The Viscosity of the Lower Mantle

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The viscosity of the mantle is important to theories of convection and continental drift and also to the understanding of the earth's external gravity field. Until recently, however, the processes causing creep in solids under the low stresses present within the earth were obscure, and there were no estimates of the viscosity of the lower mantle. In this paper the use of a stress-independent viscosity is justified, and the Navier-Stokes equation is applied to creep within the mantle, to investigate how this viscosity may vary with depth within the earth and to estimate the viscosity of the lower mantle from the nonhydrostatic equatorial bulge. The viscosity is shown to be 6×10^{26} (stokes), and this high value prevents both convection in the lower mantle and polar wandering.

1. INTRODUCTION

The viscosity of the earth's interior is important in many geophysical problems. It is essential to any calculations on convection within the mantle and must govern continental drift and tectonics of the crust. Whether polar wandering takes place will be decided by the earth's viscosity, and the harmonics of the external gravity field may perhaps be related through the viscosity to temperature differences within the mantle. However, until the nonhydrostatic bulge was discovered, there was no method of estimating the viscosity of the mantle below a depth of perhaps 1000 km. The reasons for this are (see section 6) that the deformation produced by a surface load takes place in the upper mantle and never reaches the lower mantle, however large the dimensions of the load may be. Only a body force, like rotation, is able to deform the lower mantle.

The classical method of estimating the kinematic viscosity of the mantle is to measure the isostatic uplift after a known load has been removed from the surface. *Haskell*'s [1935] calculation for the postglacial rebound of Fennoscandia, probably the most accurate of the many estimates for that region, gives a kinematic viscosity of 3×10^{21} stokes. The only other accurate calculation [*Crittenden*, 1963], on the uplift after Lake Bonneville dried out, gave a lower value of 3×10^{20} stokes. *Gutenberg* [1959] showed that the uplift of the Canadian Shield is consistent with Haskell's result.

There was no method of estimating the viscosity of the lower mantle until *MacDonald* [1963] pointed out that the nonhydrostatic bulge could only be supported by a highly viscous lower mantle. He suggested a viscosity of 10^{26} stokes, which is supported by the analysis given below. Before any calculations can be made, it is necessary to show that there is a difference between the nonhydrostatic bulge and the other harmonics of the external gravity field. This difference becomes clear in section 2, where the energy stored in each harmonic is calculated. The bulge contains more energy than any other component.

Many attempts have been made to relate the rate of strain in a solid to the stress applied, but most of the equations produced are empirical and are based on laboratory studies under conditions very different from those within the earth. *Zharkov* [1963] and *Gordon* [1965] have discussed what mechanisms can produce creep in a solid when the stress is small, and both believe diffusion creep to be the dominant mechanism within the earth.

Kaula [1963] calculated the elastic shear energy in the nonhydrostatic bulge to be 2×10^{29} ergs. Table 1 shows that the gravitational energy in the bulge is 2×10^{30} ergs. Thus the neglect of the elastic forces will introduce a 10% error, which is small compared with the other uncertainties.

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2. The External Gravity Field

The nonhydrostatic part of the equatorial bulge was not discovered until the external gravity field could be determined from the motion of satellites. In this section the gravitational energy stored in the bulge is shown to be much greater than that in any other coefficient, and thus suggests it has a different origin. Section 4 requires the gravity field of a layered sphere that has small distortions in the surface of each layer. The revelant expressions are derived here because they have not been found in the literature.

Outside the earth the gravitational potential U satisfies

$$\nabla^2 U = 0 \qquad (1)$$

and inside the earth

$$\nabla^2 U = -4\pi G\rho \qquad (2)$$

According to the usual sign convention which is followed here, U is positive everywhere.

The solution to (1) is

$$U(r, \theta, \phi) = gR_E \cdot\left(\frac{R_E}{r} + \sum \sum U_l^m\left(\frac{R_E}{r}\right)^{l+1} X_l^m(\theta, \phi)\right) \qquad (3)$$

and (2) gives

$$U(r, \theta, \phi) = gR_E\left(\frac{3R_E^2 - r^2}{2R_E^2} + \sum \sum U_l^m\left(\frac{r}{R_E}\right)^l X_l^m(\theta, \phi)\right) \qquad (4)$$

where $R_E = 6378$ km and is the mean equatorial radius and $g = GM/R_E^2 = 979.8$ cm/sec^2, M being the mass of the earth. U_l^m are the coefficients of the external field with R_E taken as a reference length; they require slight corrections if the mean radius a is used instead. At present the experimental errors are greater than the corrections.

The spherical harmonics are defined by

$$X_l^m(\theta, \phi) = (2l + 1)^{1/2}\left[\frac{(l - m)!}{(l + m)!}\right]^{1/2} P_l^{|m|}(\cos\theta)e^{im\phi} = Y_l^m(\theta)e^{im\phi}$$

then

$$\int_0^\pi \int_0^{2\pi} X_l^m(X_k^n)^* \sin\theta \, d\theta \, d\phi = 4\pi \, \delta_{lk} \, \delta_{nm}$$

The values of U_l^m are complex and are related to the real coefficients defined by

$$U(r, \theta, \phi) = ga\left(\frac{a}{r} + \sum_l C_l^0\left(\frac{a}{r}\right)^{l+1} Y_l^0(\theta)\right) + ga \sum_l \sum_{m=0}^{l} Y_l^m(\theta)\left(\frac{a}{r}\right)^{l+1} \cdot [C_l^m \cos m\phi + S_l^m \sin m\phi]$$

through the following equations

$$U_l^0 = C_l^0 \qquad U_l^{|m|} = (C_l^m - iS_l^m)/\sqrt{2}$$
$$U_l^{-|m|} = (C_l^m + iS_l^m)/\sqrt{2}$$

The coefficients C_l^m and S_l^m, calculated from the motion of satellites, are of the order of 10^{-5} to 10^{-6} except for C_2^0, the equatorial bulge. The zonal harmonic coefficients are better known than the tesseral because they produce long-term changes in the orbit parameters and thus may be measured over a long period from one station. The tesseral coefficients produce only short-term changes in the orbit, and thus the observations from different stations must be combined. Since the relative positions of the stations cannot be determined independently with sufficient accuracy, these must also be calculated from the orbits. The observations have only recently become sufficiently accurate. The geoid determined by *Izsak* [1964] has elevations and depressions in the same general regions as that of *Guier and Newton* [1965]. The geoid is the surface of mean radius a over which U is constant

$$r = a(1 + \sum \sum G_l^m X_l^m(\theta, \phi))$$
$$G_l^m = U_l^m$$

The gravitational potential due to a deformed sphere of uniform density whose surface (*not* an equipotential) is

$$r = a(1 + \sum \sum \eta_l^m X_l^m) \qquad 1 \gg \eta_l^m \qquad (5)$$

can be found by requiring U and ∇U to be continuous on the deformed surface [*Jeffreys and Jeffreys*, 1950, p. 642]

$$U_l^m = (3/(2l + 1))\eta_l^m \qquad (6)$$

The potential V_l^m on the deformed surface is found from (3), (5), and (6):

$$V_l^m = -[2(l-1)/(2l+1)]\eta_l^m$$
$$= -[2(l-1)/3]U_l^m \qquad (7)$$

When a sphere has several layers of different density, the corresponding expressions may be found by addition of expressions like (3)

$$\mathbf{U}_l^m = \mathbf{F}_l \mathbf{n}_l^m \qquad (8)$$

$$\mathbf{V}_l^m = -\mathbf{G}_l \mathbf{n}_l^m \qquad (9)$$

With two layers

$$(\mathbf{U}_l^m)^T = ({}_1U_l^m, {}_2U_l^m) \quad (\mathbf{n}_l^m)^T = ({}_1\eta_l^m, {}_2\eta_l^m)$$

$$(\mathbf{V}_l^m)^T = ({}_1V_l^m, {}_2V_l^m)$$

$$\mathbf{F}_l = \frac{3}{2l+1}\begin{bmatrix} \alpha g_1 & \alpha^l g_2 \\ \alpha^{l+2} g_1 & g_2 \end{bmatrix}$$

$$\mathbf{G}_l = \begin{bmatrix} \alpha\left(\frac{2(l-1)}{2l+1} g_1 + \alpha g_2\right) & -\frac{3\alpha^l}{2l+1} g_2 \\ -\frac{3\alpha^{l+2}}{2l+1} g_1 & \left(\alpha^2 g_1 + \frac{2(l-1)}{2l+1} g_2\right) \end{bmatrix}$$

In these equations ${}_n\eta_l^m$, ${}_nU_l^m$, and ${}_nV_l^m$ are the deformation and the potential on the undeformed and deformed surfaces, respectively, of the boundary radius a_n. Also

$$\alpha = a_1/a_2$$

$$g_1 = (4\pi G/3g)a_1(\rho_1 - \rho_2) \quad g_2 = (4\pi G/3g)a_2\rho_2$$

where ρ_n is the density within the layer n. The corresponding results for three layers are easily derived in the same way. In all models the densities were chosen to make the acceleration due to the gravity field on the surface of all layers the same as its surface value, g.

The coefficients of the external field are best compared by calculating the energy contained in each of them. This calculation cannot be accurate because the depth of the density variations responsible for the coefficients is not known. An estimate can be made by considering the earth as a uniform sphere. The energy can be calculated by adding a mass m given by

$$m = 4\pi a^2 \rho a \, dk \sum \sum \eta_l^m X_l^m \quad 0 < k < 1$$

to the surface of a sphere at potential

$$V = ga\left(1 - \sum \sum \frac{2(l-1)}{2l+1} k\eta_l^m X_l^m\right)$$

Integrating over the whole surface, over k from 0 to 1, and substituting the mass M of the sphere for $4\pi a^3\rho/3$ gives

$$E = -12\pi gaM \sum \sum \frac{l-1}{2l+1} \eta_l^m(\eta_l^m)^*$$

or

$$E = -\frac{4\pi}{3} gaM$$
$$\cdot \sum \sum (l-1)(2l+1) U_l^m (U_l^m)^* \qquad (10)$$

The negative sign occurs because U is taken as positive. The analysis above applies to a nonrotating sphere. In a rotating frame the energy is

$$E = -\frac{4\pi}{3} gaM \sum \sum (l-1)$$
$$\cdot (2l+1)(U_l^m - H_l^m)(U_l^m - H_l^m)^* \qquad (11)$$

H_l^m are the hydrostatic coefficients of the gravity field. If the axis of rotation is that of the spherical harmonics, $H_l^m = 0$ unless $m = 0$ and l is even. *Jeffreys* [1963] has calculated H_2^0 and H_4^0 to be

$$-1072.1 \times 10^{-6} \quad \text{and} \quad +2.9 \times 10^{-6}$$

for the unnormalized spherical harmonics, or -479.5×10^{-6} and $+1.0 \times 10^{-6}$ for those used here. The energies given by (11) (Table 1) are calculated from the tabulated values of C_l^m and S_l^m:

$$U_l^m(U_l^m)^* = [(C_l^m)^2 + (S_l^m)^2]/2$$

The agreement between the two determinations is not particularly good, especially for the low harmonics. However, it is clear that the energy in the X_2^0 harmonic is very much greater than that in any other.

There are several possible methods of producing slight deviations from spherical symmetry of the density distribution, but it is difficult to understand why the axis of rotation should also be the axis for these deviations. *Jeffreys* [1963] believes that the mantle has finite strength and has supported the density differences required for the gravity field since the earth was formed. However, considerations of the mechanisms involved in creep in solids at high temperatures and low stresses do not support the idea of finite strength [*Zharkov,*

TABLE 1. The Gravitational Energy in the External Gravity Field*

m	l = 2	3	4	5	6
0	112	14.5	6.8	0.0	0.7
	108	13.1	3.6	0.0	0.8
1		24.1	6.9	1.0	0.3
		5.0	1.3	1.0	0.5
2	17.8	13.6	5.0	4.1	1.6
	5.6	0.8	0.9	7.6	5.3
3		9.8	9.7	0.4	9.2
		7.8	1.1	11.6	1.6
4			1.1	6.7	11.4
			1.2	6.4	6.8
5				9.9	9.5
				4.6	7.8
6					1.7
					14.6

* Energy in units of 1.56×10^{28} ergs. Upper values, *Guier and Newton* [1965]; lower values, *Izsak* [1964].

1963; *Gordon,* 1965], nor does such a theory explain the orientation of the excess mass. Another cause of the density irregularities may be a temperature distribution with small differences from spherical symmetry, caused either by convection or by the nature of the solutions to the heat conduction equation when the conductivity is a function of temperature. Since *Tozer* [1965] shows that the Coriolis force can be neglected throughout the mantle, the only way in which the rotation can affect convection and heat flow is through the boundary conditions, which are given on a spheriod rather than on a sphere. Under these conditions a theorem due to von Zeipel [*Eddington,* 1926] prevents the surfaces of constant pressure from being isotherms, and slow circulation will take place unless the earth has finite strength. The equations which govern the flow are complicated, and no solution has been attempted. In any case it is unlikely that this effect is important, since the earth's surface features show no symmetry about the equator. Some idea of the order of magnitude of the nonhydrostatic field is obtained in appendix 2. This calculation, suggested by G. E. Backus and R. H. Dicke, gives a value of 1.6×10^{-8} for the nonhydrostatic U_2^0 that is an order of magnitude too small to explain the satellite measurement.

Another suggestion [*Wang,* 1966] is that the nonhydrostatic bulge will vanish when the rebound of formerly glaciated areas is complete. This effect (discussed in appendix 3) is also an order of magnitude too small to explain the observations.

Thus the original suggestion of *Munk and MacDonald* [1960] that the nonhydrostatic bulge is caused by the earth's angular deceleration is still the only mechanism yet discussed which can explain the observations. The difference in energy between the bulge and the other harmonics is then explained by the difference in their origins. The earth's viscosity can be calculated on the basis of this hypothesis.

3. Creep within the Earth

No further progress can be made without an equation relating stress to the rate of strain within the earth. Many complicated empirical equations have been published with very little discussion of the mechanisms by which the solid is deformed. These mechanisms will be discussed in a later paper, where it will be shown that diffusion creep is dominant if the stress is small (less than about $10^{-3}\mu$, where μ is the shear modulus). Creep of this nature in a homogeneous solid obeys the Navier-Stokes equation and thus justifies the use of a stress-independent viscosity. An earth model consistent with both the rates of postglacial uplift and the nonhydrostatic bulge is related here to the properties of mantle rocks.

The nonhydrostatic equatorial bulge produces shearing stresses which cause the earth to creep toward hydrostatic equilibrium. If the stress is insufficient to cause dislocations to move through the crystal, the only creep mechanism is diffusion of atoms or vacancies along grain boundaries or through the crystal lattice [*Gordon,* 1965]. *Herring* [1950] shows that the diffusion creep rate $\dot{\epsilon}$ depends linearly on stress σ

$$\dot{\epsilon} = DV_a\sigma/10kTR^2 \qquad (12)$$

where R is the mean crystal radius, V_a the atomic volume, and D the diffusion coefficient. D is related to the enthalpy, ΔH, required to produce a vacancy or interstitial atom

$$D = D_0 \exp(-\Delta H/kT) \qquad (13)$$

where D_0 is a constant which varies little with pressure and temperature. Thus the creep can be described by a kinematic viscosity ν, given by

$$\nu = (10kTR^2/D_0 m_a) \exp(\Delta H/kT) \qquad (14)$$

where m_a is the mass of an atom. The main pressure and temperature dependence of the viscosity is through the exponential term. In the laboratory, T is generally small and creep takes place by the movement of dislocations, a process which cannot operate below the yield stress. Under these conditions a solid has finite strength because diffusion is far too slow to be measured.

Kaula [1963] finds that the nonhydrostatic bulge produces a maximum shear stress of about 10^8 dynes/cm^2, or about $10^{-4}\mu$ within the mantle. His solution is not unique. This stress is probably too small to move dislocations, and thus creep will be by diffusion. Since both pressure and temperature are, to the first approximation, functions of r, the radius, the viscosity will also only depend on r. In this case the flow will obey equations similar to those given by *Alterman et al.* [1959] for the corresponding elastic case. These equations are given by G. E. Backus (personal communication) and could be solved for various temperature and pressure distributions. However, the uncertainties in T and in the constants in (14) are as yet too great for such an analysis to be justified. The same method as Alterman's is used for the calculations in section 4, but with a maximum of three layers only. The flow in each layer then satisfies the Navier-Stokes equation

$$\frac{D\mathbf{v}}{Dt} = \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}$$
$$= \nu\nabla^2\mathbf{v} + \nabla(U - p/\rho) \qquad (15)$$
$$\nabla\cdot\mathbf{v} = 0$$

where p is the fluid pressure and ∂_t stands for $\partial/\partial t$. *MacDonald* [1963] has shown that the viscosity required to produce the nonhydrostatic bulge is about 10^{26} stokes, which is very much greater than the value of 3×10^{21} estimated by *Haskell* [1935] for the postglacial uplift of the Baltic Shield. The movement of the earth's surface under a load of radius $\sim$ 1000 km* is determined by the creep rate in the upper mantle above about 1000 km. However, flow produced by the bulge will penetrate into the lower mantle, which therefore must have a much greater viscosity than the upper mantle.

The temperature gradient in the lower mantle is close to the adiabatic, and in the upper mantle it is very much greater. *Tozer* [1965] finds that the viscosity changes little along an adiabat; thus if ΔH remains constant throughout the mantle the viscosity should decrease steadily with depth to its value in the lower mantle. Since the viscosity increases by five orders of magnitude between the upper and lower mantle, ΔH must increase. The phase change from an olivine to a spinel lattice probably takes place between 300 and 600 km, and the movement of silicon from tetrahedral to octahedral coordination takes place between 600 and 900 km. The dense phase in the lower mantle has a higher bulk modulus and hence a higher activation energy for vacancy formation than the less dense phases in the upper mantle. Thus a good model for movements within the earth with time scales of about 10^3 years or greater probably has a surface shell 1000 km thick with a viscosity $\sim 10^{22}$ stokes overlying a lower layer of viscosity $\sim 10^{26}$ stokes, which in turn surrounds an inviscid core. There is general agreement that the boundary between the upper and lower mantle is a phase change and thus depends on pressure, whereas that between the lower mantle and the core is a composition change and hence the volume of the core is constant. These boundary conditions must be satisfied by the solutions to (15). The difference between the viscosities is sufficiently great for the lower mantle to behave as a rigid core for movements in the upper mantle. Also, flow within the lower mantle behaves as if both the core and the upper mantle were inviscid fluids; thus the transverse stress vanishes at both surfaces. A more accurate estimate of the thickness of the upper shell can probably be calculated from the uplift rate of areas with varying diameters, and it may explain the apparent difference in upper mantle viscosity as calculated from the shorelines of Lake Bonneville [*Crittenden,* 1963] and from the Baltic Shield.

Three models are considered here: model 1 is a homogeneous earth; model 2 has an inviscid core and a homogeneous mantle; and model 3 is

* $\sim$, order of magnitude of.

the one described above and is the most like the actual earth. In all models the stress vanishes at the outer surface. The next section contains the mathematical solutions to (15) with the appropriate boundary conditions.

4. Model Calculations[2]

There is no analytic solution to (15) in a rotating frame, and thus certain approximations must be made before a solution is attempted. Even the angular rotation cannot be taken as constant because the nonhydrostatic bulge is produced by the deceleration. The first part of this section justifies the approximations made, and the second contains solutions of the simplified equations for each of the models discussed in section 3. The difference between the viscosity calculated for a homogeneous sphere, model 1, and model 2, which contains an inviscid core, is small. Allowance for the phase change in model 3 requires an order of magnitude increase in the viscosity.

Fortunately, the flow toward the hydrostatic figure is very slow, and many terms in the full equations can be neglected. The equations for the flow of a viscous incompressible liquid in a rotating frame are

$$\nabla\cdot\mathbf{v} = 0$$

$$\partial_t\mathbf{v} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + 2\boldsymbol{\omega}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + (\mathbf{v}\cdot\nabla)\mathbf{v} = \nu\nabla^2\mathbf{v} + \nabla(U - p/\rho) \tag{16}$$

Since the bulge is about 10^4 cm greater than the hydrostatic, and corresponds to the hydrostatic figure of about 10^7 years ago, $|\mathbf{v}| \sim 10^{-10}$. Similarly

$$|\boldsymbol{\omega}| \sim 10^{-4} \qquad |\dot{\boldsymbol{\omega}}| \sim 10^{-22}$$

$$\partial_t\mathbf{v} \sim 10^{-24} \qquad |(\mathbf{v}\cdot\nabla)\mathbf{v}| \sim 10^{-14}$$

$$|\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})| \sim 1$$

The bulge will be governed by the largest term containing $\boldsymbol{\omega}$ or $\dot{\boldsymbol{\omega}}$, which in this case is the $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ term. Thus (16) may be written

$$0 = \nu\nabla^2\mathbf{v} + \nabla\left[U - p/\rho - \frac{\omega^2}{3\sqrt{5}}r^2X_2^0(\theta)\right]$$

$$0 = \nabla\cdot\mathbf{v} \tag{17}$$

It is more convenient to work with dimensionless quantities $\mathfrak{V}$, $\mathfrak{U}$, $\mathcal{P}$, $\mathcal{R}$, and ζ, defined by

$$\mathfrak{V} = \frac{a\mathbf{v}}{\nu} \qquad \mathfrak{U} = U/ga \qquad \mathcal{P} = r/ga\rho$$

$$\mathcal{R} = r/a \qquad \zeta = gat/\nu$$

Equations 17 may then be written in terms of $\mathfrak{J}$, a dimensionless number

$$0 = \nabla^2\mathfrak{V} + \mathfrak{J}^2\nabla\left[\mathfrak{U} - \mathcal{P} - \frac{\omega^2a}{3\sqrt{5}\,g}\mathcal{R}^2X_2^0(\theta)\right]$$

$$0 = \nabla\cdot\mathfrak{V} \tag{18}$$

where

$$\mathfrak{J}^2 = ga^3/\nu^2$$

The bulge is caused by the time dependence of ω, which may be written

$$\omega = \omega_0 \exp(-\gamma t/2) = \omega_0 \exp(-\Gamma\zeta/2)$$

$$\Gamma = \gamma\nu/ga$$

Since the hydrostatic bulge is proportional to ω^2, $1/\gamma$ is the relaxation time for the hydrostatic bulge. The value of γ may be found from

$$-\gamma/2 = \dot{\omega}/\omega$$

Equation 18 becomes

$$0 = \nabla^2\mathfrak{V} + \mathfrak{J}^2\nabla[\mathfrak{U} - \mathcal{P} + \Omega_0e^{-\Gamma\zeta}\mathcal{R}^2X_2^0(\theta)]$$

$$0 = \nabla\cdot\mathfrak{V} \tag{19}$$

where

$$\Omega_0 = -\,\omega_0^2a/3\sqrt{5}\,g$$

Equation 19 must be solved with the appropriate boundary conditions. For a homogeneous sphere which has been slightly deformed, the condition that the transverse stress should vanish on the deformed surface is, to the first approximation, the same condition as that it should vanish on a sphere, radius a. Thus

[2] This section contains no arguments or conclusions that are essential to the understanding of the problem. It may be omitted by those who do not want to study the detailed mathematical solution to the equations of viscous flow.

$$\frac{1}{\mathcal{R}} \partial_\theta\left[\frac{\mathbf{V}\cdot\mathbf{a}_\mathcal{R}}{\mathcal{R}}\right] + \partial_\mathcal{R}\left[\frac{\mathbf{V}\cdot\mathbf{a}_\theta}{\mathcal{R}}\right] = \frac{1}{\mathcal{R}\sin\theta}\partial_\phi\left[\frac{\mathbf{V}\cdot\mathbf{a}_\mathcal{R}}{\mathcal{R}}\right] + \partial_\mathcal{R}\left[\frac{\mathbf{V}\cdot\mathbf{a}_\phi}{\mathcal{R}}\right] = 0 \quad (20)$$

on $\mathcal{R} = 1$, where $\partial_\mathcal{R}$ stands for $\partial/\partial\mathcal{R}$, etc. The normal stress $\sigma_{\mathcal{R}\mathcal{R}}$ must vanish on the deformed surface, $\mathcal{R} = 1 + \eta$, and does not vanish on $\mathcal{R} = 1$.

$$\sigma_{\mathcal{R}\mathcal{R}} = -\mathcal{P}_{\mathcal{R}=1+\eta} + \frac{2}{\mathfrak{J}^2}[\partial_\mathcal{R}(\mathbf{V}\cdot\mathbf{a}_\mathcal{R})]_{\mathcal{R}=1} = 0 \quad (21)$$

The kinematic condition is that the rate at which the surface is deformed must equal the normal velocity

$$\mathfrak{J}^2\,\partial_\zeta\eta = [\mathbf{V}\cdot\mathbf{a}_\mathcal{R}]_{\mathcal{R}=1} \quad (22)$$

There is no solution to these equations with $|\mathbf{V}| \sim 1$ because the force driving the circulation is $\sim\mathfrak{J}^2$. Substituting

$$\mathbf{V} = \mathfrak{J}^2\mathbf{V}_1$$

in (18) gives

$$-\nabla^2\mathbf{V}_1 = \nabla \times (\nabla \times \mathbf{V}_1) = \nabla[\mathcal{U} - \mathcal{P} + \Omega_0 e^{-\Gamma\zeta}\mathcal{R}^2 X_2^0(\theta)]$$

$$\nabla\cdot\mathbf{V}_1 = 0 \quad (23)$$

The boundary conditions become

$$P_{\mathcal{R}=1+\eta} = 2[\partial_\mathcal{R}(\mathbf{V}_1\cdot\mathbf{a}_\mathcal{R})]_{\mathcal{R}=1}$$
$$\partial_\zeta\eta = [\mathbf{V}_1\cdot\mathbf{a}_\mathcal{R}]_{\mathcal{R}=1} \quad (24)$$

Equation 20 is unchanged. The rest of this section contains the exact solution to (23) with boundary conditions 20 and 24.

Two approximations have been made in the derivation of (23). The first is that a region of constant viscosity separated by sharp spherical boundaries is a good model for the mantle and is probably valid. The second is that buoyancy forces due to temperature variations can be neglected. It has been shown (section 2, appendix 2) that these forces are unable to produce the observed gravity field by themselves, but it is difficult to prove that they cannot interact through the two nonlinear terms, $\nu = \nu(T)$ and $\mathbf{v}\cdot\nabla T$, to produce the observed field. However, such effects are neglected.

The Laplace transforms of (23) and (24), with respect to ζ are

$$\nabla \times (\nabla \times \mathcal{L}(\mathbf{V}_1)) = \nabla\left[\mathcal{L}(\mathcal{U}) - \mathcal{L}(\mathcal{P}) + \frac{\Omega_0}{s+\Gamma}\mathcal{R}^2 X_2^0(\theta)\right] \quad (25)$$

$$\nabla\cdot\mathcal{L}(\mathbf{V}_1) = 0 \quad (26)$$

$$\mathcal{L}(\mathcal{P})_{\mathcal{R}=1+\eta} = 2[\partial_\mathcal{R}(\mathcal{L}(\mathbf{V}_1)\cdot\mathbf{a}_\mathcal{R})]$$
$$s\mathcal{L}(\eta) = [\mathcal{L}(\mathbf{V}_1)\cdot\mathbf{a}_\mathcal{R}]_{\mathcal{R}=1} \quad (27)$$

The surface of the sphere is taken as undeformed at $\zeta = 0$ because the initial conditions have no effect on the solution. The divergence of (25) gives

$$\nabla^2\left[\mathcal{L}(\mathcal{U}) - \mathcal{L}(\mathcal{P}) + \frac{\Omega_0}{s+\Gamma}\mathcal{R}^2 X_2^0(\theta)\right] = 0$$

Thus

$$\mathcal{L}(\mathcal{U}) - \mathcal{L}(\mathcal{P}) + \frac{\Omega_0}{s+\Gamma}\mathcal{R}^2 X_2^0(\theta) = \sum\sum a_l^m(s)\mathcal{R}^l X_l^m(\theta,\phi) \quad (28)$$

$a_l^m(s)$ must be found from the boundary conditions. The other solution containing $\mathcal{R}^{-l-1}$ is excluded because it is infinite at the center. The form of (25) suggests a solution of the type

$$\mathcal{L}\mathbf{V}_1 = \sum\sum\{f_{l,1}^m(\mathcal{R},s)\mathbf{P}_{ml}(\theta,\phi) + f_{l,2}^m(\mathcal{R},s)\mathbf{B}_{ml}(\theta,\phi) + f_{l,3}^m(\mathcal{R},s)\mathbf{C}_{ml}(\theta,\phi)\} \quad (29)$$

P, **B**, and **C** are the vector spherical harmonics defined in *Morse and Feshbach* [1953], but normalized so that their integrals over a unit sphere are 4π. Substitution of (29) into (25) and the traction equations shows that no solution containing **C** is possible; thus there is no solution to correspond to the toroidal magnetic fields or torsional oscillations in elastic theory. Equation 26 gives

$$\frac{1}{\mathcal{R}}\partial_\mathcal{R}[\mathcal{R}^2 f_{l,1}^m] = l(l+1)f_{l,2}^m$$

and (25) becomes

$$l(l+1)f_{l,1}^m - \partial_\mathcal{R}^2[\mathcal{R}^2 f_{l,1}^m] = a_l^m(s)\,l\mathcal{R}^{l+1} \quad (30)$$

The solution to (30) is

$$f_{l,1}{}^{m}(\mathcal{R}, s) = \left[\frac{b_l{}^m(s)}{\mathcal{R}} - \frac{a_l{}^m(s)\mathcal{R} l}{2(2l+3)}\right]\mathcal{R}^l$$

where $b_l{}^m(s)$ must be found from the boundary conditions. $\mathcal{L}(\eta)$ and the values of $\mathcal{L}(\mathcal{P})$ and $\mathcal{L}(\mathcal{U})$ on the deformed surface may be expanded in spherical harmonics

$$\mathcal{L}(\eta) = \sum\sum \eta_l{}^m(s) X_l{}^m(\theta, \phi)$$
$$\mathcal{L}(\mathcal{P}) = \sum\sum \mathcal{P}_l{}^m(s) X_l{}^m(\theta, \phi)$$
$$\mathcal{L}(\mathcal{U}) = \sum\sum V_l{}^m(s) X_l{}^m(\theta, \phi)$$

Equation 7 relates $V_l{}^m(s)$ and $\eta_l{}^m(s)$; equation 27 gives

$$s\eta_l{}^m = -\frac{2l+1}{2(l-1)} s V_l{}^m$$
$$= [f_{l,1}{}^m]_{\mathcal{R}=1}$$
$$= b_l{}^m - \frac{l}{2(2l+3)} a_l{}^m \qquad (31)$$

$$\mathcal{P}_l{}^m(s) = 2[\partial_{\mathcal{R}} f_{l,1}{}^m]_{\mathcal{R}=1}$$
$$= 2\left[(l-1) b_l{}^m - \frac{l(l+1)}{2(2l+3)} a_l{}^m\right] \qquad (32)$$

Both transverse equations give

$$\left[\mathcal{R}^2 \partial_{\mathcal{R}}\left(\frac{1}{\mathcal{R}^2} \partial_{\mathcal{R}}[\mathcal{R}^2 f_{l,1}{}^m]\right)\right]_{\mathcal{R}=1} + l(l+1)[f_{l,1}{}^m]_{\mathcal{R}=1} = 0 \qquad (33)$$

The expression in the first bracket is equal to

$$(l+1)(l-2) b_l{}^m - \frac{l^2(l+3)}{2(2l+3)} a_l{}^m$$

Elimination of $a_l{}^m$, $b_l{}^m$, and $\mathcal{P}_l{}^m$ from (28), (31), (32), and (33) leaves

$$0 = \left(1 + \frac{s}{K_l}\right) V_l{}^m(s) + \frac{\Omega_0}{s+\Gamma} \delta_{m0} \delta_{l2} \qquad (34)$$

$$K_l = \frac{1}{\tau_l} = l/[2(l+1)^2 + 1]$$

τ_l is the dimensionless decay time for a surface disturbance which only contains spherical harmonics of degree l. The true relaxation time is $\nu\tau_l/ga$. Equation 34 shows that rotation affects only the $V_2{}^0$ harmonic of the gravity field

$$V_2{}^0(s) = \frac{\Omega_0 K_2}{(s+K_2)(s+\Gamma)}$$
$$K_2 = 2/19$$

Inversion of the Laplace transform gives

$$V_2{}^0(\zeta) = -\frac{\Omega_0 K_2 e^{-\Gamma\zeta}}{(K_2 - \Gamma)} \cdot \{1 - \exp[-(K_2 - \Gamma)\zeta]\} \qquad (35)$$

Since the nonhydrostatic bulge is small compared with the hydrostatic, $K_2 \gg \Gamma$; also $\zeta K_2 \gg 1$. Hence (35) becomes

$$V_2{}^0(\zeta) \approx -\frac{\Omega_0 e^{-\Gamma\zeta} K_2}{(K_2 - \Gamma)} \qquad (36)$$

As $\nu \rightarrow 0$, $\Gamma \rightarrow 0$ and (36) gives the hydrostatic potential, $W_2{}^0(\zeta)$

$$W_2{}^0(\zeta) = -\Omega_0 e^{-\Gamma\zeta}$$

Both $V_2{}^0$ and $W_2{}^0$ are measured on the deformed surface. The corresponding coefficients, $\mathcal{U}_2{}^0$ and $H_2{}^0$, on the surface of a sphere with $\mathcal{R} = 1$, may then be found by using (7). Thus

$$\Delta = \frac{\mathcal{U}_2{}^0 - H_2{}^0}{H_2{}^0} = \frac{V_2{}^0 - W_2{}^0}{W_2{}^0}$$
$$= \frac{\Gamma}{K_2 - \Gamma} \approx \frac{\Gamma}{K_2}$$

When this equation is expressed in measured, rather than dimensionless, parameters, it becomes

$$\nu = \Delta K_2 g a \omega / 2(-\dot{\omega}) \qquad (37)$$

Where ω and $\dot{\omega}$ have their modern values. The value of Δ was obtained in section 2 and is used in section 6 to calculate the viscosity. Such a calculation is justified only if $\dot{\omega}$ has remained constant over the last 10^7 years.

The earth is not a homogeneous sphere, but contains a fluid core and a mantle whose properties vary with depth. The external gravity field can still be used in calculating a viscosity, but the theory becomes more complicated. A model with a fluid core and a homogeneous mantle is a better approximation to the earth than a homogeneous sphere is. However, the gravity field is then caused by distortions at both the earth's surface and the core-mantle boundary. Since the flow takes place in response to the gravity field, the flow at the outer surface is caused by the shape of both the outer surface and the core-mantle boundary. For this reason the model must be described in terms of its two normal modes and the rotation included as a generalized force on both. Fortunately, the decay time for both modes is short compared with that for the rotation, and therefore the normal modes need be found formally only.

Flow within the core in response to movement of the core-mantle interface obeys (19) with $\nu = 0$

$$\nabla\left[\mathcal{U} - \frac{\rho_2}{\rho_1}\mathcal{P}_1 + \Omega_0 e^{-\Gamma t}\mathcal{R}^2 X_2^0(\theta)\right] = 0 \quad (38)$$

where ρ_1 and ρ_2 are core and lower mantle densities, respectively. Thus the pressure within the core, $\mathcal{P}_1$, is hydrostatic.

$$\mathcal{P}_1 = \frac{\rho_1}{\rho_2}[\mathcal{U} + \Omega_0 e^{-\Gamma t}\mathcal{R}^2 X_2^0(\theta)] + \text{const} \quad (39)$$

The normal stress is continuous at the core-mantle boundary, $\mathcal{R} = \alpha$.

$$-\mathcal{P}_1 = -\mathcal{P}_2 + 2[\partial_{\mathcal{R}} f_{l,1}{}^m]_{\mathcal{R}=\alpha} \quad (40)$$

The kinematic and transverse stress boundary conditions are the same as in (32) and (33) but are taken at $\mathcal{R} = \alpha$. In Laplace transform space, (28) becomes

$$\mathcal{L}(\mathcal{U}) - \mathcal{L}(\mathcal{P}) + \frac{\Omega_0\mathcal{R}^2 X_2^0(\theta)}{s + \Gamma} = \sum\sum\left[{}_1a_l{}^m(s)\left(\frac{\alpha}{\mathcal{R}}\right)^{l+1} + {}_2a_l{}^m(s)\mathcal{R}^l\right]X_l{}^m \quad (41)$$

The solution to (30) now contains two more terms

$$f_{l,1}{}^m(\mathcal{R}, s) = \left(\frac{\alpha}{\mathcal{R}}\right)^{l+1}\left[\frac{{}_1b_l{}^m}{\mathcal{R}} - \frac{{}_1a_l{}^m\mathcal{R}(l+1)}{2(2l-1)}\right] + \mathcal{R}^l\left[\frac{{}_2b_l{}^m}{\mathcal{R}} - \frac{{}_2a_l{}^m\mathcal{R}l}{2(2l+3)}\right]$$

As before, ${}_1b_l{}^m$, ${}_2b_l{}^m$, ${}_1a_l{}^m$, and ${}_2a_l{}^m$ must be determined from the boundary conditions, and the rotation affects the harmonic with $l = 2$ and $m = 0$ only. The transverse stress and kinematic equations may be combined to give

$$\mathbf{A}_2\mathbf{a}_{2,0} = s\mathbf{e}_{2,0} \quad (42)$$

$$(\mathbf{a}_{2,0})^T = ({}_1b_2{}^0, {}_1a_2{}^0, {}_2b_2{}^0, {}_2a_2{}^0)$$

$$(\mathbf{e}_{2,0})^T = ({}_1\eta_2{}^0, {}_2\eta_2{}^0, 0, 0)$$

$$\mathbf{A}_2 = \begin{bmatrix} \frac{1}{\alpha^2} & -\frac{1}{2} & 1 & -\frac{\alpha^2}{7} \\ \alpha^3 & -\frac{1}{2}\alpha^3 & 1 & -\frac{1}{7} \\ \frac{8}{3}\frac{1}{\alpha^2} & -\frac{1}{2} & 1 & -\frac{8}{21}\alpha^2 \\ \frac{8}{3}\alpha^3 & -\frac{1}{2}\alpha^3 & 1 & -\frac{8}{21} \end{bmatrix}$$

The boundary conditions for normal stress give

$$\mathbf{D}\left[\mathbf{V}_2{}^0 + \frac{\Omega_0\mathbf{W}}{s + \Gamma}\right] = \mathbf{B}_2\mathbf{a}_{2,0} \quad (43)$$

$$(\mathbf{W})^T = (\alpha^2, 1), \quad \mathbf{D} = \begin{bmatrix} 1 - \frac{\rho_1}{\rho_2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{B}_2 = \begin{bmatrix} -\frac{8}{\alpha^2} & 3 & 2 & \frac{\alpha^2}{7} \\ -8\alpha^3 & 3\alpha^3 & 2 & \frac{1}{7} \end{bmatrix}$$

Equations 42, 43, and 9 can be combined to give

$$\mathbf{C}[\mathbf{V}_2{}^0 + \mathbf{W}\Omega_0/(s + \Gamma)] = s\mathbf{V}_2{}^0 \quad (44)$$

where

$$\mathbf{C} = \mathbf{G}_2(\mathbf{B}_2\mathbf{E}_2)^{-1}\mathbf{D}$$

and $\mathbf{E}_2$ is the first two columns of $(\mathbf{A}_2)^{-1}$. The motion of the core-mantle interface is coupled to that of the surface of the off-diagonal terms in $\mathbf{C}$. The normal modes can be found by diagonalizing $\mathbf{C}$. We choose $\mathbf{S}$ so that

$$\mathbf{S}^{-1}\mathbf{C}\mathbf{S} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \mathbf{\Lambda}$$

and

$$\mathbf{V}_2{}^0 = \mathbf{S}\mathbf{v}_2{}^0$$

Then (44) becomes

$$\begin{bmatrix} s + \lambda_1 & 0 \\ 0 & s + \lambda_2 \end{bmatrix}\mathbf{v}_2{}^0 = -(\Omega_0/(s + \Gamma))\mathbf{\Lambda}\mathbf{S}^{-1}\mathbf{W}$$

$$\mathbf{v}_2{}^0(s) = -\Omega_0 \cdot \begin{bmatrix} \frac{\lambda_1}{(s + \lambda_1)(s + \Gamma)} & 0 \\ 0 & \frac{\lambda_2}{(s + \lambda_2)(s + \Gamma)} \end{bmatrix}\mathbf{S}^{-1}\mathbf{W}$$

Inversion of the Laplace transform gives

$$\mathbf{v}_2^0(\zeta) = -\Omega_0 e^{-\Gamma\zeta}\begin{bmatrix} \frac{\lambda_1}{\lambda_1 - \Gamma} & 0 \\ 0 & \frac{\lambda_2}{\lambda_2 - \Gamma} \end{bmatrix}\mathsf{S}^{-1}\mathbf{W}$$

As in the case of the homogeneous sphere when $\nu \rightarrow 0$, $\Gamma \rightarrow 0$ and the hydrostatic value $\boldsymbol{\omega}_2^0(\zeta)$ is given by

$$\boldsymbol{\omega}_2^0(\zeta) = -\Omega_0 e^{-\Gamma\zeta}\mathsf{I}\mathsf{S}^{-1}\mathbf{W}$$

where I is the unit matrix. Thus

$$\mathbf{v}_2^0(\zeta) - \boldsymbol{\omega}_2^0(\zeta) = -\Omega_0\Gamma e^{-\Gamma\zeta}\mathbf{\Lambda}^{-1}\mathsf{S}^{-1}\mathbf{W}$$

$$\mathbf{V}_2^0(\zeta) - \mathbf{W}_2^0(\zeta) = \mathsf{S}[\mathbf{v}_2^0(\zeta) - \boldsymbol{\omega}_2^0(\zeta)] = -\Omega_0\Gamma e^{-\Gamma\zeta}\mathsf{C}^{-1}\mathbf{W} \quad (45)$$

and

$$\mathbf{W}_2^0(\zeta) = \mathsf{S}\boldsymbol{\omega}_2^0(\zeta) = -\Omega_0 e^{-\Gamma\zeta}\mathbf{W} \quad (46)$$

$$\frac{{}_2U_2^0 - H_2^0}{H_2^0} = \frac{\Gamma(\mathsf{F}_2\mathsf{G}_2^{-1}\mathsf{C}^{-1}\mathbf{W})_2}{(\mathsf{F}_2\mathsf{G}_2^{-1}\mathbf{W})_2} \quad (47)$$

The subscript 2 refers to the second component of the vector concerned. The expression on the right of (47) can be evaluated and is $\Gamma/1.4\ K_2$, where K_2 is as given by (34).

In the third case, another layer is introduced to represent the upper mantle. The viscosity of this layer must be $\sim 10^{21}$, or about five orders of magnitude smaller than that of the lower mantle. Thus an approximate solution can be obtained by neglecting the viscous forces due to deceleration in the upper mantle. The boundary between the upper and lower mantle is a phase change, which takes place at constant pressure. Thus (38) gives

$${}_2V_2^0(s) + (\Omega_0/(s + \Gamma)) = 0 \quad (48)$$

on the boundary. Since the normal stress is independent of θ, the pressure coefficient, $p_l^m(s)$, immediately below the boundary and within the lower mantle is

$$P_l^m(s) = 2[\partial_{\mathfrak{R}} f_{l,1}^m]_{\mathfrak{R}=1} \quad (49)$$

The gravitational and rotational potentials are continuous everywhere; hence (43) becomes

$$-2[\partial_{\mathfrak{R}} f_{l,1}^m]_{\mathfrak{R}=1} = {}_1a_2^0\alpha^3 + {}_2a_2^0 \quad (50)$$

This equation can be combined with the transverse stress equations and the kinematic boundary condition at the core-mantle boundary to give a matrix as above. When the result is written in the form of (47), the expression on the right is $\Gamma/10K_2$ for an upper mantle thickness of 300 km, and one of 1000 km gives $\Gamma/15K_2$. Thus the viscosity required to produce the external gravity field is increased by factors of 10 and 15, respectively, over that for the homogeneous sphere. The viscosities for this model are an order of magnitude greater than for either the homogeneous sphere or for the two-layer model because all the nonhydrostatic field is caused by distortion of the core-mantle boundary. Since the gravitational potential drops as $\alpha^3 \sim 0.1$, the distortion required, and hence the viscosity, is greater. The deformation from the hydrostatic ellipticity is $\sim$ 1 km, or about an order of magnitude less than could be detected by core reflections.

If the viscous stresses in the upper mantle cannot be neglected, the factor on the right of (47) must lie between $\Gamma/1.4K_2$ and $\Gamma/15K_2$, depending on the variation of viscosity with depth in the upper mantle.

5. The Angular Deceleration

The analysis in section 4 applies only if the angular deceleration has a time constant long in comparison with that of the bulge, which is $\sim 2 \times 10^8$ years in the three-layer model. Astronomical observations cover only the last 150 years, but until *Wells* [1963] found daily growth lines on corals they were the only measurement of the deceleration.

Munk and MacDonald [1960] discuss the historical variations in the length of day, which they separate into two parts—one due to a tidal exchange of angular momentum between the earth, the sun, and the moon and the other due to some internal cause, probably motions within the fluid core. The only information used to achieve this separation is astronomical observations of the orbits of the moon, the sun, and Mercury and a comparison between the heights of the sun and moon tides. They estimate the present deceleration to be 5.3×10^{-22} rad/sec² if the friction in the oceans is linear, or 5.8×10^{-22} rad/sec² if, as seems more likely, the friction depends on the square of the velocity. The second value is used for all the calculations made in section 6. The acceleration due to internal causes fluctuates with a time scale of about 20 years, but it has not produced any long-term changes since observations began.

Observations on Middle Devonian corals have been used recently to calculate the deceleration over a period of 350 million years. *Wells* [1963] and *Scrutton* [1964] have suggested that ridges on the epitheca of some Paleozoic corals correspond to daily, monthly, and annual growth and hence reveal the number of days in a month and a year. *Runcorn* [1964] finds that these observations require the earth's deceleration and moment of inertia to have remained constant since Middle Devonian time. These results, though still slightly uncertain, support the extrapolation of the modern observations.

In the analysis in section 4 it was more convenient to use an expression for the angular velocity of the form

$$\omega = \omega_0 \exp(-\gamma t/2) \qquad (51)$$

where γ and ω_0 are constant and $\gamma = -2\dot{\omega}/\omega$. Equation 51 fits all observations as well as a straight line does; however, there is no physical reason why it should be preferred.

Though Runcorn found that the coral results required no change in the earth's moment of inertia, the collapse of the equatorial bulge must produce an internal angular acceleration and a decrease in the moment of inertia. This effect is discussed here to show that it can be neglected.

If C and A are the time-dependent moments of inertia about axes passing through the pole and the equator, differentiation of MacCullagh's formula [*Jeffreys*, 1959, p. 40] gives

$$\frac{dC}{dt} - \frac{dA}{dt} = Ma^2 \frac{dJ_2}{dt} \qquad (52)$$

To the first approximation the mean moment of inertia remains constant

$$\frac{dC}{dt} + 2\frac{dA}{dt} = 0 \qquad (53)$$

Jeffreys [1959] shows that J_2 is proportional to ω^2; thus

$$(2/\omega)\dot{\omega}_t = (1/J_2)\, dJ_2/dt \qquad (54)$$

where $\dot{\omega}_t$ is the total angular acceleration due to all causes. Apart from external forces the angular momentum of the earth must remain constant

$$(1/C)\, dC/dt + (1/\omega)\dot{\omega}_i = 0 \qquad (55)$$

where $\dot{\omega}_i$ is the acceleration produced by processes in and on the earth. Equations 52 to 55 may be combined to give

$$\dot{\omega}_i = -(4J_2Ma^2/3C)\dot{\omega}_t$$

For the earth $C \approx 0.33Ma^2$; thus

$$\dot{\omega}_i = -4.0J_2\dot{\omega}_t \qquad (56)$$

$$(1/C)\, dC/dt = 4.0(J_2/\omega)\dot{\omega}_t \qquad (57)$$

Thus

$$\dot{\omega}_i = 2.3 \times 10^{-24} \quad \text{rad/sec}^2$$

$$(1/C)\, dC/dt = 3.3 \times 10^{-20} \quad \text{part/sec}$$

Since the probable error in $\dot{\omega}_t$ is $\sim 10^{-23}$, the acceleration due to the collapse of the bulge is too small to be detected. The change in C since Devonian time is only 3 parts in 10^4, which is well within Runcorn's estimate of the experimental uncertainties. These results are not changed when the finite viscosity is included.

This section shows that large changes in $\dot{\omega}_t$ or C since Devonian time are excluded by the coral results. Thus the equations derived in section 4 can be applied to the earth.

6. Damping and Viscosity in a Nonhomogeneous Earth

In previous discussions of the earth's viscosity, Haskell [1935], *Gold* [1955], and *Munk and MacDonald* [1960] have considered a homogeneous earth only. Substitution of $\Delta = 9.8 \times 10^{-3}$, $K_2 = 2/19$, $\omega = 7.3 \times 10^{-5}$ rad/sec, $\dot{\omega} = -5.8 \times 10^{-22}$ rad/sec^2, $g = 9.8 \times 10^2$ cm/sec^2, and $a = 6.4 \times 10^8$ cm into (37) shows that a viscosity of 4×10^{25} stokes is required to produce the observed equatorial bulge, and this value is not consistent with that derived from postglacial uplift. Thus a homogeneous earth is too simple a model and a layered earth must be used. Also, in a homogeneous earth the characteristic time for the damping of the Chandler wobble and for the collapse of the nonhydrostatic bulge are the same and are related to the time required for polar wandering (see *Gold* [1955] and section 7). Gold used 13 years as the decay time for the Chandler wobble and discovered that polar wandering would take place in about 1 million years, an embarrassingly short time. This difficulty is also caused by a homogeneous earth being too simple a model.

In a layered earth there is no simple relation between the damping of the Chandler wobble and the time taken for the bulge to collapse. Let us consider an earth which consists of a shell which will damp the Chandler wobble in 10 years surrounding a rigid central core, and allow the surfaces of the shell and the core to have their hydrostatic bulges caused by rotation. In this model the damping of the wobble will be rapid and will take place in the viscous shell. If the angular velocity is changed, however, the nonhydrostatic bulge produced will be permanent because it will be caused by the shape of the rigid core, even though the shell will quickly flow to make the outer surface an equipotential. If the continents are floating in the outer shell, the Eötvös force on them will not cause polar wandering because the rotational axis is fixed by the central core. This force may cause them to drift toward the equator if the viscosity of the shell is sufficiently small. Such a model will also allow isostatic adjustment to any surface load.

Present knowledge about the earth's interior from seismology shows that there is a central core, radius 3470 km, surrounded by a radially symmetric mantle. Free oscillations and body waves show the core to be inviscid over the time scales considered here, whereas postglacial uplift requires the upper mantle to have a viscosity $\sim 3 \times 10^{21}$ stokes. The discussion above shows that isostatic adjustment will be governed by the viscosity of the surface layers. The nonhydrostatic bulge will be supported by the most viscous layer within the earth, provided that the decay time within this layer is short in comparison with the rotational decay time. The viscosity required must be greater than the 4×10^{25} stokes calculated for a homogeneous earth. The only possible position for a layer of such high viscosity is the lower mantle. Solid-state considerations (section 3) support this conclusion. The outer boundary of the lower mantle is a phase change, and thus this boundary is governed by pressure. Under these conditions the analysis in section 4 shows that the shape of the boundary is governed by the least viscous of the two phases. Thus the outer surface of the lower mantle is an equipotential, and it does not contribute to the nonhydrostatic bulge (a similar argument applies to the inner core). The cause of the bulge must therefore be a distortion of the core-mantle boundary from hydrostatic equilibrium. When the value of $-\dot{\omega}_t$ obtained in section 5 is substituted into the expressions given in section 4 for the three-layer model with a phase change, the viscosity required is 4×10^{26} stokes if the upper mantle is 300 km thick and 6×10^{26} stokes if it is 1000 km thick. A two-layer model with a fluid core gives a viscosity of 6×10^{25} stokes but is not consistent with postglacial uplift. *Takeuchi and Hasegawa* [1965] ignore the gravity field due to the ellipticity of the core-mantle boundary. Their model has a rigid lower mantle; therefore the external gravity field would be dominated by the shape of the rigid lower mantle and not by the external shape.

The shape of the core-mantle boundary will fix the rotation axis to the lower mantle and hinder polar wandering. However, the Chandler wobble will be damped by the layer which can dissipate the most energy when acted on by a force $X_2^1(\theta, \phi) \exp i\omega t$. Since there is a close relationship between the mechanisms producing damping and those producing creep, the lower mantle is unlikely to cause the rapid damping of the Chandler wobble. *Anderson and Archambeau* [1964] believe that most of the damping of body waves and of free oscillations takes place in the upper mantle. It is likely that the Chandler wobble and the body tides are also damped in the same region.

Thus it is important to decide whether the process is governed by the most or the least viscous part of a nonhomogeneous earth before any simplifications are made.

7. Polar Wandering and Convection

Paleomagnetic results require the magnetic pole, and hence the rotational pole, to have moved relative to each continent during geological time. If continents have drifted relative to each other and to the pole, polar wandering is hard to define. However, the high viscosities calculated here for the lower mantle will stabilize the pole, and continental drift may then take place on an earth whose lower mantle is fixed to the rotation axis.

If a small mass m is placed on the earth's surface at a latitude θ_0, the Chandler wobble will be excited and will take place about an axis inclined at an angle of $ma^2 \sin 2\theta_0/2(C - A)$ radians to the original axis of rotation. The decay time of the Chandler wobble, though very uncertain, is probably of the order of 30 years.

It is not known where the energy concerned is dissipated, but the least likely place is the lower mantle. The dissipation will take place within the layer that has the shortest decay time for an oscillating disturbance of the form $X_2^1(\theta, \phi)$ exp $i\omega t$. However, the rate of polar wandering will be determined by the layer which has the longest decay time to a constant disturbance of the form $X_2^0(\theta)$, and the same layer will support the nonhydrostatic bulge. If the time constant for polar wandering is τ and that for the lower mantle is τ_M, where

$$\tau_M = (\nu/ga)\tau_2$$

and τ_2 is as given by (34), then *Gold* [1955] finds

$$\tau \sim 2\tau_M(C - A)/ma^2 \sin 2\theta_0 \qquad (58)$$

The force acting on the mass in the direction of the equator is called the Eötvös force. Munk and MacDonald define the excitation function due to this force as

$$|\psi| = ma^2 \sin 2\theta_0/2(C - A) \sim \tau_M/\tau \qquad (59)$$

If $\nu \sim 4 \times 10^{25}$, then $\tau_M \approx 2 \times 10^7$ years and the pole will not wander in geological time ($\tau \sim 5 \times 10^9$ years) unless $|\psi| \sim 4 \times 10^{-3}$. A value of ν of $\sim 6 \times 10^{26}$ and $a \sim 5 \times 10^8$ cm requires $|\psi| \sim 5 \times 10^{-2}$. They compute that for the present distribution of oceans and continents the value of $|\psi|$ is approximately 10^{-4}, which is insufficient to produce polar wandering. It is likely that a larger value of $|\psi|$ will result from density variations within the mantle, but since these are likely to be thermal in origin they probably have a decay time short in comparison with that of polar wandering. It is clear that the Eötvös force on the continents is not sufficient to move the pole, but this force may move the continents separately toward the equator. Geology does not support movements in latitude only, however, and thus such a force is probably not important.

Unlike polar wandering, convection need not be prevented by a highly viscous lower mantle. It is very important to know whether the nonhydrostatic bulge is caused by a high viscosity throughout the mantle or by a thin layer somewhere within it, perhaps at the core-mantle boundary. If the lower mantle has a viscosity of about 10^{26} throughout, it will affect isostatic uplift in two ways. The viscosity as calculated by *Haskell*'s [1935] method will be a function of the radius of the areas [*McConnell*, 1965], and the surface movements will differ from those calculated from the simple theory of an infinite half-space. The calculations must be made with spherical shells.

Before convection can take place in the lower mantle, the temperature gradient must exceed the adiabatic by an amount β given by

$$\beta = \kappa\nu R_c/g\alpha(a_2 - a_1)^4$$

where R_c is the critical Rayleigh number, $\sim 2 \times 10^3$, κ the thermal conductivity $\sim$ 0.01 cal/°C sec, and α the thermal expansion $\sim 2 \times 10^{-5}$/°C. If $\nu \sim 6 \times 10^{26}$, the value of β required before convection can take place is $\sim$ 10°C/km, or a temperature difference across the lower mantle of about 20,000°C. The actual temperature difference is probably between 1000°C and 2000°C and is far too small to cause convection. The adiabatic gradient is 0.5°C/km, or a temperature difference across the lower mantle of 1000°C; thus the actual temperature gradient may not even exceed the adiabatic. Even a viscosity of 4×10^{25} stokes is quite sufficient to prevent convection.

Convection in the upper mantle is not affected by these calculations, nor is there any difficulty in convecting through the phase-change region if it is spread over about 500 km [*Verhoogen*, 1965].

It is interesting that this model partly explains the occurrence of earthquakes. Below a depth of 700 km, thermal stresses are removed by creep and do not accumulate because there is no convection. Above this depth, heat is transported by movement of the rock, generating shearing stresses and hence earthquakes.

8. Conclusion

These calculations depend on the measured value of U_2^0 [*King-Hele*, 1965] and on the value calculated by *Jeffreys* [1963] for a hydrostatic earth. If either of these is in error by 1%, the energy stored in the U_2^0 harmonic may be no greater than that in any other, and this method of estimating the viscosity of the lower mantle fails. However, there is no reason to doubt either the measurements or the calculations.

The three-layer model with an inviscid core and a phase change between the upper and lower mantle is the most realistic of those considered here, and it requires a viscosity of 6 × 10^{26} stokes in the lower mantle. All three models are sufficiently viscous to prevent convection in the lower mantle and to prevent polar wandering in geological time.

The viscosity estimated from postglacial uplift will be reduced if the lower mantle is highly viscous, and thus convection within the upper mantle, 900 km thick, is likely to take place.

The only argument in favor of convection in the lower mantle is that the scale of features on the earth's surface thought to be due to convection cells is about 3000 km. Convection in the lower mantle is then required if the cells are to be circular in cross section. However, circular cells are found only at Rayleigh numbers slightly above critical, about 2 × 10^3, and there is no evidence that this occurs in the upper mantle. Thus there is no difficulty in explaining continental drift and other surface features by convection in the upper mantle.

Appendix 1

Lamb [1881] solved the equations for viscous flow in a nonrotating sphere but used a method different from that used in section 4. He retained the term $\partial_\zeta \mathfrak{V}$ and solved

$$\mathfrak{J}^2\, \partial_\zeta \mathfrak{V} = \nabla^2 \mathfrak{V} + \mathfrak{J}^2 \nabla[\mathfrak{U} - \mathcal{P}] \qquad \nabla \cdot \mathfrak{V} = 0 \tag{A1}$$

These equations have a particular integral of the form

$$\mathfrak{V}_0 = \nabla \phi$$

provided that

$$\nabla^2 \phi = 0 \tag{A2}$$

and

$$\partial_\zeta \phi = \mathfrak{U} - \mathcal{P} \tag{A3}$$

When ϕ is calculated from these equations it does not satisfy the surface traction boundary condition, and thus the general solution to (A1) is required

$$\mathfrak{V} = \mathfrak{V}_0 + \mathfrak{V}_1$$

where

$$\nabla^2 \mathfrak{V}_1 = \mathfrak{J}^2\, \partial_\zeta \mathfrak{V}_1 \qquad \nabla \cdot \mathfrak{V}_1 = 0 \tag{A4}$$

Lamb looked for solutions of the form

$$\mathfrak{V} = \mathfrak{V} \exp(-\alpha \zeta)$$

In this case (A4) becomes

$$(\nabla^2 + k^2)\mathfrak{V}_1 = 0 \qquad k^2 = \alpha \mathfrak{J}^2 \qquad \nabla \cdot \mathfrak{V}_1 = 0 \tag{A5}$$

Morse and Fesbach [1953] show that the solutions to (A5) are

$$\mathfrak{V}_1 = \nabla \times \mathbf{a}_{\mathfrak{R}} \mathfrak{R} \psi + \frac{1}{k} \nabla \times [\nabla \times \mathbf{a}_{\mathfrak{R}} \mathfrak{R} \chi]$$

$$\psi, \chi = \sum \sum a_l^m j_l(k\mathfrak{R}) X_l^m(\theta, \phi)$$

where a_l^m are complex coefficients and $j_l(k\mathfrak{R})$ are spherical Bessel functions. The χ solution may be combined with the particular integral to satisfy the boundary conditions, but the ψ solution may not. Thus the ψ solution must satisfy the boundary conditions by itself, and it can be generated only by the initial velocity field and not by gravitational flow. The ψ solution corresponds to torsional elastic oscillations and toroidal magnetic fields; it can be written in terms of the vector spherical harmonics $\mathbf{C}_{ml}$ alone. The combined ϕ and χ solution contains both $\mathbf{P}_{ml}$ and $\mathbf{B}_{ml}$ and corresponds to spheroidal elastic oscillations and poloidal magnetic fields. These solutions should be compared with (29).

Thus the solution to (A1) is

$$\mathfrak{V} = \left[\nabla \phi + \frac{1}{k} \nabla \times (\nabla \times \mathbf{a}_{\mathfrak{R}} \mathfrak{R} \chi)\right] \cdot \exp(-\alpha \zeta) \tag{A6}$$

This solution is apparently of order 1, and that in section 4 is of order $\mathfrak{J}^2$. However, when the values of a_l^m are determined from the boundary conditions and an approximate expression is used for $j_l(k\mathfrak{R})$ when $k\mathfrak{R}$ is small, the terms of order 1 cancel in (A6) and leave only those of order $\mathfrak{J}^2$ the two solutions are then the same.

Appendix 2

The effect of von Zeipel's theorem on the external gravity field can be calculated from *Bullard*'s [1948] values for the ellipticity ϵ of the surfaces of constant density. His values, though for an isothermal earth, can easily be corrected for temperature. Clairaut's theorem then gives the change in external field. In the following calculation only the order of magnitude of the effect is given.

Since the earth is not isothermal, the density is a function of both radius r and the temperature

$$\rho = \rho(r, T)$$

The original isothermal surfaces of constant density are

$$r = y[1 - \tfrac{2}{3}\epsilon P_2(\theta)] \qquad \text{(A7)}$$

where y is the mean radius of the surface. On raising the temperature by δT, we can find the change in ellipticity of such a surface from

$$\delta\rho = 0 = \left(\frac{\partial \rho}{\partial r}\right)_T \delta r + \left(\frac{\partial \rho}{\partial T}\right)_r \delta T \qquad \text{(A8)}$$

If the conductivity and rate of heat generation are constant throughout the earth, and if $\partial T/\partial t$ is zero everywhere, it is easy to show that the isotherms have the same ellipticity as the earth's surface. These assumptions are not true within the earth but do permit an order of magnitude calculation. On a surface defined by (A7) the temperature variation is

$$\delta T = -\frac{2y^2}{3a} |\nabla T|_a [\epsilon(a) - \epsilon(y)] P_2(\theta) \qquad \text{(A9)}$$

where a is the mean external radius of the earth and $|\nabla T|_a$ is the surface temperature gradient. Equation A8 and the Adams-Williamson relationship give the change in r:

$$\delta r = \frac{2}{3} \frac{\alpha K_T}{\rho a g} |\nabla T|_a \, y^2 [\epsilon(a) - \epsilon(y)] P_2(\theta)$$

The surfaces of constant density are now $r + \delta r$, and Clairaut's theorem [*Jeffreys*, 1959] gives

$$J_2 - H_2 = -\frac{8\pi}{15Ma^2} \int_0^a \rho(y) \frac{|\nabla T|_a}{a} \frac{\partial}{\partial y} \cdot \left\{ \frac{\alpha K_T}{\rho g} y^6 [\epsilon(a) - \epsilon(y)] \right\} dy$$

Integration by parts gives

$$J_2 - H_2 = -\frac{8\pi}{15Ma^3} |\nabla T|_a \cdot \int_0^a \rho(y) \alpha y^6 [\epsilon(a) - \epsilon(y)] \, dy \qquad \text{(A10)}$$

The values of $|\nabla T|_a$ is $\sim$ 20°C/km. Numerical integration using *Bullard*'s [1948] values of $\epsilon(y)$ then gives

$$J_2 - H_2 = -1.6 \times 10^{-6}$$

Thus the nonhydrostatic potential due to this effect has the wrong sign and is an order of magnitude too small to explain the observed potential.

In this calculation the slow convection caused by the rotation, which *Verhoogen* [1948] believes to be important in orogenic processes, has been neglected. Full calculations are in progress, but it will be surprising if the simple calculation above is wrong.

Appendix 3

Wang [1966] has suggested that the nonhydrostatic bulge is a relic of the last glaciation. During the last ice age most of the ice was concentrated in polar regions and remained there long enough to become isostatically compensated. When the ice caps melted, the deformation remained and now causes the nonhydrostatic external gravity field. It is shown below that this effect would indeed produce a bulge of the right sign and order of magnitude. However, isostatic rebound has reduced the deformation by a factor of about 10 since the ice melted, so that this suggestion also gives a value which is an order of magnitude smaller than that observed.

The mass of ice which caused isostatic depression can be calculated from the change in sea level, d. Any floating ice will not depress sea level or deform the mantle. If this mass formed two polar ice caps, each with an angular radius of θ_0, their thickness t would be

$$t = d \cos\theta_0/(1 - \cos\theta_0) \qquad \text{(A11)}$$

If the ice caps and the change in sea level are completely compensated by flow in the mantle of density 3 g/cm³, the polar areas will be depressed by $t/3$, and the rest of the earth (assumed all to be ocean) will be uplifted by $d/3$. The deformed surface can be written as

$$r = a(1 + \sum C_l P_l(\theta))$$

Integration over the earth's surface gives

$$C_2 = -\tfrac{5}{6}((t + d)/a) \cos\theta_0 \sin^2\theta_0 \qquad \text{(A12)}$$

The resulting value of J_2 is given by (6):

$$J_2 = -\tfrac{3}{5}C_2$$
$$= ((t + d)/2a) \cos \theta_0 \sin^2 \theta_0$$
$$= d \cos \theta_0 \sin^2 \theta_0/2a(1 - \cos \theta_0) \qquad \text{(A13)}$$

Since $d \sim 10^4$ cm, $\theta_0 \sim 35°$, and $a = 6.4 \times 10^8$ cm,

$$J_2 \sim 1.2 \times 10^{-5}$$

The observed value is 1.05×10^{-5}, which is the same order of magnitude.

The external field has changed, however, since the ice melted. In the places where isostatic uplift has been measured [*Farrand*, 1962], only about a tenth of the glacial downwarp remains. It is likely that the same is true of Siberia and Antarctica, which also carried larger ice sheets in the last glaciation. Thus isostatic adjustments have reduced J_2 from $\sim 10^{-5}$ to $\sim 10^{-6}$, or an order of magnitude too small to explain the observations.

Acknowledgments. I am grateful to Sir Harold Jeffreys, Dr. J. W. Elder, Dr. A. Gill, and Mr. J. C. R. Hunt for advice on fluid mechanics, and to Sir Edward Bullard for continuous encouragement. Dr. W. H. Munk read the manuscript and made many helpful suggestions.

The work reported was in part supported by the U. S. Air Force grant SD/IGPP/AF49(638)-1388 (seismic) and in part by a scholarship from Shell Oil Company.

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(Manuscript received January 31, 1966.)