. Waring, H. W. Straley III, Robert Thomson, Robert E. Bates, Girard E. Wheeler, Norman H. Dolloff, and Howard Scott.

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THEORY OF MECHANICAL SCALE MODELS

FAMILIAR EXAMPLES

Before going into the intricacies of the theory of models it might be pointed out that there are a large number of familiar, everyday occurrences that can be shown to be the direct consequence of the dimensional relations we are to derive and use.

1. Of two animals more or less geometrically similar but of different sizes, the smaller can fall without injury through a greater height, measured in multiples of its own length, than the larger. A mouse can fall without injury a distance many times his own length, a dog fewer, and a horse fewer still.

2. Of two animals of different sizes but similar shapes, the bodily motions, (leg motions, voice frequency, heart beat, etc.) of the smaller have the higher frequency.

3. A windmill of small radius will turn faster than one of large radius in the same wind.

4. Given any two machines having the same rate of output of the same product, the one whose parts have the higher frequency will, in general, be the smaller. For example, a high speed shaft need be only a fraction the size of a low speed shaft for identical transmission of power. This

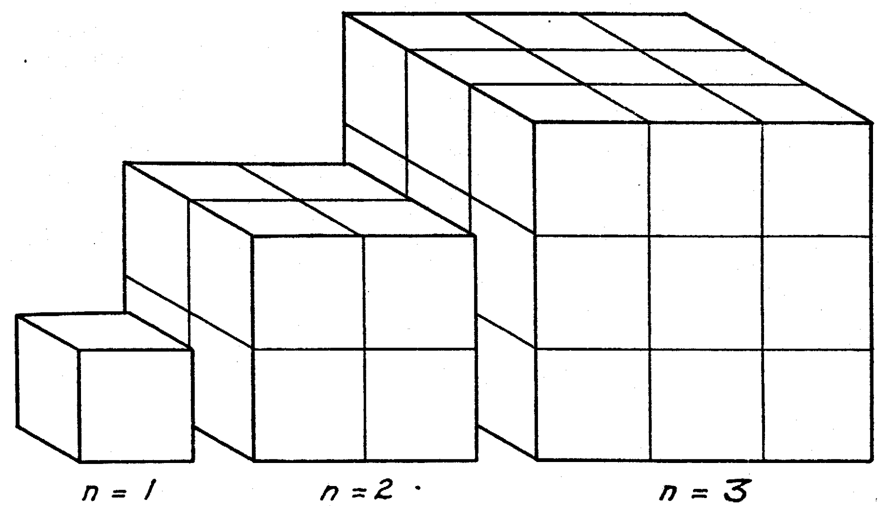


FIGURE 1.—*Change of magnitudes of a cube as the length of side is changed*

is one of the most important facts confronting the mineral industries today.

5. In electrical equipment, for the same power, the higher the frequency the smaller the transformers, generators, condensers, and motors. Short wave radio parts are very much smaller than those used for broadcast frequencies, which are smaller than audio-frequency parts, and all are smaller than 60-cycle power equipment of the same power.

Although the reasons for the above relations may not be obvious, they serve to demonstrate the fact that when the size of a body is changed its various other physical properties also change, but, in general, not proportionally to the change in size.

It shall be our immediate task to investigate how various physical properties of the body change as its size is changed. Consider a cube of lead, for example. Now let us enlarge it n diameters without changing its density. Let n be successively 1, 2, 3, 4, 5, and so on. Let us then investigate the manner in which the area, the volume, the mass, the weight, and the pressure at the base of the cube vary for the different values of n.

When n is 1, we will consider each of these quantities as unity—unit length of side, unit mass, unit area, unit volume, unit weight, and unit pressure. Now let n be 2. The length of side will be 2, the area 4, the volume 8, the mass 8, the weight 8, and the pressure, which is the weight divided by the area of the base, 2.

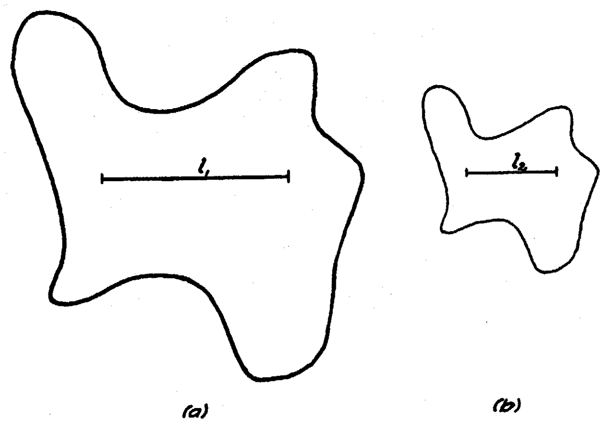


FIGURE 2.—*Two geometrically similar bodies*

Let n be 3. The side will be 3, the area 9, the volume, mass, and weight 27, and the pressure 3.

This is shown in Table 1 for values of n up to 5.

TABLE 1.—*Changes in physical properties with change in size of body*

n	Length	Area	Volume	Mass	Weight	Pressure
1	1	1	1	1	1	1
2	2	4	8	8	8	2
3	3	9	27	27	27	3
4	4	16	64	64	64	4
5	5	25	125	125	125	5

From this it is clear that the area increases as the square, the volume, mass, and weight as the cube, and the pressure at the base as the first power of the enlargement factor of the linear dimension of a cube, gravity and the density remaining constant.

DEGREES OF SIMILARITY

Geometrical Similarity.—Two bodies are said to be geometrically similar when all corresponding lengths are proportional and all corresponding angles of the two bodies are equal. This is simply a generalization

of the familiar definition of similar triangles from geometry, extended to include bodies of any shape.

If we let l_1 be a length in one of the bodies, and l_2 the corresponding length in the other, then

$$\frac{l_2}{l_1} = \lambda, \text{ or } l_2 = \lambda l_1, \tag{1}$$

where λ is the *constant of proportionality* of length of the two bodies.

If the first body is thought of as the *original* and the second as a replica of it made to scale, the latter is said to be a *scale model* of the former. The model may be either a reduction or an enlargement of the original; if a reduction, λ will be less than unity, if an enlargement, λ will be greater than unity. λ is the model ratio of length. It will be observed that λ is the familiar map scale extended to three dimensions.

In two geometrically similar bodies the ratio of corresponding areas is equal to the square of the model ratio of length.

$$\frac{A_2}{A_1} = \lambda^2, \tag{2}$$

where A_2 and A_1 are the corresponding areas of the second and first respectively. This can be seen at once if the two areas are divided into similar grid-works of n squares each. Then for each square

$$\frac{l_2^2}{l_1^2} = \lambda^2,$$

and for n such squares

$$\frac{A_2}{A_1} = \frac{nl_2^2}{nl_1^2} = \lambda^2.$$

In two geometrically similar bodies the ratio of corresponding volumes is

$$\frac{V_2}{V_1} = \frac{nl_2^3}{nl_1^3} = \lambda^3. \tag{3}$$

Kinematic Similarity.—If two geometrically similar bodies undergo geometrically similar changes of shape or positions, or both, the two bodies are said to be kinematically similar provided the time required for any given change in the one is proportional to that required for the corresponding change in the other.

$$\frac{t_2}{t_1} = \tau, \tag{4}$$

where t_1 is the time required for the original to undergo a given transformation, t_2 the corresponding time required for the model, and τ the *model ratio of time.*

Similarity of Velocities and Accelerations.—If two bodies are kinematically similar the velocities and accelerations of corresponding points must be proportional:

$$\frac{v_2}{v_1} = \eta = \frac{\frac{l_2}{t_2}}{\frac{l_1}{t_1}} = \frac{\lambda}{\tau} = \lambda\tau^{-1}, \tag{5}$$

where v_1 and v_2 are the velocities of corresponding points and η the *model ratio of corresponding velocities.*

For accelerations:

$$\frac{a_2}{a_1} = \gamma = \frac{\frac{l_2}{t_2^2}}{\frac{l_1}{t_1^2}} = \lambda\tau^{-2}, \tag{6}$$

where γ is the *model ratio of acceleration.*

For angular velocities:

$$\frac{\omega_2}{\omega_1} = \frac{\frac{\theta_2}{t_2}}{\frac{\theta_1}{t_1}} = \frac{\theta_2}{\theta_1}\cdot\frac{t_1}{t_2}, \tag{7}$$

where ω_1 and ω_2 are the angular velocities, and θ_1 and θ_2 the angles of rotation in the times t_1 and t_2 in the original and the model respectively.

But

$$\theta_1 = \theta_2.$$

Therefore

$$\frac{\omega_2}{\omega_1} = \frac{1}{\tau} = \tau^{-1}. \tag{8}$$

Dynamic Similarity.—In the discussion of geometrical and kinematic similarity we have described the relations between the forms and motions of bodies without regard to the fact that they possess mass. As all bodies possess mass, however, the presence of a gravitational field and the inertial reactions of mass to accelerated motion set up forces which have to be reckoned with.

To begin with, we require that our model have a mass distribution similar to that of the original. What we mean by this is that if dm_1 be the mass of an element of volume, dV_1, in the original, and dm_2 that of the corresponding element of volume, dV_2, of the model, then the ratio

$$\frac{dm_2}{dm_1} = \mu, \tag{9}$$

must hold point by point throughout the two bodies, μ being the *model ratio of mass.*

From the model ratios of mass and length, the ratio of density is obtained directly. Density is defined as the mass divided by the volume. Hence the model ratio of density δ is given by

$$\delta = \frac{\rho_2}{\rho_1} = \frac{\frac{dm_2}{dV_2}}{\frac{dm_1}{dV_1}} = \mu\lambda^{-3} \tag{10}$$

and is a constant throughout the two bodies.

The forces acting upon any element of mass, dm, occupying an element of volume, dV, may be divided into two separate classes, the *body forces* and the *surface forces.* The body forces, as the name implies, are forces originating inside the body of the volume. In purely mechanical systems the body forces are of two kinds, those due to gravity and those due to inertia. If we call these respectively f_g and f_i, we then have

$$f_g = dm \cdot g \tag{11}$$

and by Newton's Second Law of Motion

$$f_i = dm \cdot a \tag{12}$$

where g and a are the forces per unit of mass due to gravity, and to the local acceleration of the mass, respectively.

The surface forces are forces acting only on the external surface of the element of volume considered. They are consequently proportional to the magnitude of the area acted upon. The intensity of surface forces is measured by the ratio of the force to the area acted upon, and is known as the *stress.*

Hence,

$$\text{Stress} = \frac{\text{Force}}{\text{Area}},$$

or is the force acting per unit of area. One of the most familiar examples of stress is *pressure,* where the force is directed toward the area acted upon and normal to it.

Pressure is, however, only a special case of stress. Another special case is *tension,* where the force is directed away from the surface acted upon, and normal to it. The most general type of stress, however, is that in which the force is oblique to the surface. In this case it may be resolved into a component normal to, and another parallel to, the surface acted upon. The former per unit of area is known as the *normal component of stress,* and the latter as the *tangential* or *shearing component.*

The resultant surface force acting over any volume V is the sum obtained by adding vectorially the products of the stresses by the areas acted upon over the entire surface.

Surface forces originate in several ways. In the case of fluids at rest, they consist entirely of pressures. The shearing component is non-existent. In the case of elastic solids, surface forces result from both normal and shearing components of stress related to the elastic strain. In the case of fluids and plastic bodies in motion the surface forces result from, or are related to, the volume change and the time rate at which the body undergoes deformation. Those resulting from the latter cause may be thought of as the *resistive* forces.

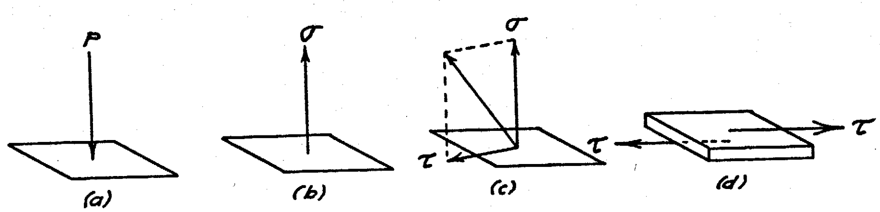


FIGURE 3.—*Types of stress*

(a) Pressure; (b) tensile stress; (c) general stress, showing normal and tangential components; (d) shearing stress.

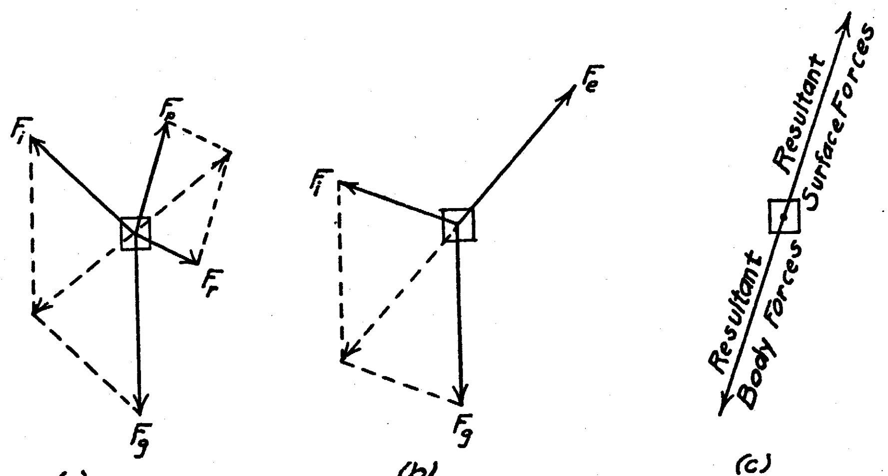


FIGURE 4.—*Equilibrium of forces acting upon an element of mass*

(a) in viscous body, (b) in elastic body, (c) general equilibrium between resultant body forces and resultant surface forces.

We now have body forces due to gravity and to inertial resistance to acceleration and surface forces arising from pressure, from elastic strains of the body, and from viscous resistance to rate of change of form. In general, each element of mass is in a state of dynamic equilibrium among as many of these forces as are acting simultaneously, and we may say that the vector sum of the body forces acting is equal and opposite to the vector sum of the surface forces.

Let F_g, F_p, F_r, F_e, and F_i be respectively the resultant vectors of forces due to gravity, to pressure, to viscous resistance, to elasticity, and to inertia, acting upon an element of mass dm contained in volume dV. As these are all the forces acting, their vector sum must be zero, so we have for bodies undergoing fluid or plastic deformation

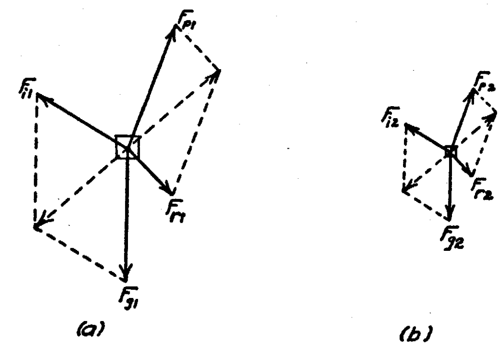


FIGURE 5.—*Similarity of forces acting on corresponding elements of mass in viscous bodies*

(a) Original, (b) model.

$$F_g + F_p + F_r = -F_i, \tag{13}$$

and for elastic solids

$$F_g + F_e = -F_i. \tag{14}$$

Dynamic similarity requires that on each element of mass dm_2 occupying volume dV_2 of the model the forces acting must be such that the motion is geometrically and kinematically similar to the corresponding motion of the element of mass dm_1 in volume dV_1 of the original. This condition is satisfied exactly if for each force F_1 acting on mass dm_1 in the original there is a corresponding vector force F_2 acting upon mass dm_2 in the model, F_2 having the same orientation as F_1, and the ratio of the magnitudes of the two forces being

$$\frac{F_2}{F_1} = \phi \tag{15}$$

where ϕ is the *model ratio of force* and is constant for all corresponding forces throughout the two bodies.

We have accordingly

$$\frac{F_{g2}}{F_{g1}} = \frac{F_{p2}}{F_{p1}} = \frac{F_{e2}}{F_{e1}} = \frac{F_{r2}}{F_{r1}} = \frac{F_{i2}}{F_{i1}} = \phi \tag{16}$$

for all corresponding forces.

We now have as the most general condition for similarity that *two massive bodies can be geometrically and kinematically similar only provided the masses of the one, point by point, are proportional to the corresponding masses of the other, and that the corresponding forces, point by point, have the same directions and proportional magnitudes.*

FORCES DUE TO VARIOUS CAUSES

Statement of Investigation.—It now remains for us to investigate somewhat more in detail the different species of force that may act upon an element of mass within a body, for, as we have seen, the ratio of each of these between the model and the original must be the constant ϕ.

Body Forces.—We begin with the body forces whose magnitudes by equations (11) and (12) are

$$F_g = dm \cdot g,$$

$$F_i = dm \cdot a,$$

and by (16) together with (11) and (12) we obtain

$$\frac{F_{g2}}{F_{g1}} = \frac{dm_2 g_2}{dm_1 g_1} = \phi = \frac{F_{i2}}{F_{i1}} = \frac{dm_2 a_2}{dm_1 a_1} \tag{17}$$

But since

$$\frac{dm_2}{dm_1} = \mu$$

and

$$\frac{a_2}{a_1} = \lambda\tau^{-2}$$

then

$$\phi = \frac{F_{i2}}{F_{i1}} = \mu\lambda\tau^{-2}. \tag{18}$$

By equations (16) we learned that all corresponding forces must have the same ratio ϕ, and now by (18) we learn that ϕ is uniquely determined by the values assigned to μ, λ, and τ, the fundamental model ratios of mass, length, and time, which, in the most general case, may be chosen independently and arbitrarily.

The general case is merely a statement of the fact that where both gravitative and inertial forces coexist, both must have the same model ratio ϕ. This fact, however, constitutes a certain duality of control in that the ratio of inertial forces must be made to fit the ratio of the gravitative forces, which lessens considerably the degree of arbitrariness that otherwise would exist. In particular, when both the original and the model are on the earth's surface

$$\frac{g_2}{g_1} = \gamma_g = 1 = \gamma = \lambda\tau^{-2}, \tag{19}$$

in which case

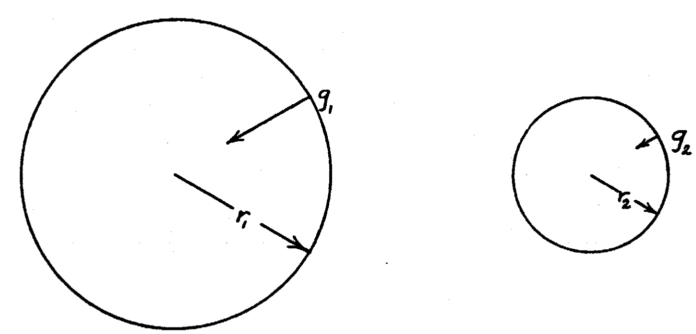
$$\tau = \lambda^{\frac{1}{2}}. \tag{20}$$

Consequently under such restraint λ and τ are no longer mutually independent, for when either is arbitrarily chosen the other is uniquely determined; only μ and λ, or μ and τ remain as completely independent model ratios.

In separate spherical astronomical bodies, however,

$$g = K \cdot \frac{m}{r^2}, \tag{21}$$

where g is the attractive force on a unit mass at any point outside the



(a) (b)

FIGURE 6.—*Similarity of gravitational forces*

body at distance r from its center, and K the universal constant of gravitation. Then if we had two such bodies, one a model of the other, we should have for the ratios,

$$\frac{g_2}{g_1} = \frac{K_2 \cdot \frac{m_2}{r_2^2}}{K_1 \cdot \frac{m_1}{r_1^2}} = \gamma = \lambda\tau^{-2},$$

or

$$\frac{K_2}{K_1} = \gamma\mu^{-1}\lambda^2 = \mu^{-1}\lambda^3\tau^{-2}. \tag{22}$$

While it is not physically possible to alter the constant of gravitation it is sometimes useful for theoretical purposes (as we shall see later) to be able to compute a model as if K could be varied. Equation (22) enables us to do this in terms of fundamental model ratios μ, λ, and τ.

Surface Forces.—We turn now to the surface forces which, as we have seen, must be such as to balance the body forces. If we take a small element of volume dV containing mass dm, the resultant surface forces over the element must be equal and opposite to the body forces acting on the mass dm. The surface forces are proportional to the stress and the area; the body forces are proportional to the mass, and hence to the volume. If l is a characteristic length of the element of volume, we have as the ratio of the body forces to the surface forces,

$$\frac{\text{Body Forces}}{\text{Surface Forces}} = k \cdot \frac{l^3}{l^2}, \tag{23}$$

which tends to the limit zero as l tends to zero. In other words the body forces diminish with size more rapidly than the surface forces, so that if the element of volume is small enough we may completely neglect the body forces for that element, and focus our attention upon its surface forces, which must then be in equilibrium among themselves. This leads us to the conception of the state of stress at a given point within the body. The criterion of dynamic similarity then tells us that at every corresponding point within the two bodies the states of stress must be similar at corresponding times.

State of Stress at a Point.—Let us take an infinitesimal element of volume with lengths of sides dx, dy, and dz, and small enough that the body forces are negligible compared with the surface forces. We then have as the condition of equilibrium that all the surface forces must be in equilibrium among themselves; that is, there must be no resultant force tending to accelerate the body, nor a torque tending to rotate it.

We resolve the stresses acting on each face into a normal component and two shearing components parallel to each of the three axes X, Y, and Z. We designate the normal component by the letter σ and we give it a subscript x, y, or z, designating that the face upon which it acts is perpendicular to the X, the Y, or the Z axis respectively. Each shearing stress component is designated by the letter τ (not to be confused with the model ratio of time) bearing two subscripts, the first

signifying the axis perpendicular to the face acted upon, and the second the axis to which the component is parallel.

For each pair of faces there is one face for which the outward directed normal (positive normal stress) is directed toward the positive end of the axis of coordinates to which it is parallel and one for which

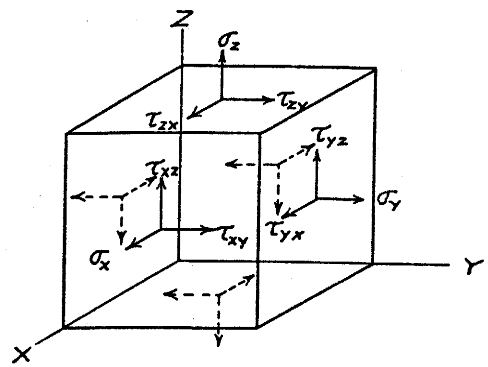


FIGURE 7.—*Stresses acting on an infinitesimal elementary cube*

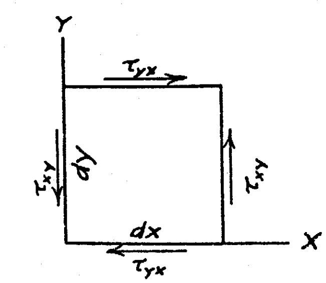


FIGURE 8.—*Equality of shearing stress couples*

Clockwise couple tending to produce rotation about the Z axis is equal and opposite to the counter-clockwise couple.

the normal is oppositely directed. We adopt the convention that on any face for which the outwardly directed normal points toward the positive end of the axis to which it is parallel, the two components of shearing stress are positive when their vectors point toward the positive ends of the axes to which they are parallel. On the opposite faces of each pair all stress directions become reversed. The normal stresses are positive when directed outward.

We then have on the three pairs of faces of the elementary parallelopiped the following components of stress:

$\sigma_x \quad \tau_{xy} \quad \tau_{xz}$ on the X face,
$\sigma_y \quad \tau_{yz} \quad \tau_{yx}$ on the Y face,
$\sigma_z \quad \tau_{zx} \quad \tau_{zy}$ on the Z face.

This number can be reduced when we consider that the turning moments about each of the axes must be zero. Then

$$\tau_{xy} = \tau_{yx};\ \tau_{xz} = \tau_{zx};\ \tau_{yz} = \tau_{zy}. \tag{24}$$

This leaves us with six fundamental components of stress:

$\sigma_x, \quad \sigma_y, \quad \sigma_z$ normal stresses,
$\tau_{xy}, \quad \tau_{yz}, \quad \tau_{zx}$ shearing stresses.

Principal Stresses.—While the proof of this will not be developed here, it can be shown that for any point in a body it is always possible to so orient the X, Y, and Z axes that all of the components of shearing stress vanish:

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0.$$

In such an instance there remain only the normal components of stress, σ_x, σ_y, and σ_z, which are then designated σ_1, σ_2, and σ_3 and are said to be the three *principal stresses* at the point in question. If the

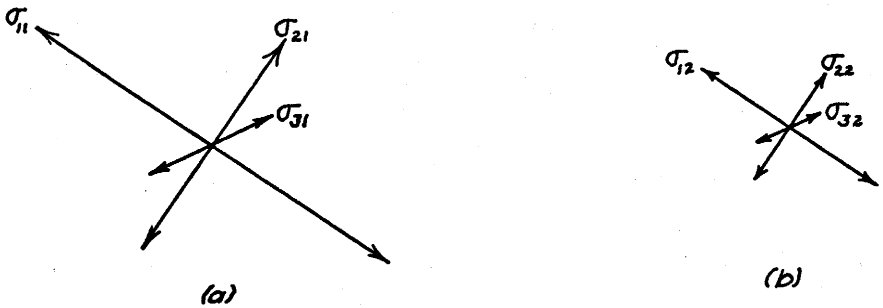


FIGURE 9.—*Similarity of principal stresses at corresponding points* (a) in original, (b) in model.

magnitudes and orientations of the three principal stresses at any point are known, the components of normal and shearing stress on any plane through that point can be computed.

In the most general case of stress in rigid bodies or in viscous fluids in motion the three principal stresses are unequal. For fluids at rest all three become equal and all shearing stresses in whatever directions become zero.

Strain.—In an elastic body the normal stresses produce elongations in the directions of the three axes, and the shearing stresses produce shearing deformation. All changes of size or of shape of a body, or both, are said to constitute *strains*. Strains are measured in terms of the ratio of the increase in a given geometrical dimension to the original value of that dimension. Thus:

$$\text{Volume strain} = \frac{\text{Increase of volume}}{\text{Original volume}}, \quad (25)$$

$$\text{Length strain} = \frac{\text{Increase of length}}{\text{Original length}}, \quad (26)$$

$$\text{Shear strain} = \frac{\text{Shear displacement}}{\text{Thickness}}. \quad (27)$$

Elasticity.—The relation between strain and stress in elastic bodies is given by Hook's Law which states that for a given material

$$\frac{\text{Stress}}{\text{Strain}} = \text{a constant.} \quad (28)$$

This constant is called the *modulus of elasticity* of the material.

As we have volume strain, length strain, and shear strain, associated with appropriate stresses, there must be as many different kinds of

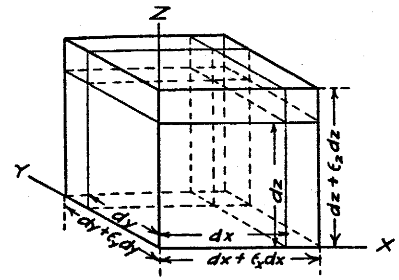


FIGURE 10.—*Elongations parallel to axes of coordinates*

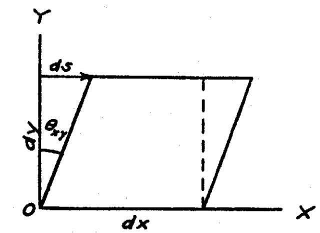


FIGURE 11.—*Shear parallel to X axis only*

moduli of elasticity, namely the *volume modulus*, *length* or *Young's modulus*, and *shear* or *rigidity modulus*.

In the case of our elementary volume we may designate the length strains parallel to the three axes respectively as ε_x, ε_y, and ε_z where ε is the amount of elongation per unit of original length.

Shear strain, from definition (27), involves a distortion of the kind obtained by shearing a deck of cards, one over the other. This involves a change of the angle at the end of the deck by an amount θ which may be taken as a measure of the strain. More strictly the shear is equal to the tangent of θ but when θ is small this becomes equal to the radian measure of θ itself. Elastic strains seldom are great enough that one need consider the tangent of θ as being different from θ itself.

Now we relate the stresses acting upon our elementary volume to the strains produced. (See Fig. 12.) ε_x, ε_y, and ε_z are the elongations; θ_{xy}, θ_{yz}, and θ_{zx} are the shears parallel to the X-Y, the Y-Z and the Z-X planes respectively. We then have

$$\frac{\sigma_x}{\epsilon_x} = \frac{\sigma_y}{\epsilon_y} = \frac{\sigma_z}{\epsilon_z} = E, \quad (29)$$

where E is the modulus of elongation, or Young's modulus; and

$$\frac{\tau_{xy}}{\theta_{xy}} = \frac{\tau_{yz}}{\theta_{yz}} = \frac{\tau_{zx}}{\theta_{zx}} = G, \quad (30)$$

where G is the modulus of rigidity.

From these it is possible to derive also the volume modulus though there is no necessity of our doing so for the present purposes.

What we are leading up to is the question of what must be done to the moduli of elasticity in order that the model may be dynamically similar to the original. We can now write that for similarity the ratio

$$\frac{E_2}{E_1} = \frac{\frac{\sigma_2}{\epsilon_2}}{\frac{\sigma_1}{\epsilon_1}} = \frac{\sigma_2}{\sigma_1} \cdot \frac{\epsilon_1}{\epsilon_2}, \quad (31)$$

and that

$$\frac{G_2}{G_1} = \frac{\frac{\tau_2}{\theta_2}}{\frac{\tau_1}{\theta_1}} = \frac{\tau_2}{\tau_1} \cdot \frac{\theta_1}{\theta_2}, \quad (32)$$

and more generally that

$$\frac{(\text{Modulus})_2}{(\text{Modulus})_1} = \frac{\frac{(\text{Stress})_2}{(\text{Strain})_2}}{\frac{(\text{Stress})_1}{(\text{Strain})_1}} = \frac{(\text{Stress})_2}{(\text{Stress})_1} \cdot \frac{(\text{Strain})_1}{(\text{Strain})_2} \quad (33)[19]$$

We have already seen that all strains of whatever kind are measured in homogeneous units of length/length, volume/volume, etc. As geometrical similarity requires all corresponding lengths to be proportional and corresponding angles to be equal in the model and the original, it follows that all strains are likewise equal. Hence the model ratio of strain is unity.

Consequently the model ratio of the elastic moduli between the model and the original is equal to that of stress. Each stress is the ratio of a force to an area. We have for the model ratio of stress, and consequently of the moduli of elasticity,

$$\frac{(\text{Modulus})_2}{(\text{Modulus})_1} = \frac{(\text{Stress})_2}{(\text{Stress})_1} = \sigma = \frac{\frac{f_2}{A_2}}{\frac{f_1}{A_1}} = \mu\lambda^{-1}\tau^{-2}, \quad (34)$$

where σ is now taken as the model ratio of stress and τ that of time, as originally.

Viscosity.—In the case of fluids we obtain the stresses at a point by identically the same reasoning as that employed for the case of elastic bodies and again the six components of stress are:

$$\sigma_x, \quad \sigma_y, \quad \sigma_z,$$
$$\tau_{xy}, \quad \tau_{yz}, \quad \tau_{zx}.$$

[19] Strictly speaking this is true only for isotropic bodies. For anistropic materials the moduli of elasticity are functions of the direction. This, however, is not incompatible with similarity provided that the model be given an anisotropy similar to that of the original.

Since our interest attaches only to changes of form in relatively incompressible materials we may consider the fluids to be incompressible. In this case we may disregard the normal components of stress, which tend to produce change of volume only, and focus our attention upon the shearing components.

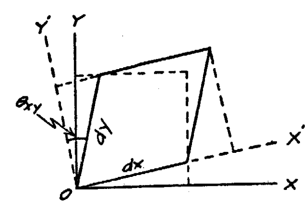


FIGURE 12.—*Shear parallel to both X and Y axes*

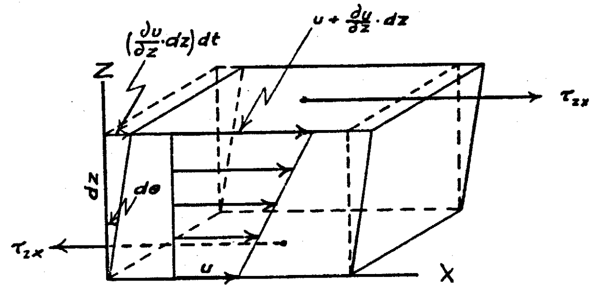


FIGURE 13.—*Relation between rate of shear, velocity gradient, and shearing stress parallel to X axis*

In its behavior under shearing stress a fluid differs from an elastic body in that whereas the shearing stress is proportional to the shearing strain in the latter, in the case of the former the shearing stress is proportional to the time rate at which the shearing strain occurs.

If we let $d\theta_{xy}$, $d\theta_{yz}$, and $d\theta_{zx}$ be the angles of the shear occurring in the fluid in planes parallel, respectively, to the *X-Y*, the *Y-Z*, and in the *Z-X* planes, in the time dt, then we have:

$$\left.\begin{aligned} \tau_{xy} &= \xi \frac{d\theta_{xy}}{dt} \\ \tau_{yz} &= \xi \frac{d\theta_{yz}}{dt} \\ \tau_{zx} &= \xi \frac{d\theta_{zx}}{dt} \end{aligned}\right\} \quad (35)$$

where ξ is the coefficient of viscosity of the fluid.

To determine the model ratio of the coefficients of viscosity, we solve (35) for ξ and take the ratio

$$\frac{\xi_2}{\xi_1} = \frac{\tau_2}{\tau_1} \cdot \frac{d\theta_1}{d\theta_2} \cdot \frac{dt_2}{dt_1} = \psi, \quad (36)$$

or

$$\psi = \sigma\tau = \mu\lambda^{-1}\tau^{-1}, \quad (37)$$

where ψ is the model ratio of viscosities and τ, when used as a model ratio, signifies that of time.

Reynold's Number.—In the case of fluid motion somewhat more light is thrown on the requirements for similarity if we focus our attention upon the ratio of the body force due to inertia acting upon the mass contained in an element of volume of the fluid, to the force due to viscosity acting upon the surface of that volume. In this case we choose our element of volume large enough that the body forces are no longer negligible in comparison with the surface forces.

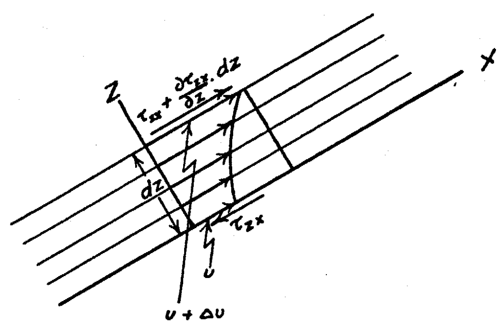


FIGURE 14.—*Shearing stress gradient and velocity gradient*

Gradient of shearing stress will produce a net force in the X direction.

The force due to inertia is equal to the mass multiplied by the acceleration and is directed opposite to that of the acceleration. The net force due to viscous resistance of the fluid results from the fact that the shearing stress is greater on one side of the element than on the other. This in turn is due to a progressive change with distance of the velocity gradient of the fluid flow. This net force due to viscous resistance has the same direction as the fluid velocity.

We orient our element of volume with its sides parallel to the flow lines of the fluid, and its ends perpendicular to these flow lines in such a manner that the maximum rate of fluid shear occurs between one pair of its faces.

We choose axes such that the maximum shearing stress is parallel to the X-axis and acts upon the plane perpendicular to the Z-axis.

Let τ_{zx} be the shearing stress on one face, and let this increase to $\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \cdot dz$ on the opposite face. The resultant force will be the difference between the products of these two stresses by the areas of the faces acted upon, and will be directed toward the positive end of the X-axis.

$$f_r = \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \cdot dz\right) dxdy - \tau_{zx} dxdy$$
$$= \frac{\partial \tau_{zx}}{\partial z} \cdot dxdydz = \frac{\partial \tau_{zx}}{\partial z} \cdot dV, \quad (38)$$

where f_r is the net force due to viscous resistance acting upon the volume dV.

By equation (35) we can express the shearing stress in a fluid in terms of the viscosity and the gradient of the velocity, since

$$\tau_{zx} = \xi \frac{d\theta_{zx}}{dt}.$$

$d\theta$ equals the ratio of the displacement in time dt of the upper layer over the lower, to the distance separating the two. Hence

$$d\theta_{zx} = \frac{dudt}{dz}, \quad or \quad \frac{d\theta_{zx}}{dt} = \frac{du}{dz},$$

where u is the velocity in the X direction. Consequently

$$\tau_{zx} = \xi \frac{du}{dz}. \quad (39)$$

By combining (39) and (38) we obtain for f_r:

$$f_r = \frac{\partial \tau_{zx}}{\partial z} \cdot dV = \xi \frac{\partial^2 u}{\partial z^2} \cdot dV. \quad (40)$$

The force due to inertia is

$$f_i = dm \cdot a = \rho \cdot dV \cdot a, \quad (41)$$

and from (40) and (41) we obtain the ratio

$$\frac{f_i}{f_r} = \frac{\rho a \cdot dV}{\xi \cdot \frac{\partial^2 u}{\partial z^2} \cdot dV} = \frac{\rho a}{\xi \cdot \frac{\partial^2 u}{\partial z^2}}$$

If we resolve the members of the right hand term into their constituent elements and arrange them in a slightly different manner we get

$$\frac{f_i}{f_r} = \frac{\rho a}{\xi \cdot \frac{\partial^2 u}{\partial z^2}} = \frac{\rho \cdot \frac{v}{t}}{\xi \cdot \frac{v}{l^2}} = \frac{lv}{\frac{\xi}{\rho}} = \frac{lv}{\nu} = R, \quad (42)$$

where l is any characteristic length, v a characteristic velocity, ν the ratio of the viscosity to the density, known as the *kinematic viscosity*, and R, the numerical value so determined, is a constant known as Reynold's Number.

Reynold's Number, which derives its name from Osborne Reynolds,[20] who in his pioneer experiments first appreciated its significance, is one of the most important quantities in fluid mechanics. If we have a fluid model dynamically similar to the original we get for the model ratio

$$\frac{R_2}{R_1} = \frac{\lambda\eta}{\frac{\psi}{\delta}} = \frac{\lambda^2\tau^{-1}}{\lambda^2\tau^{-1}} = 1. \tag{43}$$

Consequently, for a geometrically similar model of a fluid, the canons of dynamic similarity are entirely satisfied when the Reynold's Number for the model is the same as that for the original, the Reynold's Number being, of course, based upon corresponding velocities and lengths in the two cases.

The Reynold's Number, in any given case, increases with the velocity of the fluid, and experiment has shown that in each case at a certain critical value of Reynold's Number there is a transition from laminar to turbulent motion. This transition occurs in the model employing a totally different fluid from the original at identically the same value of Reynold's Number.

This criterion enables us to perform model experiments using quite different fluids in the model from that in the original. A model experiment of an airplane, for example, can be made using water for the fluid and still obtain results that are quantitatively correct for an original in air.

The smaller the Reynold's Number in any particular instance, the greater the stability of the fluid motion.

Strength and Plasticity.—So far we have spoken of the properties of elasticity and of viscosity. Elastic deformation is characterized by the fact that when a body is subjected to a stress it undergoes a finite strain proportional to the stress applied, and that when the stress is released the strain disappears and the body regains its original form. A body is said to be viscous, on the other hand, if when a shearing stress is applied the body deforms continuously at a rate proportional to the applied shearing stress, approaching zero as the stress approaches zero.

It has already been stated that the most general state of stress at a point inside a body is characterized by the three principal stresses, σ_1, σ_2, and σ_3, at that point. It can be shown that the maximum shearing stress occurs on a pair of conjugate planes parallel to the axis of intermediate stress, σ_2, and bisects the angles between the greatest and the least principal stresses, σ_1 and σ_3. The magnitude of the maximum shearing stress is equal to half the difference between the greatest and the least principal stresses.

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}.$$

When

$$\sigma_1 = \sigma_2 = \sigma_3,$$
$$\tau_{\max} = 0,$$

and all shearing stresses vanish.

When an elastic solid is subjected to unequal principal stresses it deforms elastically, but as the difference between the greatest and the least principal stresses is made continuously larger a state of shear strain is reached beyond which the body will not return to its original form upon release of the stress. The body is then said to have reached its *elastic limit*. If the stress continues to be further increased the body will fail by slip, by rupture, by brittle fracture, or else a *yield point* will be reached and the body will flow *plastically*.

The particular stress combination under which a given material fails by fracture or flows plastically is said to be its *strength*. Strength is not a constant but is a dependent variable; it is a function of the three principal stresses and of the temperature. The strength of a given material decreases as the temperature is increased, becoming zero in the vicinity of the melting point. As the intermediate principal stress, σ_2, is increased the strength of a material as measured by the difference between the greatest and the least principal stresses, σ_1 and σ_3, at which fracture or plastic deformation occurs, increases.

The most common tests made in testing laboratories to determine the strengths of materials are the application of simple tension, longitudinal compression, and shear stress. These give the tensile strength, the crushing strength, and the shear strength, respectively. By a variety of more elaborate experiments Adams and associates, von Kármán, Bridgman, Griggs, and others have measured the strengths of rocks under high confining pressures. The strengths of rocks have been increased by a factor of 10 times or more in this manner.

It is thus clear that the strength of a material is neither a simple nor a well-defined property. For our purposes it is sufficient, however, that strength is measured in terms of stress. If in the original, failure occurs

[20] Osborne Reynolds: *An experimental investigation of the circumstances whether the motion of water will be direct or sinuous, and of the law of resistance in parallel channels,* Royal Soc. London, Philos. Tr. (1883) *or* Sci. Pap., vol. 2, p. 51.

in a given manner at a certain stress, then in the model, for similarity, failure must occur in the same manner at the corresponding value of the stress. Hence the model ratio of strength must be the same as that of stress.

$$\frac{(\text{strength})_2}{(\text{strength})_1} = \frac{(\text{stress})_2}{(\text{stress})_1} = \sigma = \mu\lambda^{-1}\tau^{-2}. \tag{44}$$

Many substances, the malleable metals for example, deform plastically at ordinary temperatures and zero intermediate stress. Most rocks, however, fail under similar circumstances by brittle fracture. The above-mentioned experiments with rocks under high confining stresses have all shown that rocks, brittle under ordinary conditions, become plastic under high enough confining stress.

The most important experiments of this kind are those recently reported by Bridgman.[21] A bar of hardened steel called the "anvil" was placed between two opposed hardened steel pistons. Powdered specimens of materials to be tested were placed between the pistons and the anvil. The pressure on the pistons was increased and the anvil was periodically rotated.

For low pressures of the pistons the specimen would slip at the surface of contact and behave as a rigid body. As the pressure on the pistons was increased, however, a point would be reached at which the specimen would shear internally as the anvil was rotated. By knowing the area of the specimen and measuring the torque necessary to cause shear, the shearing strength of the material could be computed. Bridgman found that the shearing strength of the materials tested increased as the pressure on the pistons was increased. The maximum pressure obtained was 50,000 kg./cm.², which is equivalent to a depth of 166 km. beneath the earth's surface. At this highest pressure the shearing strength of most materials was increased by a factor of 10 times or more over its value under ordinary conditions.

The manner of failure or of flow is also important. Many substances, once they reached the plastic state, would flow continuously and could be made to shear an indefinite amount by continued rotation of the anvil. Other substances could never be made to flow continuously. In the latter, shearing stress would build up to a maximum and then be released by sudden jumping and crackling as the anvil was rotated.

On a few of the substances treated the shear stress increased about 10 times as the velocity of shear was increased 10,000 times. In most of the materials, however, there was no measurable change of the shearing strength as a function of the rate of shear.

[21] P. W. Bridgman: *Shearing phenomena at high pressure of possible importance to geology,* Jour. Geol., vol. 44 (1936) p. 653-669.

The fact that in plastic deformation the shear stress necessary to produce plastic shear is very nearly independent of the rate of shear differentiates this type of deformation from viscous shear, where the stress is proportional to the rate of shear. Plastic bodies undergo elastic deformation until, under the conditions obtaining, the shear stress equals the shear strength; then they shear plastically at constant values of the shear stress, which is essentially independent of whether the shear movement is slow or rapid.

This independence between the shear stress and the rate of shear in plastic deformation makes it possible to increase the speed of a plastic process without materially altering the magnitudes of the stresses involved. This enables us to distort the time factor in models of mountain making so that in the model an amount of deformation may be made in a few hours which otherwise should have required very much longer time, without any effective loss of similarity—a point that we shall return to later.

Elastico-viscosity.—Besides solid, fluid, and plastic bodies there are materials such as sealing wax which deform elastically to stresses of short duration and yet deform viscously to extremely minute shearing stresses of long duration. Maxwell[22] has called these *elastico-viscous* materials. Under large enough stresses these bodies fail like ordinary solids. Sealing wax fractures brittlely.

In such materials the coefficient of viscosity is quite high. The rate of deformation cannot be speeded up beyond a certain amount because when the stress reaches a certain critical value the substance fails by fracture. By analogy with other substances it is highly probable that under suitable stress an elastico-viscous material can be made to fail by plastic flowage.

There is much evidence to indicate that rocks are elastico-viscous materials of very high viscosity. The transition to the plastic state occurs under suitable stress differences and we then get more rapid deformations such as mountain making. Viscous movements are exemplified in the post-Glacial uplift of the Great Lakes region and of Scandinavia.

RELATION BETWEEN MODEL RATIOS AND PHYSICAL DIMENSIONS

Fundamental Units.—In mechanics, all quantities are measured by appropriate combinations of three fundamental units: *mass, length,* and *time.* Quantities other than these fundamental ones are said to be *derived quantities.* The *physical dimensions*[23] of a quantity are expressed by the number of times and manner in which the fundamental quanti-

[22] J. Clerk Maxwell: *On the dynamical theory of gases,* Philos. Mag., vol. 35 (1868) p. 134.
[23] P. W. Bridgman: *Dimensional analysis,* revised ed. (1931) Yale Univ. Press.

ties are involved in the measurement of that quantity. A *dimensional expression* is an expression, usually housed in brackets, of the symbols M, L, and T, representing mass, length, and time, respectively, where each symbol has an exponent signifying the number of times it occurs. A positive exponent signifies that the symbol belongs in the numerator; a negative exponent signifies the denominator; and a fraction signifies a root. For example

$$\begin{aligned} &[\text{Dimension of quantity}] = [M^a L^b T^c]; \\ &[\text{Velocity}] = [M^0 L T^{-1}] = [L T^{-1}]; \\ &[\text{Force}] = [M L T^{-2}]. \end{aligned} \quad (45)$$

A physical equation is valid only provided every term has the same dimensions as every other term. It is physically not permissible, for example, ever to do the equivalent of adding horses to apples—a procedure employed regularly and as a matter of course in economic computations.

It will be noted, and can be shown to be of general validity, that when an expression for the model ratio, in terms of μ, λ, and τ is of the form $\mu^a \lambda^b \tau^c$, the dimensional expression in terms of M, L, and T is of the form $M^a L^b T^c$. This important fact enables us to write by inspection the model ratios for all mechanical quantities from their dimensional formulas. This is done in Table 2 for the more commonly used quantities.

Alternative Fundamental Units.—Frequently it is convenient to express the model ratio in terms of ratios of density, velocity, viscosity, acceleration, or strength, taken as fundamental rather than in terms of those of length, mass, and time. Any three such ratios will serve equally well provided only that they contain the ratios of mass, length, and time.

Transition from one set of ratios, taken as fundamental, to any other may be made quite readily by setting up the appropriate equations of transformation. For example, suppose we wish to take the ratios of length, density, and acceleration as fundamental. For equations of transformation, we have

$$\begin{aligned} \lambda &= \lambda \\ \delta &= \mu\lambda^{-3} \\ \gamma &= \lambda\tau^{-2} \end{aligned} \qquad \text{or} \qquad \left.\begin{aligned} \lambda &= \lambda \\ \mu &= \delta\lambda^{3} \\ \tau &= \lambda^{\frac{1}{2}}\gamma^{-\frac{1}{2}} \end{aligned}\right\}. \quad (46)$$

By means of these equations any quantity may be transformed from either set of units to the other.

SPECIAL CASES

General Statement.—We have now covered the essential theory necessary to enable us to compute the properties of any kind of a mechanical model—rigid, plastic, or fluid—provided the properties of the original are known. Before we proceed to the computation of actual models, however, there are a number of aspects of the general theory to which attention paid now will forestall serious difficulties later on.

In setting up our basic criterion of dynamic similarity—that all forces of like kinds must be proportional—we chose the ratio of the forces due to inertia, $\mu\lambda\tau^{-2}$, as the standard to which all other forces have been made to conform. We encountered at once, however, a certain duality in the controlling factor of the force ratio owing to the fact that where both the model and the original are on the earth's surface, the same gravity, g, acts in both cases. This at once forces us to set the ratio of acceleration, $\lambda\tau^{-2}$, equal to unity, whereupon λ becomes equal to τ^2 for all such experiments. Consequently under such restrictions λ and τ are no longer independent ratios, but rather are so related to each other that the choice of a value for either uniquely determines that for the other.

Suppose, for example, that we wish to make a model of some earth phenomenon which required a million years. We wish to perform our

TABLE 2.—*Model ratios of mechanical quantities*

Quantity	Dimensional Formula	Model Ratio
Angle	L^0	1
Area	L^2	λ^2
Volume	L^3	λ^3
Curvature	L^{-1}	λ^{-1}
Frequency	T^{-1}	τ^{-1}
Velocity	LT^{-1}	$\lambda\tau^{-1}$
Acceleration	LT^{-2}	$\lambda\tau^{-2}$
Angular velocity	T^{-1}	τ^{-1}
Angular acceleration	T^{-2}	τ^{-2}
Density	ML^{-3}	$\mu\lambda^{-3}$
Momentum	MLT^{-1}	$\mu\lambda\tau^{-1}$
Moment of momentum	ML^2T^{-1}	$\mu\lambda^2\tau^{-1}$
Angular momentum	ML^2T^{-1}	$\mu\lambda^2\tau^{-1}$
Force	MLT^{-2}	$\mu\lambda\tau^{-2}$
Torque	ML^2T^{-2}	$\mu\lambda^2\tau^{-2}$
Work and energy	ML^2T^{-2}	$\mu\lambda^2\tau^{-2}$
Power	ML^2T^{-3}	$\mu\lambda^2\tau^{-3}$
Action	ML^2T^{-1}	$\mu\lambda^2\tau^{-1}$
Stress	$ML^{-1}T^{-2}$	$\mu\lambda^{-1}\tau^{-2}$
Strain	L^0	1
Elastic modulus	$ML^{-1}T^{-2}$	$\mu\lambda^{-1}\tau^{-2}$
Viscosity	$ML^{-1}T^{-1}$	$\mu\lambda^{-1}\tau^{-1}$
Kinematic viscosity	L^2T^{-1}	$\lambda^2\tau^{-1}$
Gravitational constant	$M^{-1}L^3T^{-2}$	$\mu^{-1}\lambda^3\tau^{-2}$

experiment within a few hours. The time ratio would be of the order of 10^{-9}.

In such an instance, λ, which is equal to τ^2, becomes 10^{-18}, and if the original were the earth itself, the model would have to be about the size of a molecule.

Suppose that on the other hand the original is of the order of 100 kilometers in length and we wish the model to be a convenient size for laboratory use. A value for λ of 10^{-5} would give a length of 100 centimeters for the model. Then τ, which is $\lambda^{1/2}$, would become $10^{-5/2}$. If the original time were 10^6 years, the corresponding time for the model would be $10^{3.5}$ or 3,160 years.

A somewhat similar difficulty is encountered in a problem such as this: It is known that a small flywheel made of a certain material will explode at a certain angular velocity. It is desired to know the corresponding velocity of a geometrically similar, larger flywheel made of the same material.

Our strict theory of similarity tells us nothing whatever about such a problem. It tells us instead that if we change the size of the wheel we also must change its strength, and hence its materials, or else the two cases will not be dynamically similar.

It is this objectionable inflexibility of the strict theory that we now must find a way to circumvent, for otherwise in models of geological phenomena either length or time will always be getting out of bounds. If we choose the time ratio to suit our experiment the length ratio becomes such as to render the model sub-microscopic in size; or if we choose the length ratio for convenience, the time ratio becomes such that the time required is much longer than that available for making the experiment.

A key to the solution of the difficulty is found when we return to first principles and scrutinize carefully the requirement for dynamic similarity—that all forces of a like kind must be proportional. We have already remarked that we have a duality of control in that there are two body forces, one due to inertia and the other due to gravity, both of which must simultaneously be satisfied. In general, both these forces do exist and, if they exist, both must be satisfied. The ratio of the forces due to inertia is $\mu\lambda\tau^{-2}$ and that due to gravity is $\mu \frac{g_2}{g_1}$ which on the earth's surface is equal to μ. Consequently, in such cases

$$\mu\lambda\tau^{-2} = \mu, \quad \text{or} \quad \lambda\tau^{-2} = 1,$$

as remarked before.

Inertial Forces Negligible.—Suppose, however, that in certain instances both forces do not exist simultaneously or, if they do, that one of them is negligible in magnitude as compared with the other. Consider for example a static structure in which the acceleration and hence the force due to inertia is zero; or a slowly moving viscous or plastic body in which the acceleration is so nearly zero as to produce forces of only infinitesimal magnitude. In such cases, while there is no incorrectness involved in basing the ratio ϕ of the forces upon the inertial ratio determined by $\mu\lambda\tau^{-2}$, dynamic similarity no longer depends upon our doing so. In this case the equilibrium of forces for elastic bodies becomes

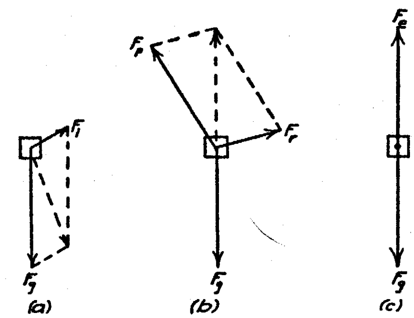


FIGURE 15.—*Special cases of force equilibrium*

(a) As force due to inertia approaches zero, the resultant body force approaches that due to gravity; force equilibrium without inertia in (b) fluid and (c) elastic body.

$$F_g + F_e = 0, \tag{47}$$

and

$$F_g + F_r + F_p = 0, \tag{48}$$

for viscous and plastic bodies.

Obviously in such cases the real controlling factor is not the body force due to inertia, for this is practically non-existent, but is instead that force due to gravity which on the earth's surface is quite independent of the model ratio of time employed in the particular experiment. Consequently, in cases where the acceleration is so small that forces due to inertia are negligible the canons of dynamic similarity are entirely satisfied if we choose the model ratios μ, λ, and τ arbitrarily and make all forces conform to the ratio

$$\phi = \mu\gamma_g, \tag{49}$$

where γ_g is the model ratio of the acceleration due to gravity and may, for the earth's surface, be set with negligible error equal to unity. Then

$$\phi = \mu = \delta\lambda^3. \tag{50}$$

The same result is achieved if we factor $\lambda\tau^{-2}$ out of all the general model ratios for the derived quantities, such as stress, strength, viscosity, and the like when and where needed.

For example:

Quantity	Strict model ratio	Model ratio neglecting inertia
Force	$\mu\lambda\tau^{-2}$	μ
Stress	$\mu\lambda^{-1}\tau^{-2}$	$\mu\lambda^{-2}$
Strength	$\mu\lambda^{-1}\tau^{-2}$	$\mu\lambda^{-2}$
Viscosity	$\mu\lambda^{-1}\tau^{-1}$	$\mu\lambda^{-2}\tau$

Since Reynold's Number was derived from the ratio of the inertial to the resistive forces it will be noted that as the inertial forces become negligible, Reynold's Number approaches zero. We have already seen that for the general case, dynamic similarity requires that the Reynold's Number for the model be equal to that for the original. This placed us under the severe restriction that

$$\frac{\lambda\eta}{\frac{\psi}{\delta}} = 1,$$

as a necessary requirement for similarity. In those cases where the inertial forces are negligibly small, we can now violate this restriction and obtain dynamic similarity when the Reynold's Numbers are no longer equal in the two cases, though quite small.

Gravitative Forces Negligible.—Another special case is the one exemplified by the flywheel in which the acceleration due to gravity is negligible compared to that of the motion of the body. In such an instance we may completely neglect the forces due to gravity and base the force ratio entirely upon the ratio of the forces due to inertia. This gives us throughout equations identical to those we have already derived from the strict theory with the exceptions that the relation

$$\lambda\tau^{-2} = 1,$$

which was formerly imposed by gravity, is now no longer required and can, when gravity is negligible, be completely ignored, λ and τ being taken as independent variables.

Resistive Forces Negligible.—We have still another special case when the forces due to resistance are so small in comparison with those due

to gravity, to inertia, and to pressure, that their effect is negligible. Such cases are to be found in the flow of fluids of small viscosity through large orifices, in the flow of water over a weir, and in the movement of boats at high velocities.

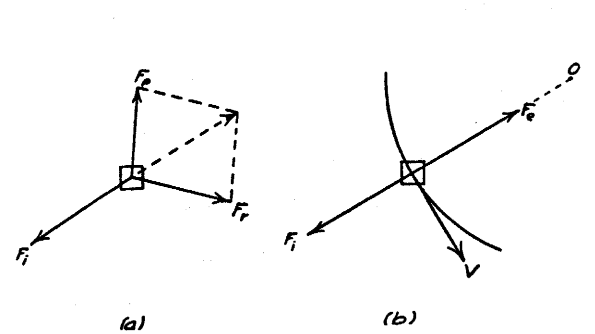


FIGURE 16.—*Inertial force large compared with gravitational force*
(a) fluid, (b) rotating solid.

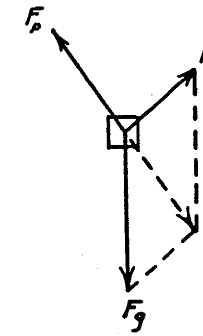


FIGURE 17.—*Force due to resistance negligible*

Whereas, for the general case, dynamic similarity could be achieved only when the proper viscosity ratio

$$\psi = \mu\lambda^{-1}\tau^{-1},$$

was satisfied, in this case the ratio of the viscosities may be ignored, or by using the same fluid in both cases, be made equal to unity.

Similarity Based on Resultant Body Forces.—An effective way of changing the force ratio of static structures is to base the model ratio of force on the vector sum of the forces due to accelerated motion and those due to gravity, without regard to the ratio between these two kinds of body force. This enables one to obtain similarity for the case in which the original is a static structure acted upon by gravity, and the model is a geometrically similar static structure being whirled in a centrifuge.

The model ratio of force then is

$$\phi = \frac{(\text{body forces})_2}{(\text{body forces})_1} = \frac{m_2(\vec{g} + \vec{a})}{m_1 g} \doteqdot \mu \cdot \frac{a}{g}.$$

The length ratio λ is arbitrary. The time ratio τ is not involved. The stress ratio is given by

$$\sigma = \phi\lambda^{-2} = \mu\frac{a}{g} \cdot \lambda^{-2} = \delta\lambda \cdot \frac{a}{g}.$$

In particular if the stress in the model is made equal to that in the original, the same materials can be used in both. Then

$$\sigma = 1,$$
$$\delta = 1,$$

and

$$a = \frac{g}{\lambda}.$$

This tells us that if the model is reduced by a length ratio of 1/1000, a must be made equal to $1000g$, if the same materials are used. This is the basis of Bucky's[24] experiments employing a centrifuge to test mine models.

Summary of Special Cases.—To summarize, we find that the mechanical properties of any model can be completely specified, when the properties of the original are known, in terms of the three fundamental model ratios μ, λ, and τ of mass, length, and time. We distinguish the general case and three special cases as follows:

1. General case.
   Independent fundamental ratios: μ, λ, and τ.

$$\phi = \frac{f_{i2}}{f_{i1}} = \frac{f_{g2}}{f_{g1}} = \mu\gamma = \mu\gamma_g.$$

Therefore,

$$\gamma_g = \gamma = \lambda\tau^{-2};$$
$$\phi = \mu\lambda\tau^{-2}.$$

2. Special cases.
   A. *Both the model and the original are at the earth's surface.*
   Fundamental model ratios: μ, λ, and τ.

$$\frac{g_2}{g_1} = \gamma_g = 1 = \lambda\tau^{-2}$$

Therefore,

$$\tau = \lambda^{\frac{1}{2}}.$$

Hence, the independent model ratios reduce to two: μ and λ, or μ and τ.

   B. *The forces due to inertia are negligible compared with those due to gravity; Reynold's Number is small.*
   Fundamental model ratios: μ, λ, and τ.

$$\phi = \mu\gamma_g;$$
$$\lambda\tau^{-2} \neq \gamma_g;$$
$$\gamma_g = 1 \text{ at earth's surface};$$
$$\lambda \text{ independent of } \tau.$$

[24] P. B. Bucky: *The use of models for the study of mining problems*, Am. Inst. Min. Metall. Eng. Tech. Pub. 425 (1931).

   C. *Force due to gravity is negligible compared with that due to inertia, or else is eliminated from consideration by the arrangement of the experiment.*
   Fundamental model ratios: μ, λ, and τ.

$$\lambda \text{ is independent of } \tau;$$
$$\phi = \mu\lambda\tau^{-2}.$$

   D. *Force due to viscous resistance negligible as compared with other forces.*
   Fundamental model ratios: μ, λ, and τ.

$$\tau = \lambda^{\frac{1}{2}} \text{ at earth's surface};$$

ψ independent of μ, λ, and τ; may be equal to unity.

   E. *Similarity based upon model ratio of total body forces.*
   Fundamental model ratios, μ, λ, and ϕ.

$$\phi = \frac{a_2}{g_1}$$

when

$$a_1 = 0,$$
$$g_2 = g_1,$$

and

$$a_2 >> g_1.$$

ILLUSTRATIVE EXAMPLES

Introduction.—Before proceeding with geologic problems it perhaps would be useful to demonstrate the theory we have developed by applying it to a few simple illustrative examples.

Viscous Fluid.—Suppose we have a barrel of tar and a cannon ball. The cannon ball is placed upon the tar and sinks slowly to the bottom. Using this for our original, we wish to construct a dynamically similar model on a reduced scale. What must the properties of the model be?

We proceed first by means of the general theory. μ, λ, and τ are our fundamental model ratios. We derive the remaining ratios that are of interest to us:

Viscosity ratio,	$\psi = \mu\lambda^{-1}\tau^{-1}$;
Velocity ratio,	$\eta = \lambda\tau^{-1}$;
Density ratio,	$\delta = \mu\lambda^{-3}$.

We now make use of two special circumstances: that gravity is the same in the model and in the original, and that the motion is so slow that the forces of inertia may be neglected. This corresponds to Special Case B already discussed. μ, λ, and τ are independent, and ϕ is equal to μ. So long as the inertial forces may be neglected in the model, dynamic similarity is achieved when

$$\psi = \mu\lambda^{-2}\tau,$$
$$\eta = \lambda\tau^{-1},$$
$$\delta = \mu\lambda^{-3}.$$

If the density of the model is made equal to that of the original then δ becomes equal to unity, and

$$\mu = \lambda^3,$$

whence

$$\psi = \lambda\tau,$$
$$\eta = \lambda\tau^{-1}.$$

Thus for a given length scale λ we can vary the ratio of viscosities ψ widely and still obtain dynamic similarity. This enables us to adjust the time scale to our convenience.

Flywheel.—A small flywheel explodes at a certain critical angular velocity. At what velocity would a dynamically similar large flywheel explode?

If we solve this for the case where gravity is the same in the model and the original we choose model ratios μ and λ and determine τ by the fact that

$$\lambda\tau^{-2} = 1.$$

For the derived model ratios we obtain for:

Strength: $\sigma = \mu\lambda^{-1}\tau^{-2} = \mu\lambda^{-2};$

Angular velocity: $\dfrac{\omega_2}{\omega_1} = \tau^{-1} = \lambda^{-\frac{1}{2}}.$

That is, the angular velocity at which the wheel will explode changes with the size of the wheel as $1/\sqrt{\lambda}$, but by the strict theory this is true only when the strength of the material has been changed by a factor $\mu\lambda^{-2}$.

In this case, however, the forces due to gravity are negligible and so by neglecting gravity, the ratios λ and τ again become independent.

The model ratios then become

Strength: $\sigma = \mu\lambda^{-1}\tau^{-2};$

Angular velocity: $\dfrac{\omega_2}{\omega_1} = \tau^{-1}.$

Consequently, we may use the same materials in both wheels, giving for the strength and density ratios:

$$\sigma = \mu\lambda^{-1}\tau^{-2} = 1;$$
$$\delta = \mu\lambda^{-3} = 1;$$

from which

$$\lambda = \tau,$$

and the ratio of angular velocities becomes

$$\frac{\omega_2}{\omega_1} = \tau^{-1} = \frac{1}{\lambda}.$$

Static Structure.—For a static structure all the accelerations due to motion are zero. The model is completely determined by the model ratios μ and λ,

$$\phi = \mu,$$

and

$$\sigma = \mu\lambda^{-2},$$

determine the ratio of the forces, and of the stresses and the strengths of the materials.

If the ratio δ for density is employed instead of μ,

$$\delta = \mu\lambda^{-3},$$
$$\phi = \delta\lambda^3,$$
$$\sigma = \delta\lambda.$$

When δ is unity, corresponding to the fact that the density is the same in both the model and the original,

$$\sigma = \lambda.$$

This tells us that if a structure is increased, for example, 100 times in linear dimensions the strength of its materials must also be increased 100 times if it is to support a corresponding load. Since in actual structures the strengths of the materials rarely are increased by such an amount, the resulting weakness must be compensated by departure from geometrical similarity. For this reason, small animals have spindley legs and large animals have massive legs.

Flow of Water Over Weir.—The constants of a model of water flowing over a weir are to be determined.

Solving for the general case first, we choose μ, λ, and τ, the latter two being related by $\lambda\tau^{-2} = 1$. The derived model ratios that are of interest then become:

Velocity:	$\eta = \lambda\tau^{-1} = \lambda^{\frac{1}{2}}$,
Density:	$\delta = \mu\lambda^{-3}$,
Viscosity:	$\psi = \mu\lambda^{-1}\tau^{-1} = \mu\lambda^{-\frac{3}{2}}$,
Discharge:	$\dfrac{D_2}{D_1} = \lambda^3\tau^{-1} = \lambda^{\frac{5}{2}}$.

Since in this case it is impossible to have both δ and ψ set equal to unity simultaneously, this tells us that we can have the model dynamically similar to the original only provided a liquid of different kinematic viscosity from that of the original is used in the model.

We may avoid this restriction by noting that the frictional forces are negligible compared with the remaining forces (Special Case D). Then we can ignore the viscosity altogether and use water in the model as well as in the original. We obtain in this manner:

Velocity:	$\eta = \lambda\tau^{-1} = \lambda^{\frac{1}{2}}$,
Density:	$\delta = \mu\lambda^{-3} = 1$,
Discharge:	$\dfrac{D_2}{D_1} = \lambda^3\tau^{-1} = \lambda^{\frac{5}{2}}$,
Viscosity:	$\psi = 1$.

This tells us that for fluids of low viscosity flowing over geometrically similar weirs (or through orifices), the discharge increases as the length ratio raised to the power 5/2, where the *same* fluid is used in both instances.

MODELS OF GEOLOGIC STRUCTURES

PROCEDURE

General Statement.—In the light of the foregoing development we are now prepared to answer a question propounded earlier in this paper: What should the mechanical properties of the materials of a geological model be in order that the model may give results which are dynamically similar to the original? We shall derive the answer to this question for a number of cases of geological importance. By applying a similar procedure anyone can solve any other particular problem that may be of interest to himself.

Before proceeding with this, however, perhaps it should be made clear that in this paper it is only intended to show how to determine the necessary specifications for the geological models in order that they may behave in a manner similar to the original. In many cases it may be difficult or impossible to obtain materials having the properties specified. Fortunately the theory enables us to vary the specifications somewhat widely, so that with a variety of materials it will frequently

be possible to adjust the model ratios until an approximate fit is obtained. The present paper is only a guide to the experimenter to give him a means of knowing to what extent his model materials do satisfy the requirements of similarity. It does not purport to foresee or to solve the experimental difficulties to be encountered in actually building a model.

There is still another thing about geologic models that perhaps should be mentioned. That pertains to boundary conditions, and the manner in which the deformation is to be induced. The geologic model is likely to have boundaries that do not exist in the original. This may not be serious if these boundaries are remote from the region of the model being studied. The method of inducing deformation in the model necessarily involves assumptions regarding the stress relations in the earth, and the model accordingly will vary with what the experimenter assumes to be the case. Variation of these assumptions should then lead to variations of the experiment. These, however, are problems that the individual experimenter must solve for himself.

Granite Cube.—For purely illustrative purposes let us begin with the simplest possible kind of problem. Let us imagine a cube of flawless granite, 20 kilometers to the side, resting upon a plane, horizontal, rigid base. We wish to determine the properties of a dynamically similar model 20 centimeters to the side.

This is a static problem and hence a special case of the more general theory. The model ratios of interest to us are:

Fundamental ratios:	μ and λ
Acceleration:	$\gamma = 0$; $\gamma_g = 1$
Density:	$\delta = \mu\lambda^{-3}$
Force:	$\phi = \mu\gamma_g = \mu$
Stress and strength:	$\sigma = \phi\lambda^{-2} = \delta\lambda$

The ratios of length, density, and strength are the ones that concern us most, for they determine the size, the density, and the strength of the model materials. The length ratio λ is determined from the problem as given to be 10^{-5}. The density of granite is 2.7; that of convenient model materials would range between about 1.5 and 2.7 which would determine the value of δ to be in the range from 0.5 to 1.0. From these the model ratio of strength may be determined. If δ is 1.0, σ will be equal to λ, or 10^{-5}. If δ is 0.5, σ will be 5×10^{-6}.

For simplicity let us make the density of the model the same as that of granite. Then in this case δ will be unity and σ will be equal to λ, or to 10^{-5}. This means that the strength of the model materials will have to be 10^{-5} of the strength of granite. Taking the crushing strength of granite to be 2×10^9 dynes/cm² or 29,000 lbs./in² we find that the model material must be a brittle substance that will crush under a stress of 2×10^4 dynes/cm² or 0.29 lbs./in².

Since the pressure at the base of the model would be 5.3×10^4 dynes/cm², which is about 2.7 times as great as the crushing strength of the material, it follows that the model would be too weak to stand unsupported and would collapse under its own weight.

That this is not an absurd result may readily be seen when one computes the pressure at the base of the original cube of granite, which will also be found to be 2.7 times as great as its crushing strength so that the original would likewise collapse under its own weight. Therefore the strength we have chosen is the proper one if the model is to behave in a manner dynamically similar to the original.

Mountain Range.—We take up now a larger unit. We choose a mountain range for the original and again we wish to know the model ratios and essential model constants. This is the same kind of a special case as that discussed for the granite cube. Aside from occasional earthquake motions of small magnitude, the accelerations are sensibly zero and may be neglected. The model ratios are the same (algebraically) as in the case of the cube, and in particular the ratio for strength is

$$\sigma = \delta\lambda.$$

For the original, suppose the width is 200 kilometers, the density 3.0, and the strength of the materials 2×10^9 dynes/cm², or the strength of granite under surface conditions. For the model let the width be 1 meter and the density 1.5. This gives for λ a value of 5×10^{-6}, and for δ a value of 0.5. Then σ, which is $\delta\lambda$, becomes 2.5×10^{-6}. Consequently the strength of the model would have to be:

$$\text{Model strength} = \sigma \times (\text{Strength of original}) = 5 \times 10^3 \text{ dynes/cm}^2.$$

Since on such a scale rocks are known to deform principally by plastic flowage, the model material would need to be more plastic than brittle.

The above figure for strength is taken merely as an approximation and for illustrative purposes. Actually the strength of rocks is known to increase with increase of confining stress—that is, with depth in the earth. The strength also, however, decreases with increase of temperature, which also increases with depth. Consequently the actual strength of rocks at a given depth within the earth is unknown except by rather indirect considerations.

When one is actually making a model each part of it should have the strength so chosen as to give the model the same strength inhomogeneities as are known or assumed to exist in the original. Neglecting for present purposes the inhomogeneities it is significant that the value of

5.0×10^3 C. G. S. for the over-all strength of the model materials corresponding to the case given is the right order of magnitude. This corresponds to a material so weak that a cube of it (density, 1.5) larger than 3.3 centimeters to the side could not support its own weight. Butter at ice box temperature would be much too strong. Vaseline or very soft clay would be more nearly correct.

Post-Glacial Uplift.—A somewhat different kind of problem is that exemplified by the gradual uplift that has been occurring in the regions of the Great Lakes and of the Scandinavian Peninsula since the retreat of the last ice sheet. Nansen [25] and Daly [26] have presented data which appear to indicate that both in North America and in Europe this uplift has occurred in a manner similar to the uplift of the surface of a barrel of tar following the removal of a weight. If this be so then the movement would occur as true viscous flow in an elastico-viscous medium having very high viscosity.

For our purposes we shall make the assumption that the departure from sphericity in the areas affected by post-Glacial uplift is slight enough that they may be represented by a model having a plane surface. We shall also assume that the earth has uniform viscosity to great depth.

Our model will then consist of a large vessel of viscous material, the surface of which has been depressed and is later returning to equilibrium. If we knew the over-all viscosity of the earth we could, from the model ratio of viscosity, determine the proper viscosity for the model. If, however, we regard the earth viscosity as an unknown to be determined, the model theory should tell us how to compute the earth viscosity from that of our model. The results in either case will not be exact because of the simplifying assumptions made, but they should give us the correct order of magnitude.

We choose in this case the fundamental model ratios μ, λ, and τ of length, mass, and time. We note that we are dealing with a fluid type of motion having an extremely small Reynold's Number and that all accelerations are sensibly zero. Consequently the inertial forces are negligible as compared with the resistive forces. Under these circumstances μ, λ, and τ are to be regarded as mutually independent model ratios from which we write the desired derived ratios:

Acceleration:	$\gamma = 0; \gamma_g = 1,$
Force:	$\phi = \mu\gamma_g = \mu,$
Velocity:	$\eta = \lambda\tau^{-1},$
Stress:	$\sigma = \mu\lambda^{-2},$
Density:	$\delta = \mu\lambda^{-3},$
Viscosity:	$\psi = \sigma\tau = \mu\lambda^{-2}\tau = \delta\lambda\tau.$

[25] Fridtjof Nansen: *The earth's crust, its surface, forms, and isostatic adjustment* (1928) Oslo.

[26] R. A. Daly: *Our mobile earth* (1926) New York.

The ratios concerning us most are those of density, length, time, and viscosity, and we see that that of viscosity is determined when the other three are chosen.

The radii of the depressed regions both in North America and in Europe are of the order of 1,000 kilometers. In the model, to avoid boundary effects, it is necessary that the viscous medium be of much greater extent than just the depressed area. In order to do this and not have the model inconveniently large, suppose we give the depressed area a radius of 10 centimeters. This would give a value for λ of 10^{-7}.

The total time elapsed since the beginning of the ice retreat is unknown but is about 25,000 to 35,000 years. What is more important for the present purposes is, however, the time interval between successive amounts of uplift as shown by old shore lines. This appears to be known with some degree of accuracy in certain cases. The ice retreat itself required some 10,000 to 15,000 years.

To keep the model time within limits we may set τ equal to 10^{-7}. For this time ratio an amount of uplift requiring 10,000 years in the original would require slightly less than 9 hours in the model.

The density of the original may be taken as 3.0; that of the model materials will probably be near 1.0, giving a value of 0.33 for δ. From these ratios that of viscosity is obtained:

$$\psi = \delta\lambda\tau = 3.3 \times 10^{-15}$$

An experimental difficulty will be recognized at once when the amount of uplift is considered in comparison with horizontal distances involved. The total uplift is of the order of 250 meters as compared with a radial distance of about 1,000 kilometers. For a model with a 10-centimeter radius the depression, to scale, would be only 2.5×10^{-3} centimeters. This is a difficulty that would remain for any size of a model not too large to work with. Direct measurements of such small displacements would require probably some optical device such as an interferometer.

To avoid the difficulty inherent in small displacements, let us consider the possibility of introducing distortion by depressing the model by an amount equal to the correct depression multiplied by a factor β. So long as the amount of depression remains a small fraction of the radial distance the flow lines in the viscous material remain essentially unchanged. The velocity of flow, however, and also the rate of shear are directly proportional to the shearing stress and inversely to the viscosity. Since the viscosity in the model with distortion is unchanged, the shearing stress varies with the distortion.

Shearing stress in this case results from the fact that the model is not in static equilibrium. It is proportional to the displacement from the

equilibrium position. Consequently, if the amount the surface is depressed is increased by a factor β, with the radius of the depression kept constant, the shearing stress throughout the model will be increased by the factor β and the velocity of displacement will likewise be increased by the factor β.

Consequently, if the uplift in the undistorted model is a given amount in a certain time, the uplift in the model having distortion will be β times that amount. In particular, if in the undistorted model an uplift equal to any given fraction of the initial amount of the depression occurs in time t, an amount of uplift equal to the same fraction of its original depression will occur in the same time t in the model having distortion. In other words, the depressed surface approaches its equilibrium position as a negative exponential function of time, or

$$h = h_o e^{-at},$$

where h_o is the initial amount the surface was depressed, h the amount of depression still remaining at time t, e the base of natural logarithms, and a a constant.

When the distortion factor β is known, all that is necessary in order to obtain the correct displacement is to divide the observed displacement by β.

Once the model ratios are determined we may proceed in either of two directions, depending upon what is considered as known and what is unknown. Suppose that for the earth something is known of the amount of uplift and the time required, and it is desired to know the earth's coefficient of viscosity. Then, since

$$\psi = \frac{\xi_2}{\xi_1} = \delta\lambda\tau,$$

the viscosity of the earth is obtained by

$$\xi_1 = \frac{\xi_2}{\psi} = \frac{\xi_2}{\delta\lambda\tau}.$$

In this case δ and λ are chosen and τ and ξ_2, the model ratio of time, and the viscosity of the model, are determined by direct measurement.

If, on the other hand, τ and ξ_1 are regarded as known, ξ_2, the viscosity of the model, may be computed.

For the case given, we have already taken the model ratio of viscosity to be 3.3×10^{-15}. By semi-theoretical, semi-experimental considerations Haskell [27] concludes that the earth has a kinematic viscosity of 2.9×10^{21} C. G. S. Gutenberg cites figures of the same order of magnitude, or

[27] N. A. Haskell: *The motion of a viscous fluid under a surface load*, Physics, vol. 6 (1935) p. 265.

slightly less. This would give for the viscosity itself about 10^{22} C. G. S. Taking this figure, we would obtain for the model

$$\xi_2 = \xi_1\psi = 3.3 \times 10^7 \text{ C.G.S.}$$

This is about the viscosity of asphalt at 40 degrees Centigrade.

A lower viscosity for the model can be obtained by reducing the time required by the model, and hence further reducing the model ratio.

Salt Dome.—Although rock salt is crystalline and not fluid, there is much evidence that it deforms under long-time stresses very much as if it were an elastico-viscous material of high viscosity. Assuming that salt flowage is a viscous-like phenomenon we may derive the model ratios for a salt dome.

This is a special case of the general theory like that of the post-Glacial uplift. The movement is assumed to be viscous in character. The forces of inertia are sensibly zero. μ, λ, and τ may be taken as independent model ratios. The derived model ratios of interest then are:

Acceleration:	$\gamma = 0;\ \gamma_g = 1,$
Velocity:	$\eta = \lambda\tau^{-1},$
Force:	$\phi = \mu\gamma_g = \mu,$
Density:	$\delta = \mu\lambda^{-3},$
Stress:	$\sigma = \mu\lambda^{-2} = \delta\lambda,$
Viscosity:	$\psi = \sigma\tau = \delta\lambda\tau.$

The appropriate physical constants of a salt dome in the Gulf Coast region are given by Nettleton [28]:

Depth to mother salt:	20,000 feet
Density of salt:	2.2

Density of sediments:

Depth (feet)	Density
0	1.9
2,000	2.2
4,000	2.27
6,000	2.32
8,000	2.36
10,000	2.39
20,000	2.47

The average area of salt domes in this region is between 1 and 2 square miles.

The age of the mother salt bed is as yet uncertain. Barton [29] presents evidence that it is at least as old as Lower Cretaceous, and possibly older. He also shows that there is evidence that uplift of the domes has been occurring from early Eocene to the present.

[28] L. L. Nettleton: *Fluid mechanics of salt domes*, Am. Assoc. Petrol. Geol., Bull., vol. 18 (1934) p. 1175-1204.

[29] D. C. Barton: *Mechanics of formation of salt domes wth special reference to Gulf Coast salt domes of Texas and Louisiana*, Am. Assoc. Petrol. Geol., Bull., vol. 17 (1933) p. 1025-1083.

According to present radioactive time determinations the beginning of the Cenozoic was about 60,000,000 years ago.

The viscosity of salt is a somewhat uncertain quantity, depending upon whether flowage takes place predominantly by plastic deformation

TABLE 3.—*Properties of original and model salt dome*

	Original	Model Ratios	Model
Radius	10^5 cm.	$\lambda = 5 \times 10^5$	5 cm.
Height	6×10^5 cm.	" "	30 cm.
Thickness of mother salt	To be determined	" "	Assume different values
Time of formation	6×10^7 years or 2×10^{15} sec.	$\tau = 10^{-11}$	2×10^4 sec. (5.5 hours)
Viscosity	Salt: 10^{18} C.G.S.	$\psi = \delta\lambda\tau = 5 \times 10^{-16}$	5×10^2 C.G.S.
	To be determined	$\psi = \delta\lambda\tau$; τ to be measured experimentally	Viscosity of model to be chosen arbitrarily.
Shearing strength	Sediments: 10^8 C.G.S.	$\sigma = \delta\lambda = 5 \times 10^{-5}$	5×10^3 C.G.S.
Density	Salt: 2.2 Sediments: 1.9—2.5	$\delta = 1$	Same as original

within the crystals themselves or by a slow process of solution at points of higher stress and reprecipitations at points where the stress is less. Laboratory experiments are likely to give values due to the former types of deformation, while the infinitely slower natural movement may be to a great extent due to the latter cause.

Here, as in the case of the post-Glacial uplift, we may do either of two things: we may consider the viscosity of salt as known and solve for the proper viscosity for the model, or we may choose a known viscosity for the model and, by determining experimentally the model ratio of time, solve for the viscosity of salt as an unknown. If the laboratory values for salt viscosity are the same as those effective in salt dome flow these two alternative procedures should give the same results.

Gutenberg [30] cites for rock salt viscosity a value of 2×10^{18} C. G. S.

[30] Beno Gutenberg: *Handbuch der Geophysik*, vol. 2, pt. 1 (1931) p. 539. Berlin.

at 18° C. and 2.5×10^{17} C. G. S. at 81° C. Taking an intermediate value, we may consider 10^{18} to be the correct order of magnitude.

There remains one major unknown—the thickness of the mother salt layer. On this there is no direct information. Barton, by an indirect method, arrives at a figure of about 700 feet. This might be taken as

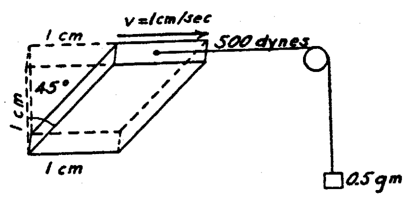


FIGURE 18.—*Physical meaning of a viscosity of 5×10^2 C. G. S.*

a trial thickness in the model, but it would be wise to try different experiments in which the salt thickness is varied from one to the other.

We are now able to set up the dimensions for an "original" salt dome and determine the constants for the model. Rather than taking the initial conditions before there was a salt dome, we will define our model ratios in terms of the end-product—the domes as we now see them.

For the conditions given the significant results are that the model equivalent of salt must have viscosity of 5×10^2 C. G. S. The sediments being intruded, if regarded as plastic, must have a shearing strength of about 5×10^3 C. G. S.

To give physical meaning to these figures, imagine in each case a one-centimeter cube of the material, one pair of whose faces are acted upon by a shearing stress couple produced by a weight suspended from a string passing over a pulley. A liquid of viscosity of 5×10^2 C. G. S. means that a shearing stress couple of 500 dynes will produce in the material an amount of shear of unity (shear angle equal to 45 degrees) in one second. The force per square centimeter required to do this is 500 dynes, or approximately the pull of gravity over a 0.5 gram mass.

A strength of 5×10^3 C. G. S. means that plastic flow would occur under a shearing stress equivalent to the pull of a 5-gram weight per square centimeter applied tangentially. If the model is made smaller, or the time shorter, the strength and viscosity of the materials will be correspondingly reduced.

Knowing the physical properties of the model, the experimental procedure would consist in setting up a suitable arrangement of parallel layers, with the equivalent of the mother salt at the bottom. By some manner a doming motion is initiated and allowed to continue.

The more direct experimental procedure is to regard the viscosity of salt and the thickness of the mother layer as unknowns to be determined. The experiment is then set up and varied to determine what model constants are necessary to give the best kinematic representation of a true salt dome. From these models, the length, the time, and the density model ratios are then computed. From the latter the viscosity, *et cetera* of the original may be computed.

The quantities taken here are only approximate and for illustrative purposes. They are, however, the correct order of magnitude. They indicate that of the experiments which have been performed on model representation of salt dome formation, those of Nettleton, employing two viscous liquids, or a liquid and a wax, are the most nearly dimensionally correct.

Impacting Meteorite.—A problem of some contemporary interest is that of the nature of the hole that a large meteorite of given size and velocity would make upon impact with the earth. This is not a problem that lends itself readily to mathematical solution, and the occurrences of such impacts are too few to afford adequate observational data. It remains to be seen if the model technique can be applied. That is, we wish a model of a meteorite to impact a model of the earth's surface in such a manner that the model (including, of course, the hole) is dynamically similar to the original.

This case is the most general possible where inertia, gravity, and resistive forces all play an important part. We choose μ, λ, and τ as fundamental model ratios and obtain the desired ratios:

Acceleration:	$\gamma = \lambda\tau^{-2} = \gamma_g = 1,$
	or $\lambda = \tau^2,$
Velocity:	$\eta = \lambda\tau^{-1} = \lambda^{\frac{1}{2}},$
Density:	$\delta = \mu\lambda^{-3},$
Force:	$\phi = \mu\lambda\tau^{-2} = \mu = \delta\lambda^3,$
Stress:	$\sigma = \mu\lambda^{-1}\tau^{-2} = \delta\lambda,$
Strength:	$\sigma = \delta\lambda,$
Elastic moduli:	$\sigma = \delta\lambda.$

This gives us the necessary model ratios to enable us to set up the model once the properties of the original are assumed. We are here concerned only with large meteorites, so for convenience let us take the original to be a nickle-iron sphere 100 meters in diameter. The velocity of such a meteorite may be expected to be of the order of magnitude of the earth's orbital velocity. At such a velocity a meteorite of this magnitude would possess so great an amount of kinetic energy that it would not be appreciably slowed down by the earth's atmosphere. Meteorites have been observed with velocities as high as 75 kilometers per second. Let us assume a velocity of 50 kilometers per second.

For the model let us use for the meteorite a sphere of lead or of amalgam (to obtain the proper strength) 0.5 centimeters in diameter. This would give for λ the value 5×10^{-5}. The model ratio for velocity then would be

$$\eta = \lambda^{\frac{1}{2}} = \sqrt{5 \times 10^{-5}} = 7 \times 10^{-3}.$$

Taking the velocity v of the original to be 5×10^6 cm/sec, we may solve for the velocity of the model projectile,

$$v_2 = \eta v_1 = 7 \times 10^{-3} \times 5 \times 10^6 = 3.5 \times 10^4 \text{ cm/sec}.$$

This is about 1/3 of the velocity of an army rifle bullet, or somewhere near that of a pistol bullet.

The strength of the model material is $\delta\lambda$ times that of the original. Taking δ to be 1.5 we get

$$\sigma = \delta\lambda = 1.5 \times 5 \times 10^{-5} = 7.5 \times 10^{-5}.$$

If we take the surface materials of the earth to have a shearing strength of 10^9 dynes/cm², the material for the model will have a strength of 7.5×10^4 dynes/cm², and a density of about 3.5.

The strength of the projectile would have to be correspondingly reduced. A nickle-iron meteorite would have a strength of the order of 5×10^9 dynes/cm² and a density of about 8. The model projectile should have a strength of about 4×10^5 dynes/cm² and a density of 12. Some forms of amalgam having these properties might be found. Since the inertial forces are dominant there is little difference at high velocities between the impact of a liquid and of a solid. Hence the effect should be nearly the same if lead were used.

For strict dynamic similarity the modulus of compressibility of the model should be reduced from that of the original by the factor σ. This is probably impossible to do experimentally, but it is doubtful if the error introduced by not doing so is great.

The effect of the air has been neglected. It is known that in the case of the Siberian meteorite the air wave was destructive to local forests. For velocities several times that of sound the air wave is distinctly an after-effect whose magnitude is small as compared with that of the meteorite itself, and may be neglected for present purposes.

For an approximate model of the impact of the meteorite specified we obtain the following: a ball of lead 0.5 centimenter in diameter fired with a velocity of 350 meters per second into the horizontal surface of a material having a density of about 3.5 and a shearing strength of 7.5×10^4 dynes/cm². A material of the latter properties can be made using clay loaded with a proper amount of lead oxide to provide the correct density.

An idea of what might be expected is obtained from the fact that when bird shot from a shot-gun is fired into a bank of soft mud, an individual shot will make a hole about the size of a man's fist. The physical constants in this case are not far from correct for a model of the impact of a large meteorite.

Glacier.—It would be of great importance if a model could be specified and the materials found that would behave in a manner dynamically similar to the movement of ice in glaciers. Insofar as the movement of ice in glaciers is a purely mechanical phenomenon, this can be done and the model ratios for strength, and for viscosity are identically the same as those employed elsewhere.

Strength:	$\sigma = \delta\lambda,$
Viscosity:	$\psi = \delta\lambda\tau.$

Unfortunately, however, the peculiar movement of ice appears to derive its unique characteristics from the unique properties of water. This takes us into the realm of thermodynamics, which, while not outside the domain of model theory or technique, is beyond the scope of the present paper, which is confined to purely mechanical systems. Accordingly, no attempt will be made to specify the properties required for a glacial model. A few remarks regarding the nature of the problem will, however, be made.

The two properties of ice that distinguish it most markedly from ordinary solids are its low melting point and the fact that it expands upon solidification. The internal bodily deformation of glacial ice is not a fluid phenomenon, but rather a plastic flowage involving in part shear displacement within the ice crystals and, in part, the actual melting and freezing by infinitesimal amounts of the ice crystals.

Flow by shearing within the crystals is the familiar plastic deformation of ordinary crystalline solids. The melting and freezing between the crystals is a process peculiar to ice and is the result of both the low melting point and the fact that ice expands on freezing.

Imagine two pieces of ice to be pressed together in a heat-insulated container at 0° C. At the points of greatest pressure the melting point will be depressed below the temperature of the box; melting will occur and water will be produced. The water will flow to the region of reduced pressure where the freezing point is again normal. This may be an infinitesimal distance. The thawing, however, can take place only provided there is a proper supply of heat available—79.7 calories per gram—which is greatly in excess of the heat produced by the mechanical work done on the system. This heat is most readily provided by re-freezing the water produced, thereby obtaining the required 79.7 calories per each gram of water so frozen.

Consequently we have a closed thermodynamic system wherein melting takes place due to the depression of the melting temperature at points of high compressive stress, the water so released flowing to points of lower pressure, and for each gram of ice melted the latent heat being supplied by the re-freezing of one gram of water. The work done, in case the pressure is due to weight, is derived from the loss of gravitational potential energy by the system as a whole.

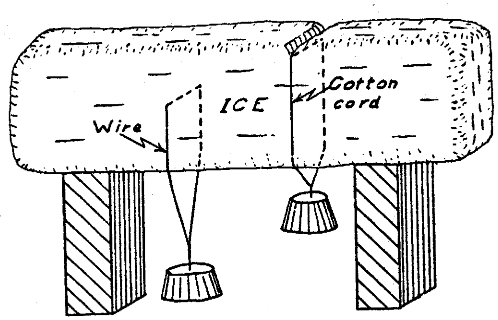


FIGURE 19.—*Block of ice cut by wire and by a cotton cord*

In order for this process to occur it is necessary only that the temperature of the ice be above the depressed melting point. The ice need not be in a heat-insulated container for the process to occur partially, as is shown by a familiar experiment. The experiment consists in taking a large block of ice and bridging it across two end supports. Two metal weights are suspended from the ice, one by a wire passing over the top of the ice and the other by a cotton cord of the same diameter as the wire. The room is at ordinary temperature.

After an hour or so it will be observed that the wire has cut some distance into the ice, but that the water has re-frozen behind the wire leaving no trench. This may be continued until the wire has passed all the way through the ice without cutting it into two pieces. In the case of the cotton cord, the movement is much slower, leaving an open cut behind, for in this case *the water does not re-freeze.*

The explanation is that the metal wire is a good conductor of heat. The ice melts beneath the wire; the water flows around to the top of the wire; the latent heat of melting beneath the wire is supplied by metallic conduction from the freezing water above the wire. This produces a closed thermodynamic system.

Were the wire non-conductive of heat the system could not work. Consequently when the wire is replaced by a cotton cord having heat-

insulating properties, the results are quite different and no re-freezing occurs.

In glacier ice at or near melting temperature this process necessarily takes place. The pressure is supplied by the weight of the ice, each

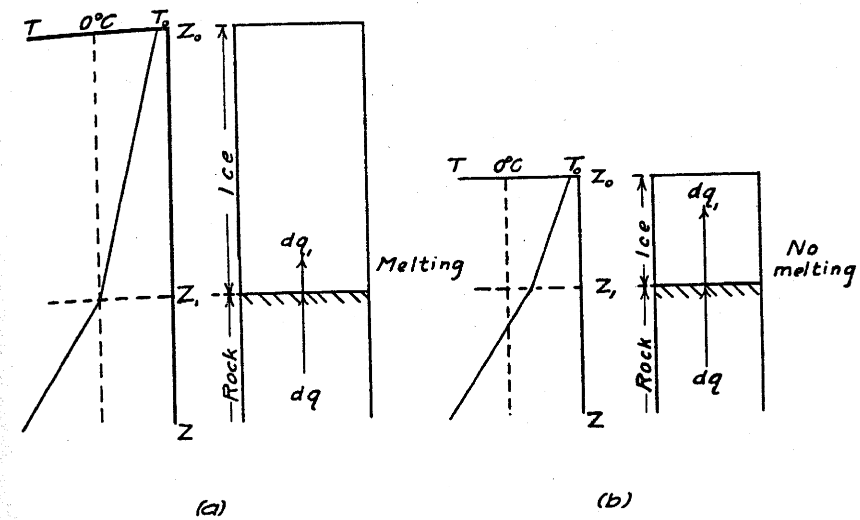


FIGURE 20.—*Temperature gradients*

(*a*) In rock and in glacier with melting at bottom. (*b*) Without melting.

infinitesimal amount of melting resulting in a shortening of some length, always in such a direction as to lower the center of mass of the system. The integral of this effect throughout the body of a glacier would of itself constitute glacial flowage.

Another characteristic of glacial motion rendering it unique is the distinct surface of slip between the moving ice and the bounding bed-rock. Glacial motion is essentially laminar, and one usual characteristic of laminar flow is that the velocity is zero at the boundary.

This surface of slip is due largely, if not wholly, to the low melting temperature of ice. There is an important effect, discussed by Lagally,[31] though first called to my attention by T. S. Lovering, due to the flow of heat from the earth's interior.

Consider an ice-sheet of thickness Z_1. Let T_0 be the temperature at the top of the ice and T_1 that at the bottom of the ice. Let dq be the amount of heat coming out of the ground in time dt per unit of area, and dq_1 the amount passing through the ice. Let K_1 be the thermal conduc-

[31] M. Lagally: *Mechanik und Thermodynamik des stationaren Gletschers,* Gerlands Beiträge zur geophysik, supp. vol. 2 (1933) p. 1-90. Leipzig.

tivity of ice and K_2 that of the rock beneath. Let the whole system be in a steady state of flow, that is, not changing with time.

The amount of heat escaping from the ground per unit area in time dt will then be

$$dq = -K_2 \cdot \left(\frac{dT}{dZ}\right)_2 \cdot dt,$$

where Z is positive downward. And the amount passing through the ice will be

$$dq_1 = -K_1 \cdot \left(\frac{dT}{dZ}\right)_1 \cdot dt = -K_1 \cdot \frac{T_1 - T_o}{Z_1} \cdot dt.$$

$\left(\frac{dT}{dZ}\right)_1$ and $\left(\frac{dT}{dZ}\right)_2$ are the thermal gradients in the ice and in the ground respectively.

Now we differentiate two cases: (1) all of the heat flowing out of the ground passes through the ice and

$$dq_1 = dq;$$

(2) part of the heat from the ground does not flow through the ice, and

$$dq_1 < dq.$$

The limiting upper temperature of the ice is 0° C. so that

$$T_o \leqq 0° \text{ C., and } T_1 \leqq 0° \text{ C.};$$

also

$$T_1 \geqq T_o.$$

In other words the temperature gradient in the ice will always be from a higher temperature at the bottom to a lower one at the top, the limiting case being when both temperatures are the same. The bottom temperature may be equal to or less than 0° C.

Since the amount of heat flowing through the ice is given by

$$dq_1 = -K_1 \cdot \frac{T_1 - T_o}{Z_1} \cdot dt,$$

and since this cannot be greater than the amount of heat coming from the ground, it follows that

$$dq_1 \leqq dq,$$

or that

$$\frac{T_1 - T_o}{Z_1} \leqq \frac{K_2}{K_1} \cdot \left(\frac{dT}{dZ}\right)_2.$$

Since the thermal conductivities are constant and the geothermal gradient is constant by hypothesis, it follows that the thermal gradient in the ice can never exceed a certain critical quantity. If it is equal to this critical value, the bottom temperature of the ice will be equal to or less than freezing temperature; if it is less than this critical value, the bottom temperature of the ice will be at freezing temperature and the heat from the earth will undergo fractionation, one fraction being conducted by the ice and the other serving as the latent heat of melting for the ice at the surface of contact with the bedrock.

This latter condition is favored by either great ice thickness or high surface temperature. Taking the thermal conductivity of granite to be .008 C.G.S. and of ice to be .006, and the geothermal gradient as 1° C. per 30 meters, we obtain as the maximum value for ice a gradient of 1° C. per 25 meters. This means that for an ice thickness of one kilometer the mean annual temperature would have to be below —40° C. if the bottom of the ice is to be below freezing. For surface temperatures greater than —40° C. the bottom of the ice will be at freezing temperature and a part of the heat from the earth will be spent in producing melting.

This leads us to the conclusion that in most actual glaciers during most of the time the bottom temperature is actually at freezing and that at such times the ice is continuously melting at the contact.

This fact is sufficient to account for the surface of slip at the contact of a glacier with bedrock. In the absence of contrary evidence it appears doubtful that a glacier would slip on the bedrock were the temperature low enough to prevent bottom melting.

It is these unique properties of water in its liquid and solid phases that make it difficult if not impossible to construct a satisfactory model of a glacier.

Owing to the fact that there is a strong similarity between the dissolving and re-precipitation of salts in a saturated solution to relieve local stress, and the melting and re-freezing of water in the manner described, there is a possibility that a suitable glacial model might be devised using some soluble solid and properly controlled interstitial water. Even if possible, the working out of the details presents a research problem of considerable magnitude.

Entire Earth.—This brings us to the last of our models—that of the earth as a whole. As we have remarked before, the strength of the earth is not too certainly known, but the density and rigidity moduli are moderately well known as a function of depth. The over-all modulus of rigidity of the earth as determined in various ways is of the order of twice that of steel. The mean density of the earth is 5.52, ranging from about 2.7 at the surface to about 10 at the center.

The strength of the earth materials increases with increase of pressure and decreases with increase of temperature. Bridgman's recent work shows that at confining pressures equivalent to a depth of 166 kilometers in the earth, ordinary solids increase in shearing strength commonly by a factor of ten times or more. This is at ordinary temperature, so we do not know how much this would have been reduced at temperatures probably existing at such a depth.

When the shearing stress was equal to the strength of the material, in Bridgman's experiments, any amount of shear could be produced at a stress approximately independent of the rate of shear.

The over-all strength of the earth is probably not greater than that of ordinary steel, so we should not be greatly in error by assuming it to have such a strength.

With this assumption, our problem becomes essentially that of determining how a body having the density distribution and motion of the earth together with the strength of steel should behave under the influence of its own forces, or of such other forces as we might imagine to be impressed upon it. Steel balls of our everyday experience, for example, rebound elastically when dropped upon a steel anvil. Would an elastic earth having the strength of steel rebound if it should collide with a rigid plate at a velocity corresponding to that of the steel ball dropped on the anvil?

To answer these and similar questions we must bring the earth into the domain of our direct observation. We must define the properties of a body having shape, velocity, density, strength, etc., which are strictly similar to the corresponding properties of the real earth, except that the model must be of proper size to be conveniently observed, the criterion of similarity being that of dynamic similarity as previously defined.

For convenience we wish the model to be of a size readily observed—about that of a large terrestrial globe. We wish it to rotate at a rate fast enough to be readily observed, and yet not so fast that one has not time to see the surface details. This would mean a rate of about one or two revolutions per minute.

From such general specifications we determine our fundamental model ratios and derive the remaining ratios that are of interest. Instead of taking μ, λ, and τ to be fundamental, let us choose δ, λ, and γ of density, length, and acceleration. Then we set up the appropriate equations of transformation:

$$\lambda = \lambda \qquad \lambda = \lambda$$
$$\delta = \mu\lambda^{-3} \quad \text{or} \quad \mu = \delta\lambda^3$$
$$\gamma = \lambda\tau^{-2} \qquad \tau = \lambda^{\frac{1}{2}}\gamma^{-\frac{1}{2}}$$

In this case it should be noted that we imagine our model to be completely removed from the present earth's gravitational field, and to be

influenced only by its own field and that of the corresponding model of the solar system. There will then be a model value of gravity to be determined by the value chosen for γ, which in the general case need not be equal to that on the earth.

Taking δ, λ, and γ as fundamental, the derived model ratios become:

Velocity:	$\eta = \lambda\tau^{-1} = \lambda^{\frac{1}{2}}\gamma^{\frac{1}{2}}$,
Angular velocity:	$\frac{\omega_2}{\omega_1} = \tau^{-1} = \lambda^{-\frac{1}{2}}\gamma^{\frac{1}{2}}$,
Gravity:	$\gamma_g = \gamma$
Gravitational constant:	$\frac{K_2}{K_1} = \mu^{-1}\lambda^3\tau^{-2} = \delta^{-1}\lambda^{-1}\gamma$,
Force:	$\phi = \mu\lambda\tau^{-2} = \delta\lambda^3\gamma$,
Stress:	$\sigma = \mu\lambda^{-1}\tau^{-2} = \delta\lambda\gamma$,
Strength:	$\sigma = \delta\lambda\gamma$.

For the magnitudes of the fundamental model ratios, the mean radius of the earth is 6.37×10^8 cm. so it is convenient to give the model a radius of 63.7 cm. (about 2 feet) corresponding to a value for λ of 10^{-7}. It also is convenient to let the model have the same density as the original, making δ equal to unity. The proper angular velocity will be obtained for the model if we also choose γ equal to unity; that is, a one-gram mass would weigh the same on the surface of the model as it does on the surface of the earth.

With the fundamental ratios given, the values—

$$\delta = 1,$$
$$\lambda = 10^{-7},$$
$$\gamma = 1,$$

we can now write the numerical values of the derived ratios:

Period:	$\tau = \lambda^{\frac{1}{2}}\gamma^{-\frac{1}{2}} = 10^{-3.5} = 3.16 \times 10^{-4}$,
Velocity:	$\eta = \lambda^{\frac{1}{2}}\gamma^{\frac{1}{2}} = 10^{-3.5} = 3.16 \times 10^{-4}$,
Angular velocity:	$\frac{\omega_2}{\omega_1} = \lambda^{-\frac{1}{2}}\gamma^{\frac{1}{2}} = 10^{3.5} = 3.16 \times 10^3$
Gravity:	$\gamma_g = 1$,
Gravitational constant:	$\frac{K_2}{K_1} = \delta^{-1}\lambda^{-1}\gamma = 10^7$,
Force:	$\phi = \delta\lambda^3\gamma = 10^{-21}$,
Stress and strength:	$\sigma = \delta\lambda\gamma = 10^{-7}$.

Then if we know the magnitude for a property of the earth we can determine the appropriate corresponding magnitude for the model. This is done in Table 4.

We now have the specifications for our model of the earth. It is a spheroid of mean radius 63.7 cm. It moves through space under the influence of its own gravitational field and the fields of a model sun, moon, etc., of a model solar system. The mean radius of its orbit is

15.0 km. It rotates once upon its axis in 27.3 seconds and moves in its orbit with a mean translational velocity of 9.44 meters per second, making one complete revolution in 2.77 hours.

The density of the model varies from 2.7 at the surface to about 10 at its center. The over-all shearing strength of its materials is 4×10^2

TABLE 4.—*Properties of the original and a model earth*

Property	Original	Model Ratio	Model
Mean radius	6.37×10^8 cm.	10^{-7}	63.7 cm.
Mean radius of orbit	1.50×10^{13} cm.	10^{-7}	1.50×10^6 cm.
Period of rotation (Sidereal Day)	8.6164×10^4 sec.	3.16×10^{-4}	27.3 sec.
Period of revolution (Sidereal Year)	3.156×10^7 sec.	3.16×10^{-4}	9.98×10^3 sec. (2.77 hours)
Mean orbital velocity	2.98×10^6 cm./sec.	3.16×10^{-4}	9.44×10^2 cm./sec.
Mean density	5.52	1	5.52
Mass	5.98×10^{27} gm.	10^{-21}	5.98×10^6 gm.
Pressure at 100 km. depth	3×10^{10} dynes/cm²	10^{-7}	3×10^3 dynes/cm²
Shear strength	4×10^9 dynes/cm²	10^{-7}	4×10^2 dynes/cm²
Gravitational constant	$6.67 \times 10^{-8} \frac{cm^3}{gm.\ sec^2}$	10^7	$0.667 \frac{cm^3}{gm.\ sec^2}$

dynes/cm². Its gravity will be the same as that of the earth but the gravitational constant of its materials will have to be 10^7 times as large as that for the solar system, or 0.667 cm³/gm sec².

The matter of most concern to us is the strength which has been taken to be the model equivalent of an earth having the strength of cold steel. A material having a shearing strength of 4×10^2 dynes/cm² would begin to shear plastically under a shearing stress equivalent to the force due to the weight of a 0.4-gram mass applied tangentially per square centimeter of surface. It is difficult to think of a substance having such a strength but, as a first approximation, very soft mud would be somewhere near correct. That is to say that a sphere of a substance having a density of 5.52 and the strength of cold steel would, if the sphere were the size of the earth, exhibit properties very similar to those of a sphere of soft mud of radius 63.7 cm.

Perhaps a better way to visualize this is by means of dynamic effects. It was remarked earlier that a sufficiently small sphere of cold steel rebounds elastically when dropped upon a steel anvil. Imagine such a sphere the size of the earth to impact against a rigid plane surface while moving at the earth's orbital velocity of 29.8 km/sec. Would it then rebound elastically?

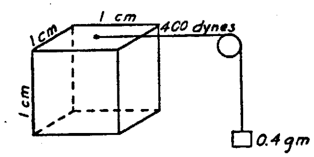


FIGURE 21.—*Physical meaning of a shear stress of 400 dynes/cm²*

Our model enables us to answer this question. We can imagine the model to impact against such a rigid plane while moving at its orbital velocity of 9.44 meters/sec. From the definition of dynamic similarity the result for the model impact would be similar to what would happen if the earth itself were to have such an accident. A velocity of 9.44 meters/sec. is equivalent to the terminal velocity attained by a body on the earth's surface falling freely from a height of 4.54 meters (14.9 feet).

Now if we can conceive what would happen if a 6.6-ton sphere of soft mud having the density of hematite and a diameter of 4 feet were dropped from a height of 15 feet onto a concrete pavement, we will have a very good idea of what would happen if our earth were to have a similar collision while moving at its orbital velocity of 29.8 km/sec. Instead of rebounding elastically like a small steel ball, it would splatter, although having the strength of cold steel.

SUMMARY AND CONCLUSIONS

At the beginning of this paper we pointed out certain of the difficulties inherent in obtaining a proper idea of geologic structures when these occur on a scale of tens or hundreds of kilometers. Our direct observation of rocks (at any one time) are limited to small specimens possessing great strength, hardness, and rigidity. Yet our geological observations show us that on a large scale these same rocks have been deformed like putty.

Other data confirm the small-scale observations that the magnitudes of the specific physical properties of the earth as a whole are about the same as those obtained by direct measurement on small specimens, embodying great strength and rigidity.

On the other hand, numerous attempts have been made by means of small-scale model experiments to produce results resembling observed geologic structures such as occur in mountain ranges and elsewhere. Invariably it has been found that the experimental results only resembled the original structures in those cases for which the model materials possessed great weakness instead of great strength. This has led to a paradox of an earth whose materials by every known test show great strength, yet whose repeated deformations suggest a material of great weakness.

The resolution of this paradox has been one of the principal purposes of the present paper. To do this, extensive development and use has had to be made of the methods of dimensional analysis. While the present analysis has been confined strictly to mechanical systems, the methods outlined here are equally applicable to thermodynamical, to chemical, and to electro-magnetic systems.

Should this not have been made sufficiently clear heretofore, let it again be emphasized that the methods of analysis employed here are neither original nor new. In his *Two New Sciences* Galileo [32] has an excellent treatise upon the subject of the manner in which the various physical dimensions of a body change with size. That Galileo's understanding of problems such as we have been discussing was by no means superficial can be seen most easily by quoting his own summary.

"From what has already been demonstrated, you can plainly see the impossibility of increasing the size of structures to vast dimensions either in art or in nature; likewise the impossibility of building ships, palaces, or temples of enormous size in such a way that their oars, yards, beams, iron-bolts, and, in short, all their other parts will hold together; nor can nature produce trees of extraordinary size because the branches would break down under their own weight; so also it would be impossible to build up the bony structures of men, horses, or other animals so as to hold together and perform their normal functions if these animals were to be increased enormously in height; for this increase in height can be accomplished only by employing a material which is harder and stronger than usual, or by enlarging the size of the bones, thus changing their shape until the form and appearance of the animals suggest a monstrosity. . . .

". . . Clearly then if one wishes to maintain in a great giant the same proportion of limb as that found in an ordinary man he must either find a harder and stronger material for making the bones, or he must admit a diminution of strength in comparison with men of medium stature; for if his height be increased inordinately he will fall and be crushed under his own weight. Whereas, if the size of a body be diminished, the strength of that body is not diminished in the same proportion;

[32] Galileo Galilei: *Dialogues concerning two new sciences* (1914) p. 130-131. Macmillan Co., New York.

indeed the smaller the body the greater its relative strength. Thus a small dog could probably carry on his back two or three dogs of his own size; but I believe that a horse could not carry even one of his own size."

Sir Isaac Newton [33] contributed further to the subject. Since the time of Sir George Stokes,[34] Helmholtz,[35] and Reynolds,[36] dimensional methods have come into wide use in studies of fluid mechanics. Today the methods are being more and more widely used in all the domains of engineering and of applied physics and chemistry. In the present paper the tools that have been evolving for 300 years are merely borrowed for use in solving problems of a slightly different kind.

By the use of the methods of dimensional analysis the paradox of an earth, apparently both strong and weak, vanishes completely. We see that strength and weakness are purely relative terms devoid of meaning unless the size of the body is specified. Quantitatively, strength is a specific term describing only a specimen of material of a given size; the strength of a body of another size must be determined by analysis. We have made the analysis and learned that quite generally, for a body of material having a given specific strength, the over-all strength of the body taken as a whole decreases with increase of size. Thus small bodies of a given material are strong; large bodies of the same material are weak, and the larger the body the greater its weakness. Conversely, if we are to obtain an idea of the weakness of a large body by observations made on a small one we must artificially weaken the smaller body to bring it into similarity with the larger one. Otherwise we will make the mistake of substituting for an earth whose specific strength is of the order of that of steel our experiences with small steel balls. Steel, as we have seen, when it occurs in the form of small steel balls, is strong; steel, if it occurred in a ball the size of the earth, would be very, very weak.

In the light of what we have now seen it is instructive to appraise some of the better-known model experiments that have been made in attempts to represent mountain-like structures. Most of the experimenters have made the mistake of using materials whose specific strengths were much too great. Bailey Willis,[37] for example, in his discussion of his experiments on Appalachian structures showed plainly

[33] Sir Isaac Newton: *Principia*, Bk. 2, Prop. 32.

[34] Sir George Stokes: *Physical and mathematical papers*, vol. 1 (1880). Cambridge Press.

[35] H. von Helmholtz: *Ueber ein Theorem, geometrisch ähnliche Bewegung flüssiger Körper betreffend, nebst Anwendung auf das Problem, Luftballons zu lenken*, Monatsberichte der Königl. Akademie der Wiss., Berlin (June 1873) p. 501-514.

[36] Osborne Reynolds: *An experimental investigation of the circumstances whether the motion of water will be direct or sinuous, and of the law of resistance in parallel channels*, Royal Soc. London, Philos. Tr. (1883) *or* Sci. Pap., vol. 2, p. 51.

[37] Bailey Willis: *The mechanics of Appalachian structure*, U. S. Geol. Surv., 13th Ann. Rept., pt. 2 (1891-1892) p. 211-289.

that he understood qualitatively the necessity of using materials of less specific strengths than the originals. He failed however to deduce this reduction factor quantitatively, with the result that most of his materials were much too strong. This accounts for the fact that his model structures had to be loaded with an overburden of buckshot equivalent to a depth of rock 1.5 to 2.5 times the length of the model, or if the original were 100 kilometers long the overburden equivalent to that of the model would have been from 150 to 250 kilometers—a geologically impossible amount. Even then the more brittle materials broke into rigid slabs.

Experiments using materials somewhat more nearly correct have recently been made by Kuenen. In some of these experiments Kuenen has employed paraffin floating on warm water, and certain mixtures of paraffin, vaseline, and mineral oil. Nettleton's experiments with salt domes using two viscous liquids of different densities are very nearly dimensionally correct.

The best experiments of all that have come to the author's attention, however, are those of Hans Cloos [38] representing a wide variety of tectonic structures. Cloos actually built his models to conform to simply deduced criteria for similarity. He reasoned that if an original earth feature were composed of rocks strong enough to support a column 10 to 20 kilometers high, a model reduced by a length factor of 1/50,000 would have to be composed of a material that would support a column 1/50,000th as high as that supported by the original materials, or a column 20 to 40 cm. high. It will be noted that this result coincides with the one we have deduced for a similar case, provided the density ratio is equal to unity.

The models made by Cloos display an uncanny resemblance to structures one actually sees in the field. The material used was wet, half-liquid clay too soft to stand without lateral support. In one place Cloos describes it as having the consistency of thick cream. In spite of this it deformed by fracturing into minute blocks so small that the integrated deformation appeared continuous.

The earliest explicit application of the method of dimensional analysis to tectonic structures so far discovered is a paper written in 1912 by Koenigsberger and Morath.[39] Taking their clew from a study by Helmholtz on the theory of hydrodynamic and aerodynamic models, they developed the theory in a simplified form for application to geologic models. They arrived at the conclusion that for a length ratio of 10^{-5}, and a density ratio of unity, the model materials for a mountain structure would need to be reduced in strength by a factor of 10^{-5} from the strength of the original.

Koenigsberger and Morath also performed some model experiments, employing for their model materials a mixture of paraffin, vaseline, gutta-percha, Ramsay-fat *(Ramsayfett)*, and machine oil. For loading materials iron filings and lead powder (?) *(Bleipulver)* were used. Iron oxide and chromium oxide were added to give different colors to the separate layers. Experiments were made on scales of 1/100,000, 1/50,000, and 1/25,000.

In an abstract presented to the International Geological Congress in Washington, Koenigsberger [40] again made brief mention of the application of model theory to a study of the size of folds.

The most significant thing about this earlier work is that it seems to have gone very nearly unnoticed. There is evidence that it had the misfortune of being 20 years ahead of its time. The theory developed then was limited in scope but was entirely correct as far as it went.

Bucky's work on mine models, employing a centrifuge to increase the body forces has been mentioned. This is still in a state of evolution, but shows great promise as an experimental technique. It is practically difficult to have a centrifuge that will handle a model more than a foot or so in length, with an acceleration greater than a few thousands of times gravity. The models, however, can be observed by means of a stroboscope, and photo-elastic models can also be used. The method is unique, and may be expected ultimately to yield results of great importance, which are otherwise unobtainable.

Maillet and Blondel [41] have made a more extensive use of this method in working out the properties of a model of the earth. By allowing some unnecessary restrictions and by an unfortunate choice of fundamental ratios they obtained a model of the earth having a radius of 0.04 millimeters which, though dimensionally correct, is much too small to be of use.

There may be others who have contributed to this problem who have been overlooked. If so the oversight is accidental and unintentional. In any case the evidence is that in remote parts of the world the geological professional is already awaking to the importance of so powerful a tool as that afforded by the method of dimensional analysis and correctly made scale models, for the solving of problems that have not yielded satisfactorily to methods of attack previously employed.

[38] Hans Cloos: *Kunstliche Gebirge*, Nat. u. Mus. Senckenbergische Naturforschende Gesellschaft, Frankfort, pt. 1 (1929) p. 225-243; pt. 2 (1930) p. 258-269.

[39] G. Koenigsberger and O. Morath: *Theoretische grundlagen der experimentallen tektonik*, Zeitschrift der Deutschen Geologischen Gesellschaft, Monat. vol. 65 (1913) p. 65-36.

[40] G. Koenigsberger: *An experimental tectonic rule proved in the field*, 16th Intern. Geol. Cong. (1936) p. 1000.

[41] R. Maillet and F. Blondel: *Sur la similitude en tectonique*, Soc. géol. France, Bull., sér. 5, t. 4, no. 6-7 (1934) p. 599-602.

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