Mon. Not. R. astr. Soc. (1974) **168**, 603–637.

THE EVOLUTION OF VISCOUS DISCS AND THE ORIGIN OF THE NEBULAR VARIABLES

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(Received 1974 March 18)

SUMMARY

The evolution of discs under the action of viscosity is studied by both similarity solutions and Green's functions. The angular momentum is steadily concentrated onto a small fraction of the mass which orbits at greater and greater radii while the rest is accreted onto the central body.

We assume that the angular momentum excess of a proto-star is initially concentrated onto one-third of the total mass which forms a disc orbiting the new-born star. Viscous dissipation in this disc will cause it to shine with a luminosity greater than the final main sequence star for a period of 10^5 yr or so. Most of the properties of T Tauri stars can be explained as a consequence of disc evolution. Flares in Flare stars are interpreted as the entry of blobs of an old disc into the late type stellar atmospheres. On this hypothesis flaring activity could be observed in M stars of up to 5×10^8 yr old, and planetary systems will be common. Disc solutions appropriate to dwarf novae and X-ray sources are also given.

1.1 *Introduction*

Discs have played a large part in astrophysical thought ever since it was discovered that the solar system is flat. Saturn's ring and the shape of the Galaxies likewise added interest to such problems, while rings around Be stars, U Gem stars, accreting binary X-ray sources and possibly about black holes and quasars all make the subject especially topical.

Protostellar material on condensing from 10^{-24} g cm^{-3} will decrease its moment of inertia by a factor of 10^{16}. Were angular momentum conserved this would lead to ridiculous spin rates, so it has long been known that angular momentum must be lost. Here we consider initial situations in which some of this excess angular momentum has been stored in a massive disc which is left orbiting the star. We show that for young stars the dissipation in such a disc leads to its shining with a sizeable fraction of the total light of the system. The light from the disc consists of two parts:

(i) from the disc itself we predict a slowly variable infrared spectrum with approximate form* $F_\nu \propto \nu^{4/3}/(\exp(h\nu/kT)-1)$ which becomes $F_\nu \propto \nu^{1/3}$ for small ν, and

(ii) from the boundary layer where the material in the disc grazes and enters the surface of the star we predict a rapidly variable blue continuum, with emission peaking in the ultraviolet. These phenomena are typical of the T Tauri stars. We find that, in general, the smaller the mass of the underlying star, the longer the disc dominates the luminosity output from the system. In addition the flaring of flare

* Here and wherever it is explicitly mentioned ν is a frequency in the electromagnetic spectrum. However, elsewhere in the bulk of the paper ν is a kinematic viscosity.

stars can be readily understood as the entry of blobs of gas from an elderly disc into the stellar atmosphere. We estimate that the discs could produce flaring activity in M stars with the observed intensity for about 5×10^8 yr after formation.

We are encouraged to believe that most of the nebula variables can be explained in this way and it is possible that some of the infrared and ultraviolet colour anomalies in the Orion Be stars etc are similarly attributable to shining orbiting discs and their boundary layers. If this interpretation of T Tauri stars is correct, their study will provide new and important evidence on the conditions under which the planets in the solar system were formed.

Some years ago now (1) we worked out the basic similarity solutions for the evolution of time-dependent Newtonian discs under the action of viscosity but failed to find any solutions that evolved under their own self-gravity. However, in many of the more recent applications the self-gravity is negligible so these solutions are now of greater interest. This paper is an elementary one devoted to the behaviour of time-dependent discs and showing that, whatever the dissipation mechanism, the basic form of evolution is the expansion of the outermost parts to carry all the angular momentum together with the collection of an ever increasing fraction of the mass towards the centre. This process is much slower in systems of larger scale and this fact encourages us to see analogies between the present state of the Galaxy and a very much earlier stage of the solar system when that was a spinning disc of gas and dust.

Section 2 gives the mathematics of the similarity solutions for viscous discs and shows, by Green's, functions, that all discs evolve towards these similarity discs.

Section 3 applies the solutions to galaxies and the different types of star that may have equatorial discs surrounding them.

1.2 *Does dissipation lead to rigid rotation?*

We start our discussion with a well-known theorem which is perhaps the basis of the supposition that dissipation will lead to bodies in uniform rotation.

Theorem. For a given density distribution and total angular momentum the motion of least energy is uniform rotation.

Since the proof is very direct we give it:

The density being fixed the internal and gravitational energies are fixed so we have only to minimize the kinetic energy

$$T = \tfrac{1}{2}\int \mathbf{u}^2\rho\, dV = \tfrac{1}{2}\int \mathbf{u}^2\, dm.$$

The angular momentum is constrained to be H so

$$\int R u_\phi\, dm = H$$

where ϕ is azimuth measured about an axis through the mass centre and parallel to H and R is distance from that axis.

Define the moment of inertia I.

$$I = \int R^2 \rho\, dV = \int R^2\, dm.$$

By Schwartz's inequality

$$\int u_\phi^2\, dmI = \int u_\phi^2\, dm \int R^2\, dm \geqslant \left[\int Ru_\phi\, dm\right]^2 = H^2$$

where equality only holds down when $u_\phi \propto R$ and then

$$u_\phi = (H/I)\, R$$

which is uniform rotation.

Evidently

$$T = \tfrac{1}{2}\int \mathbf{u}^2\, dm \geqslant \tfrac{1}{2}\int u_\phi^2\, dm \geqslant \tfrac{1}{2}H^2/I$$

where the second equality only holds when all the motions are about the axis and the third when the motions further reduce to uniform rotation.

As a corollary of this theorem we see that all states of minimum energy must rotate uniformly whether or not the density is fixed; for if there were a non-uniformly-rotating configuration of minimum energy we could fix its density distribution and obtain an even lower energy state by constraining it to rotate uniformly with the same angular momentum. Thereby we prove that the original state was not a true minimum energy state but an imposter.

The theorem further suggests a ruse to help us find minimum energy states. Instead of varying the whole energy we may always take the rotation energy to be $\frac{1}{2}H^2/I$ and the angular velocity to be H/I where I is the moment of inertia of the configuration considered. The minimum energy problem is then reduced to minimization with respect to all possible density distributions since I is known once ρ is known.

Notice that the theorem does not require that the gravitational energy arises from the body itself rather than from external sources.

By contrast with the above theorem consider what will happen to a disc, rotating in centrifugal force versus gravity balance, when there is a small viscosity to dissipate the energy. In the astronomically important cases the rotation is such that the angular velocity $\Omega(R)$ decreases outwards while the specific angular momentum $h = R^2\Omega$ increases outwards. Since Ω decreases outwards the inner parts of the disc shear past the outer parts, so the viscosity causes a couple. To the outer parts of the disc (that already have large specific angular momentum) more angular momentum is given, and from the inner parts (that have little) is taken away even that which they have. On acquiring their gain the outer parts rather than increasing their angular velocity and losing their tactical advantage increase their size instead and thereby actually decrease their angular velocity. Likewise the inner parts can only increase their angular velocity, consequent on their shrinkage as a result of angular momentum loss. Thus the shearing is not reduced as a result of the angular momentum flow.

At first sight it seems obvious that this system is not moving towards any uniformly rotating state because the difference between the angular velocities of the parts is accentuated by the angular momentum transference. How do we square this with the idea that since energy is dissipated by the viscosity the system ought to be moving towards a minimum energy state?

The theorem also lies somewhat incongruously with a similar theorem that for a star of fixed angular momentum about the galactic centre the orbit of least energy is circular. At first sight this would seem to imply a differential rotation for a system of stars, for in the galaxy circular motions of different radii have different periods.

To elucidate this problem let us consider the minimum energy state of two particles of masses m_1 and m_2 in the fixed potential of a galaxy, ψ.

Many times we shall wish to refer to the angular momentum per unit mass and the energy per unit mass. These names are too long so we propose the terms 'specific angular momentum' and 'specific energy' for them. For a particle of specific angular momentum h the specific energy may be written in the form

$$\tfrac{1}{2}(u_R{}^2+u_z{}^2)+\tfrac{1}{2}h^2R^{-2}-\psi(R, z) = \epsilon$$

where ψ has a maximum for given R on $z = 0$.

The minimum value of ϵ for given h is attained with $u_R = u_z = 0$ and $z = 0$ in the circular orbit of radius R_h which is the value of R at which

$$\tfrac{1}{2}h^2R^{-2}-\psi(R, 0)$$

attains its minimum value, $\epsilon(h)$.

Thus R_h is the solution of $(\partial/\partial R)\{\tfrac{1}{2}h^2R^{-2}-\psi(R, 0)\} = 0$.

The minimum specific energy is in obvious notation

$$\epsilon(h) = \tfrac{1}{2}h^2R_h{}^{-2}-\psi(R_h, 0) = \tfrac{1}{2}V^2-\psi.$$

We shall presently need the quantity $d\epsilon/dh$. Since R_h is a function of h we would expect to vary both h and R_h in the expression for $\epsilon(h)$, but the defining equation for R_h shows that $\epsilon(h)$ is locally stationary for variations of R_h with h fixed. Thus

$$\frac{d\epsilon}{dh} \equiv \epsilon'(h) = \partial\epsilon(h)/\partial h = hR_h{}^{-2} = \Omega$$

where Ω is the angular velocity around the circular orbit.

We wish to minimize the energy of our two particles, keeping the total angular momentum constant. Clearly we may first minimize the energy of each separately, keeping its angular momentum constant, and only then consider whether the energy may be further lowered by exchange of angular momentum. Thus the two particles may be taken to move in circles and their energy will be

$$E = m_1\epsilon(h_1)+m_2\epsilon(h_2)$$

and their angular momentum will be

$$H = m_1h_1+m_2h_2.$$

Now consider a small change in the angular momentum keeping the total H constant

$$dE = m_1\, dh_1\epsilon'(h_1)+m_2\, dh_2\epsilon'(h_2)$$

where

$$m_1\, dh_1+m_2\, dh_2 = 0$$

hence

$$dE = m_1\, dh_1[\epsilon'(h_1)-\epsilon'(h_2)] = m_1\, dh_1(\Omega_1-\Omega_2).$$

Thus energy can be reduced by exchanging angular momentum in such a way that the orbit of least angular velocity gains angular momentum. Since in practical cases Ω decreases outwards this means that the energy is lowered if the angular momentum flows outwards. It is likely that the ultimate driving energy for the spiral structure of galaxies arises because the spiral structure can transfer angular momentum outwards and lower the energy of the whole configuration (2).

To get a fuller knowledge of the minimum energy states we now consider the minimum problem when only the sum of the masses of the two particles is fixed. If a mass dm_1 is transferred from m_2 in circular orbit to m_1 while the orbits of m_1 and m_2 are changed to keep H constant, then the change in energy is

$$dE = d[m_1\epsilon(h_1)+m_2\epsilon(h_2)]$$

where

$$dm_1 = -dm_2$$

and

$$dH_1 = d(m_1h_1) = -dH_2 \equiv d(m_2h_2).$$

Using these relationships

$$dE = dm_1[\epsilon(h_1)-h_1\epsilon'(h_1)]+d(m_1h_1)\ \epsilon'(h_1)+(\text{similar terms in 2})$$

thus

$$dE = dm_1\{[\epsilon(h_1)-h_1\Omega_1]-[\epsilon(h_2)-h_2\Omega_2]\}+dH_1(\Omega_1-\Omega_2).$$

Now $\epsilon(h)-h\Omega$ increases outwards because

$$\frac{d}{dR}(\epsilon(h)-h\Omega) = \frac{d}{dR}(-\tfrac{1}{2}V^2-\psi) = -V\left(\frac{dV}{dR}-\frac{V}{R}\right) = -RV\frac{d}{dR}\left(\frac{V}{R}\right) > 0.$$

Not only is the energy lowered by moving the angular momentum outwards towards smaller Ω but also by moving mass inwards towards smaller values of $\epsilon - h\Omega$.

Thus the minimum energy configuration is a limit in which one particle of infinitesimal mass carries all the angular momentum in a circular orbit at infinity while all the remaining mass aggregates at the centre. This conclusion is reached on energy and angular momentum arguments alone. In the next section we give some specific models of frictional evolution of discs which elucidate detailed behaviour. Here we remark that although we fixed the gravity field in the above calculations nevertheless the final configuration arrived at with almost everything at the middle is the configuration of the greatest possible binding energy for a self-gravitating system; so this configuration must be the state of least energy for such systems too.

We may summarize this discussion by saying that energy dissipation will in the long term transfer angular momentum outwards but while the radius-outside-which-half-the-angular-momentum-is-stored will increase, the radius containing half the mass will normally decrease.

1.3 *Angular momentum transport by shearing stresses*

Consider what will happen to a disc, rotating in centrifugal force versus gravity balance, when there is a small friction to dissipate the energy. We shall assume that the velocities in the disc are independent of height above the central plane. We shall further assume that the frictional force per unit length of circumference at radius R is dependent on the local rate of shear and the density so that the couple g exerted on the stuff outside R by the stuff inside R can be written

$$g = R2\pi R\nu\sigma 2A \quad (1)$$

where $A(R)$ is the local rate of shearing

$$A = -\tfrac{1}{2}R\, d\Omega/dR \tag{2}$$

Ω is the angular velocity about the centre, and $\nu(R)$ is the friction coefficient per unit surface density $\sigma(R)$. ν is not assumed constant; it might depend on R, A, σ or time but it will be assumed to be positive. Notice that had the law been the law of viscous friction then ν would be kinematic viscosity.

An element of mass $dm = 2\pi R\sigma\, dR$ which lies initially between R and $R+dR$ will normally have its angular momentum increased by the couple from the faster rotating material inside it, and will have its angular momentum decreased by shearing past the material further out. Thus if we write

$$h = \Omega R^2$$

then

$$\frac{D}{Dt}(h\,\delta m) = -\delta R\,\frac{\partial g}{\partial R}$$

where $D/Dt = \partial/\partial t + \mathbf{u}.\nabla$ is the rate of change following the elements of the fluid.

Since $D\,\delta m/Dt = 0$ we deduce

$$2\pi R\sigma\frac{Dh}{Dt} = -\frac{\partial g}{\partial R} = \frac{\partial}{\partial R}\left(2\pi R^3\sigma\nu\frac{\partial\Omega}{\partial R}\right). \tag{3}$$

This equation tells us where the angular momentum is deposited. Consider for instance the outer parts of the disc. If the disc is mainly supported by centrifugal force then Ω will be decreasing there. Furthermore the couple must tend to zero at the edge of the disc so g will be positive and decreasing. We therefore deduce that the outer parts of the disc, those with the most specific angular momentum, will secularly get still more angular momentum as a result of friction. The gain in angular momentum leads to movement to greater R and a resultant net loss of angular velocity. A similar effect causes the central regions to rotate more rapidly on loss of the angular momentum except where the system is mainly pressure supported. While in the latter case uniform rotation may be achieved, the configuration normally consists of a slowly rotating central mass which is pressure supported, plus a small satellite at a large distance carrying most of the angular momentum in an orbit so large that it is only traversed at the same slow rotation rate. In the special case when the gravity field is fixed and h is a function of R, but not t, then equation (3) takes the simpler form

$$F\frac{dh}{dR} = -\frac{\partial g}{\partial R} = -\frac{\partial}{\partial R}(4\pi R^2\nu\sigma A) \tag{4}$$

where F is the outward flux of material through radius R. A special solution that will concern us later is the steady state with F constant. That solution is given by

$$\begin{aligned} g &= (-F)\,h + g_0 \\ \sigma &= (4\pi\nu)^{-1}\,[-F(\Omega/A) + g_0/(R^2A)]. \end{aligned} \tag{5}$$

Notice that: (1) when $g_0 = 0$, F must be negative so material flows into the origin, (2) when F is positive then a central couple of at least $g_0 > Fh_{\max}$ is needed to drive a steady flux F out to a sink at that R where $h = h_{\max}$.

ENERGY FLOW

For the general case it is instructive to look at the equation of energy flow. We derive this by taking the equations of motion in the form

$$\rho \frac{D\mathbf{u}}{Dt} = \operatorname{div} \boldsymbol{s} + \rho \boldsymbol{\nabla} \psi \tag{6}$$

multiplying by u, and integrating them over the time dependent volume $\int d\tau$ occupied by a chosen mass of fluid.

In the above the constant mass of fluid considered is $M = \int \rho \, d\tau$, $\boldsymbol{s}$ is the stress tensor describing both the pressure and the viscous stresses, and ψ is the (positive) gravitational potential. The stress tensor $\boldsymbol{s}$ is given by

$$\boldsymbol{s} = -p\boldsymbol{I} + 2\eta(\boldsymbol{e} - \tfrac{1}{3}e\boldsymbol{I}) + \zeta e \boldsymbol{I}$$

where $\boldsymbol{I}$ is the unit tensor with components δ_{ij}, $\boldsymbol{e}$ is the rate of straining tensor $e_{ij} = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ and e is its trace which is also div $\mathbf{u}$. η is the coefficient of shear viscosity and ζ is the coefficient of bulk viscosity.

Performing the integration over τ we find the energy equation:

$$\frac{D}{Dt}\left(\int \frac{1}{2} \mathbf{u}^2 \, dm\right) = \int \mathbf{u}.(\boldsymbol{\nabla}.\boldsymbol{s}) \, d\tau + \int (\mathbf{u}.\nabla) \, \psi \, dm$$

where

$$dm = \rho \, d\tau.$$

Now

$$\frac{D}{Dt}\left(\int \psi \, dm\right) = \int \frac{\partial \psi}{\partial t} \, dm + \int (\mathbf{u}.\nabla) \, \psi \, dm,$$

and

$$\begin{aligned}\int \mathbf{u}.(\boldsymbol{\nabla}.\boldsymbol{s}) \, d\tau &= \int \nabla.(\boldsymbol{s}.\mathbf{u}) - (\boldsymbol{s}.\nabla).\mathbf{u} \, d\tau = \int (\boldsymbol{s}.\mathbf{u}).d\mathbf{S} - \int \boldsymbol{s} : \boldsymbol{e} \, d\tau \\ &= \int (\boldsymbol{s}.\mathbf{u}).d\mathbf{S} + \int [p \operatorname{div} \mathbf{u} - 2\eta(\boldsymbol{e} - \tfrac{1}{3}e\boldsymbol{I}) : \boldsymbol{e} - \zeta e^2] \, d\tau \\ &= \int (\boldsymbol{s}.\mathbf{u}).d\mathbf{S} - \int \frac{p}{\rho}\frac{D\rho}{Dt} \, d\tau - \int [2\eta(\boldsymbol{e} - \tfrac{1}{3}e\boldsymbol{I}) : (\boldsymbol{e} - \tfrac{1}{3}e\boldsymbol{I}) + \zeta e^2] \, d\tau.\end{aligned}$$

These terms are (1) the rate at which viscous and pressure forces are conveying energy into our chosen volume, (2) the rate at which the internal energy is being increased by increased compression, and (3) the rate of viscous dissipation of energy. Substituting these expressions in our energy equation it becomes

$$\frac{D}{Dt}\left[\int (\tfrac{1}{2}\mathbf{u}^2 - \psi) \, dm\right] = \int (\boldsymbol{s}.\mathbf{u}).d\mathbf{S} + \int p \frac{D(1/\rho)}{Dt} \, dm - \int D_1 \, d\tau - \int \frac{\partial \psi}{\partial t} \, dm$$

where D_1 is the viscous dissipation rate per unit volume.

To apply this equation to our disc it is useful to put the equation in terms of the couple g of the inside on the outside. This is in the $\hat{z}$ direction

$$g\hat{z} = -\int \mathbf{R} \times \boldsymbol{s}.d\mathbf{S}$$

where the integration extends over the surface of a cylinder of radius concentric with the axis. Since $d\mathbf{S}$ is the parallel to $\mathbf{R}$ it is simple to see that the isotropic part of s does not contribute to the integral and so

$$g = -2\hat{z}.\int \mathbf{R}\times(\eta e).d\mathbf{S} = -2\pi\int_{-\infty}^{\infty} R^3\eta\,\frac{\partial\Omega}{\partial R}\,dz$$

where $\Omega = u_\phi/R$ is the angular velocity of the fluid about the axis. Notice that this equation agrees* with equation (1) with $\nu\sigma = \int \eta\,dz$. In cylindrical coordinates we likewise find

$$\int (s.\mathbf{u}).d\mathbf{S} = 2\pi\int_{-\infty}^{\infty} Ru_R[-p+\operatorname{div} u(\zeta-\tfrac{2}{3}\eta)]\,dz + 2\pi\int_{-\infty}^{\infty}\Omega R^3\eta\,\frac{\partial\Omega}{\partial R}\,dz$$

$$D_1 = \left\{2\eta\left[\left(\frac{\partial u_R}{\partial R}\right)^2+\left(\frac{u_R}{R}\right)^2+\frac{1}{2}\left(R\,\frac{\partial\Omega}{\partial R}\right)^2\right]+(\zeta-\tfrac{2}{3}\eta)(\operatorname{div}\mathbf{u})^2\right\}$$

if $\partial u_R/\partial R$ is negligible compared with Ω then div $\mathbf{u}$ is likewise very small. If further we take the velocities to be independent of height above the central plane of the disc at least over the bulk of the matter then

$$\int D_1\,d\tau = 2\pi\int\!\!\int \eta R^3\left(\frac{\partial\Omega}{\partial R}\right)^2 dz\,dR = -\int g\,\frac{\partial\Omega}{\partial R}\,dR$$

finally our energy equation for the ring currently between R_1 and R_2 is

$$\frac{D}{Dt}\left[\int\!\!\int(\tfrac{1}{2}\mathbf{u}^2-\psi)\,dm\right] = \left[-2\pi Ru_R\int p\,dz - g\Omega\right]_{R_1}^{R_2} + \int_{R_1}^{R_2} g\,\frac{\partial\Omega}{\partial R}\,dR - \int\frac{\partial\psi}{\partial t}\,dm + \int p\,\frac{D(1/\rho)}{Dt}\,dm. \quad (7)$$

The terms involving p describe the work done on this bit of the disc by pressure and the rate of change of the internal energy. The terms that concern us are rather the expression for the dissipation $\int g(-\partial\Omega/dR)\,dR$ and the convection of energy $g\Omega$ due to the viscous couple. When p is negligible and the flow is steady

$$\frac{D}{Dt}\int(\tfrac{1}{2}\mathbf{u}^2-\psi)\,dm = \int 2\pi R\sigma u_R\,\frac{\partial}{\partial R}(\tfrac{1}{2}\mathbf{u}^2-\psi)\,dR = F\left[\frac{u^2}{2}-\psi\right]_{R_1}^{R_2}$$

and so equation (7) reads

$$-F\left[\frac{\mathbf{u}^2}{2}-\psi\right]_{R_1}^{R_2} = [g\Omega]_{R_1}^{R_2} + \int_{R_1}^{R_2} D2\pi R\,dR \quad (8)$$

the left-hand side is the energy generation caused by the material flowing into the gravitational well; this energy is convected by $g\Omega$ and dissipated by D, the frictional dissipation rate per unit area. The equation of angular momentum flow can be derived simply, by operating with $\hat{z}.\int r\times$(equation (6)) $d\tau$ however this yields exactly

$$\frac{D}{Dt}\int h\,dm = g_1 - g_2$$

which is equivalent to equation (3) once we assume velocities independent of height.

* When $\partial\Omega/\partial R$ is independent of z.

2.1 Application to steady discs

For the steady state disc with no central couple the energy source is

$$\tfrac{1}{2}F\frac{\partial\psi}{\partial R} = \tfrac{1}{2}(-F)\,V\Omega$$

while the energy dissipated is

$$2\pi RD(R) = (-F)\,V\left(-R\frac{\partial\Omega}{\partial R}\right) = -FV2A.$$

The difference is caused by the energy transport due to the viscous couple.*

A more realistic steady state disc has a central body of radius $R = R_*$ which rotates more slowly than break-up speed at the equator. Since the disc will be going faster than the stellar equator a boundary layer will form in which the viscosity slows down the disc to the equatorial velocity. Thus Ω, which was increasing inwards, now decreases to that defined by the star's equator. This decrease can only occur in the region in which centrifugal force no longer balances gravity, that is in the viscous boundary layer. We define the critical point $R = R_c$ as that radius at which $\partial\Omega/\partial R = 0$. By integrating our exact equation (4) and using equation (1)

$$g = 2\pi R^3\nu\sigma\frac{\partial\Omega}{\partial R} = (-F)\,h+g_0$$

$\partial\Omega/\partial R$ changes sign in the boundary layer and this will be very close to the star's surface, so just outside the star we have $g = 0$. At this critical point we must have therefore $R = R_c$, $g_c = 0$ and hence $(-F)\,h_c+g_0 = 0$. Where a suffix c indicates the values at the critical point. Now starting just outside the boundary layer and with R a little greater than R_c, h will have the value given by centrifugal force gravity balance. The boundary layer is thin and Ω continues to increase inwards down to R_c, so h_c will be very close to, although very slightly less than, the value that would be given by centrifugal force gravity balance. This is in turn negligibly different from the value h_* that would produce centrifugal force gravity balance in a star-grazing orbit. Thus with an error that vanishes with the viscosity we may take

$$g_0 = -(-F)\,h_*$$

this gives us for our general solution

$$g = (-F)(h-h_*).$$

This argument depends critically on the star rotating more slowly than the disc; otherwise it is indeed untrue! Thus when a magnetosphere is being rotated so fast that the disc is being dragged around by the star this solution is not the correct one.

* For a Newtonian point mass $\Omega \propto R^{-3/2}$ and so $2A = \tfrac{3}{2}\Omega$. Thus the power liberated at radii between R and $R+dR$ is three times larger than the energy generated there. In such a model there is actually a large viscous transport of energy $g\Omega$, out of the origin, that makes up the difference. $g\Omega$ vanishes at $r = 6m$ for a Schwarzschild black hole so in that case $g\Omega$ only redistributes the same *total* power. Although the power in the outer parts of the disc is three times greater than that generated locally this is compensated by a slightly smaller power close to the Schwarzschild mouth. These phenomena were inadvertently omitted from the relativistic discussion in (3). For solutions that are fully realistic down to the origin the energy flux $g\Omega$ out of the origin should vanish.

Notice that g vanishes at the critical point so $g\Omega$ vanishes and there is no flux of energy coming out of the critical point into the disc. Thus the $g\Omega$ term merely redistributes the energy generated in the disc outside the critical point. In particular if a point mass provides the gravity field then

$$\tfrac{1}{2}V^2 = \tfrac{1}{2}GM/R, \quad h = (GM)^{1/2}R^{1/2}, \quad \Omega = (GM)^{1/2}R^{-3/2}$$

we write equation (8) in the differential form

$$-F\frac{\partial}{\partial R}(\tfrac{1}{2}u^2-\psi) = \frac{\partial}{\partial R}(g\Omega)+2\pi RD$$

and so we find

$$(-F)\frac{\partial}{\partial R}\left(-\frac{GM}{2R}\right) = \frac{\partial}{\partial R}\left\{(-F)\frac{GM}{R}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]\right\}+2\pi RD.$$

Thus the dissipation D between R and $R+dR$ is

$$D = (-F)\frac{3}{4\pi}\frac{GM}{R^3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right] \quad \text{per unit area.} \tag{9}$$

Furthermore since $g\Omega$ vanishes both at R_* and at infinity the total integral of $(\partial/\partial R)(g\Omega)$ is zero, as it should be for any term that merely redistributes the energy and does not make any. The surface density and couple in this steady solution are given by equation (5) with $g_0 = -(-F)(GMR_*)^{1/2}$. Although the couple acting through the critical radius R_c is zero, nevertheless the star does acquire the accreted angular momentum and mass of the material falling on it. Thus the rate of increase of the star's mass and angular momentum are $(-F)$ and $(-F)\,h_*$ respectively. Further, in the transition through the boundary layer considerably more energy is dissipated through the viscosity during what in space-age parlance would be called ' entry ' into the atmosphere. The total energy dissipated must be such as to bring the material down from orbit to rest on the star rotating with angular velocity Ω_0. Thus the rate of energy dissipation in the boundary layer is approximately

$$(-F)\left[\frac{1}{2}\frac{h_*^2}{R_*^2}-\frac{1}{2}(\Omega_0 R_*)^2\right] = \frac{1}{2}(-F)\left(\frac{GM}{R_*}\right)\left[1-\frac{\Omega_0^2 R_*^3}{(GM)}\right].$$

Notice that for $R \gg R_*$ the above solution reduces to the $g_0 = 0$ solution obtained by setting $R_* = 0$.

2.2 *Time-dependent examples of interest*

The general nature of the evolution of a flat disc under friction is the expansion of the outer part and the contraction of the main body, as we have discussed. Nevertheless it is interesting to have some mathematical examples of this process. Readily solvable problems of this nature are obtained when, as in the Galaxy, the gaseous material subject to friction is not the main source of the gravity field. We may then consider frictional evolution in a fixed gravitational field.

Equation (4) may be re-written

$$2\pi R\sigma u_R = F = -\frac{\partial g}{\partial h} \tag{10}$$

where u_R is the slow radial drift velocity brought about by the friction. The continuity equation for the fluid reads

$$\frac{\partial \sigma}{\partial t}+R^{-1}\frac{\partial}{\partial R}\left(\frac{F}{2\pi}\right) = 0. \tag{11}$$

If we substitute for F and σ in terms of g by using equations (1) and (10) we find

$$\frac{\partial}{\partial t}\left[\frac{g}{(2A\nu R)}\right]-\frac{\partial}{\partial R}\left(\frac{\partial g}{\partial h}\right) = 0$$

that is

$$\frac{\partial^2 g}{\partial h^2} = \frac{\partial}{\partial t}\left[\frac{g}{\left(2A\nu R\frac{dh}{dR}\right)}\right]. \tag{12}$$

For further progress we need to know the behaviour of $2A\nu R(dh/dR)$. In this we shall be guided by two important applications.

(1) When the gravity field is that of a point mass M then

$$\Omega = (GM)^{1/2}\,R^{-3/2}, \quad A = -\tfrac{1}{2}R\,\frac{d\Omega}{dR} = \tfrac{3}{4}\Omega, \quad h = (GMR)^{1/2}$$

and hence

$$2A\nu R\,\frac{\partial h}{\partial R} = \tfrac{3}{4}GM\,\frac{\nu}{R} = \tfrac{3}{4}(GM)^2\,\nu h^{-2}.$$

(2) When the gravity field is that corresponding to Mestel's law (4) of galactic rotation $V = V_0$

$$\Omega = \frac{V_0}{R}, \quad A = \frac{\tfrac{1}{2}V_0}{R}, \quad h = V_0 R$$

and hence

$$2A\nu R\,\frac{dh}{dR} = V_0{}^2\nu.$$

Both these special cases give

$$2A\nu R\,\frac{dh}{dR} \propto h^{-\delta}$$

whenever ν is constant or varies as a power of R. We shall adopt the form

$$2A\nu R\,\frac{dh}{dR} = 4l^2\kappa^{-2}h^{2-1/l} \tag{13}$$

where κ and l are constants and the precise form in which it is written is chosen for later convenience. All power-law rotation curves $V \propto R^\gamma$ are included in the form (13) and we do not need ν to be independent of position but only that it vary as some power of h. For ν constant the above special cases have $l = \tfrac{1}{4}$, $\kappa^{-2} = 3(GM)^2\,\nu$ for the point mass and $l = \tfrac{1}{2}$, $\kappa^{-2} = V_0{}^2\nu$ for the $V = V_0$ case.

Using equation (13), equation (12) takes the form

$$\frac{\partial^2 g}{\partial h^2} = \tfrac{1}{4}\left(\frac{\kappa}{l}\right)^2 h^{1/l-2}\,\frac{\partial g}{\partial t}. \tag{14}$$

Method of solution for the time dependent case. We resolve equation (14) into modes in which the variation of g with time is $g \propto e^{-st}$. On writing $k^2 = \kappa^2 s$ we find for such a mode

$$g'' + \tfrac{1}{4}\left(\frac{k^2}{l^2}\right) h^{1/l-2} g = 0. \quad (15)$$

If we now set $x = h^{1/2l}$ and $g_1 = x^{-l} g$ we find

$$\frac{\partial^2 g_1}{\partial x^2} + x^{-1}\frac{\partial g_1}{\partial x} + \left(k^2 - \frac{l^2}{x^2}\right) g_1 = 0.$$

Thus equation (15) is a transformation of Bessel's equation and its solution is

$$g = e^{-st}(kx)^l \left[A(k)\, J_l(kx) + B(k)\, J_l(kx)\right]$$

where as before $x = h^{1/2l}$.

The general solution of equation (14) is a superposition of such modes so

$$g = \int_0^\infty \exp\left(-\frac{k^2 t}{\kappa^2}\right)(kx)^l \left[A(k)\, J_l + B(k)\, J_l\right] dk. \quad (16)$$

Once this solution has been fitted to the boundary conditions one may recover the flux $F = -\partial g/\partial h$, the surface density $\sigma = g/(4\pi R^2 A\nu)$ and the radial velocity $u_R = F/(2\pi R\sigma)$.

The elementary solutions under the boundary condition that g vanishes at $x = x^*$ are

$$\left[J_l(kx) - \frac{J_l(kx^*)}{J_{-l}(kx^*)} J_{-l}(kx)\right](kx)^l \exp - \left(\frac{k^2 t}{\kappa^2}\right)$$

and the general solution under this boundary condition is a super-position of these. Any distribution over the range $x > x^*$ can be analysed into such a superposition initially, the $A(k)$ determined from the initial condition, and the whole temporal behaviour follows from expression (16).

The solutions for which x^* is the origin have $B(k) \equiv 0$ and are therefore a little more amenable to handle. We use the Fourier–Bessel theorem to analyse the given initial distribution into modes. The theorem states that

$$\int_0^\infty \int_0^\infty A(k)\, J_\nu(kx)(kx)^{1/2}\, dk J_\nu(k'x)(k'x)^{1/2}\, dx = A(k').$$

Hence applying this to equation (16) with $t = B(k) = 0$ we find

$$\int_0^\infty g(h, 0)\, J_l(k'x)(k'x)^{1-l}\, dx = A(k'). \quad (17)$$

Substituting these $A(k)$ back into equation (16) yields the solution for $g(h, t)$. Certain special forms of $g(h, 0)$ give particularly simple solutions that we can analyse in detail; we begin by such a study and later turn to Green's function solution of the general case.

2.3 *Special solutions with no central couple*

Consider the case when $g(h, 0) = Ch \exp(-ah^{1/l}) = Cx^{2l} \exp - ax^2$ then applying equation (17) and using ((**30**) formula 8.6–10)

$$A(k) = \int_0^\infty C \exp(-ax^2)\, J_l(kx)\, k^{1-l} x^{1+l}\, dx = Ck(2a)^{-(l+1)} \exp - \left(\frac{k^2}{4a}\right).$$

Returning to equation (16) with these $A(k)$, and $B(k) \equiv 0$, we have

$$g(h, t) = \int_0^\infty \exp\left[-k^2(t\kappa^{-2}+\tfrac{1}{4}a^{-1})\right] k^{l+1}x^l J_l(kx)\, dk C(2a)^{-(l+1)}.$$

The integral is in essence the same as the last one so we obtain

$$g(h, t) = CT^{-(l+1)}x^{2l} \exp -\left(\frac{ax^2}{T}\right),$$

where $T = 4a\kappa^{-2}t+1$ which is a dimensionless time. In terms of the physical variables this solution is

$$\left.\begin{aligned} g &= CT^{-(l+1)}h \exp\left(-\frac{ah^{1/l}}{T}\right), \\ -F &= \frac{\partial g}{\partial h} = CT^{-(l+1)} \exp -\left(\frac{ah^{1/l}}{T}\right)\left[1-\frac{ah^{1/l}}{(lT)}\right], \\ \sigma &= \frac{g}{(4\pi R^2\nu A)}, \\ u_R &= \frac{F}{(2\pi R\sigma)} = -\left(\frac{\partial \log g}{\partial h}\right)(2R\nu A) = -\frac{2R\nu A}{h}\left\{1-\frac{ah^{1/l}}{(lT)}\right\}. \end{aligned}\right\} \quad (18)$$

For the special case of the point mass constant ν solutions we have $l = \frac{1}{4}$, $\kappa^{-2} = 3(GM)^2\,\nu$, $h^{1/l} = (GMR)^2$ so the above solutions become

$$\left.\begin{aligned} g &= CT^{-5/4}(GMR)^{1/2} \exp\left[-\frac{a(GMR)^2}{T}\right], \\ -F &= CT^{-5/4} \exp\left[-\frac{a(GMR)^2}{T}\right]\left\{1-\frac{4a(GMR)^2}{T}\right\}, \\ \sigma &= \frac{CT^{-5/4}}{3\pi\nu} \exp\left[-\frac{a(GMR)^2}{T}\right], \\ u_R &= -\tfrac{3}{2}\nu R^{-1}\left\{1-\frac{4a(GMR)^2}{T}\right\}, \end{aligned}\right\} \quad (18')$$

where $T = 12(GM)^2\,\nu a t+1$. This solution is illustrated in Fig. 1.

We see from the formulae for F and u_R that the flux and velocity are outwards for $h > (lT/a)^l$ and inwards inside that radius. This point of velocity reversal moves out as T increases and overtakes more and more material. Furthermore we see that near the centre $g \simeq -Fh$ so that the solution has the same form as the steady state solution with the outward flow of angular momentum due to the couple balanced by the angular momentum convected inwards with the material.

The angular momentum distribution in this solution is interesting. The total angular momentum within R is

$$H = \int_0^R 2\pi R\sigma h\, dR = \int_0^{h(R)} \frac{g h\, dh}{2R\nu A \dfrac{dh}{dR}} = \frac{\kappa^2}{4l^2}\int_0^h g h^{1/l-1}\, dh \quad (19)$$

$$H = C\int_0^{x(R)} \frac{\kappa^2}{4l^2} T^{-(l+1)}x^2 \exp -\left(\frac{ax^2}{T}\right) 2lx^{2l-1}\, dx = \frac{k^2 C}{4la^{l+1}}\int_0^{h(R)} y^l \exp(-y)\, dy$$

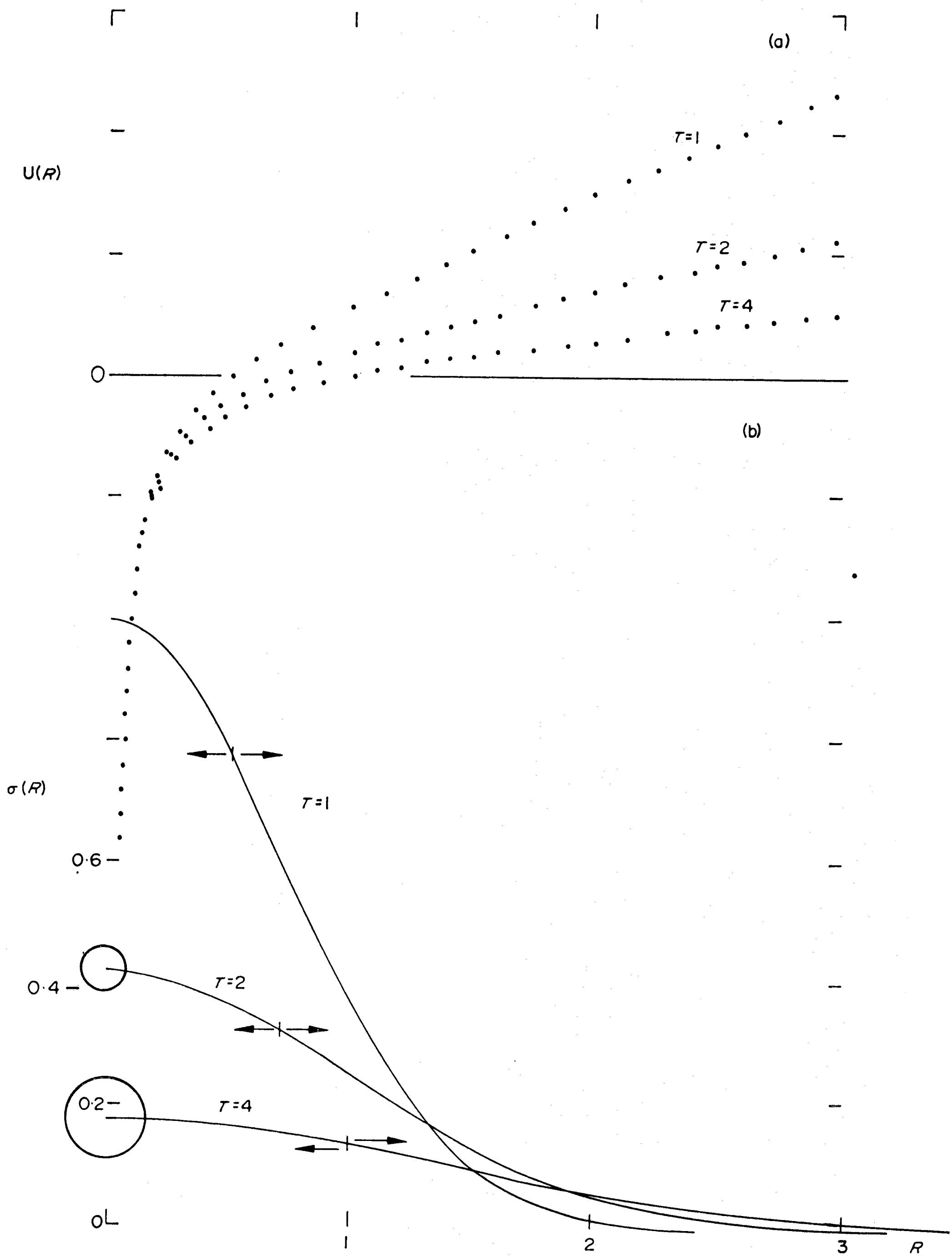

FIG. 1. *Radial velocity, u, and surface density, σ, in the disc as functions of radial distance from the centre for the similarity solutions with no central couple. The unit of distance is $\sqrt{2}R_1$. The distributions are plotted out three times and the spheres at $R = 0$ have volumes corresponding to the mass that has fallen from the disc onto the central object.*

where

$$y = \frac{ax^2}{T} = \frac{ah^{1/l}}{T}. \tag{20}$$

This solution is a similarity solution, for the distribution of H with y remains invariant. For the point mass solution this variable $\propto R^2/T$. Notice that the points with y constant move outwards in time and that the angular momentum between two such rings remain constant. Similarly the mass between two specified values of y will be

$$M_{12} = \frac{\kappa^2 C}{4lT^l a}\int_{y_1}^{y_2} \exp(-y)\,dy = \frac{\kappa^2 C}{4laT^l}\{\exp(-y_1)-\exp(-y_2)\}. \qquad (21)$$

Notice that the total mass in the disc decreases like T^{-l} but that its distribution spreads out such that corresponding points have $h \propto T^l$. The decrease in mass is accounted for by the mass flux into the origin

$$-F(0) = CT^{-(l+1)} = -\frac{dM_D}{dt}$$

where C is the initial flux onto the star and

$$M_D = \frac{\kappa^2}{4la} CT^{-l}.$$

While the 'wave form' of the density expands the material in the central parts is moving inwards and draining into the origin. Thus the real matter distribution is a growing central point mass surrounded by a density distribution which grows in size but decreases in mass. A particular case is illustrated as Fig. 1.

As we have seen, the point of radial velocity reversal moves outwards overtaking the material. Thus material that starts inside will move towards the origin whereas material that starts outside will begin by moving outwards only to be overtaken (if it is not on the extreme edge). Once overtaken it moves inwards and ends in the ever-hungry sink at $R = 0$. The trajectories of material particles illustrate the behaviour rather well. If we follow the trajectory of a single particle we have $\dot{h} = (dh/dR)\,u_R = -4l^2\kappa^{-2}h^{1-1/l}\{1 - ah^{1/l}/lT\}$ that is

$$\frac{a}{l}\frac{dh^{1/l}}{dT} = -\left\{1 - \frac{ah^{1/l}}{lT}\right\},$$

which integrates to give

$$h^{1/l} = \left(\frac{T}{a}\right)[ah_0^{1/l} - l\log T] \qquad (22)$$

this gives the paths traced out by the particles as a function of time. They turn around when

$$h = \left(\frac{lT}{a}\right)^l \quad \text{(see Fig. 2).}$$

The convected energy flux is

$$g\Omega = CT^{-(l+1)}\Omega h \exp -\left(\frac{ah^{1/l}}{T}\right) \qquad (23)$$

while the energy dissipated per unit area is $D = (2\pi R)^{-1} g(-\partial\Omega/\partial R)$ and so

$$D = (4\pi R^2)^{-1}\, CT^{-(l+1)}Ah \exp -\left(\frac{ah^{1/l}}{T}\right). \qquad (24)$$

The effective temperature of the disc, i.e. the temperature it would have if it radiated like a black body, is $[D/(2\sigma_0)]^{1/4}$ where σ_0 is Stefan's constant. For a disc about a point mass $Ah \propto R^{-1}$ and so this temperature is proportional to $R^{-3/4}$

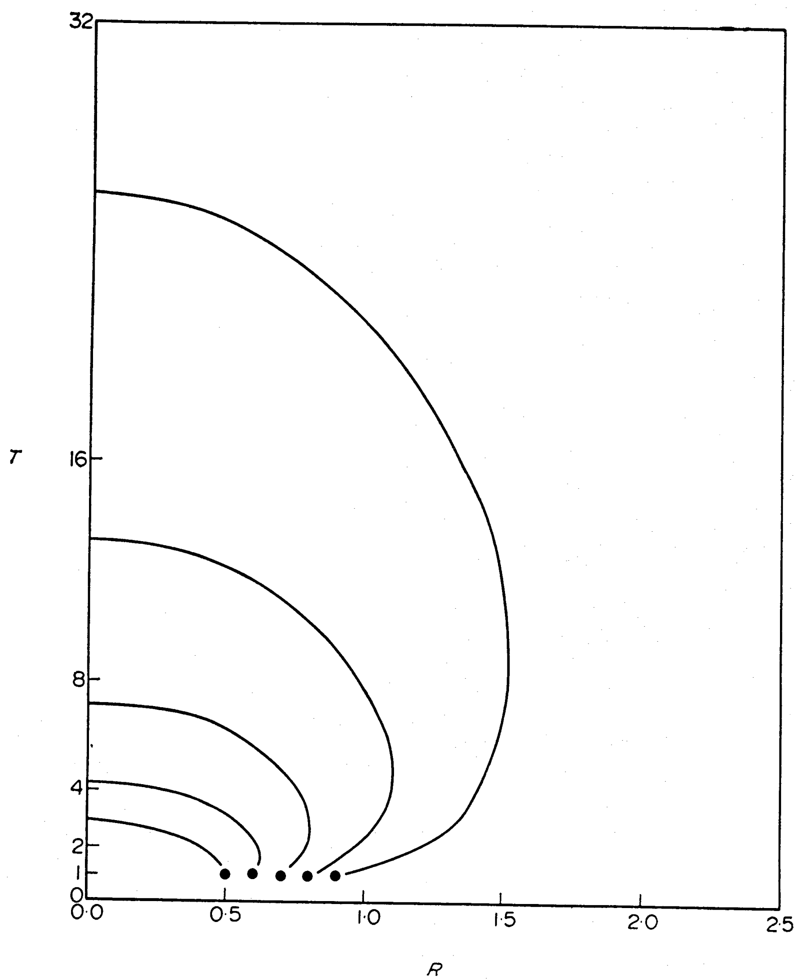

FIG. 2. *Trajectories of different fluid elements of the disc as functions of time. The unit of* R *is* $\sqrt{2}R_1$.

just as for the steady discs, provided we restrict ourselves to the energy producing region at small R.

Approximate solution with a central star (Fig. 3). We have seen that in the central regions the solution has the steady state structure for zero couple. We also know that the steady state structure for a central star only differs from the $g_0 = 0$ solution near the centre. We deduce that a good approximation to the time dependent solution with a central star will be found by modifying the solution in its central region to agree with the known form of the steady solution there. Thus a good approximate solution will be

$$g(h, t) = CT^{-(l+1)}(h-h_*) \exp - \left(\frac{ah^{1/l}}{T}\right) \tag{25}$$

provided $ah_*^{1/l}/T$ is small. Once again $D = (2\pi R)^{-1} g(-d\Omega/dR)$.

2.4 *General solution with no central couple* (Figs 4 and 5)

Our equation (12) for g is linear so different solutions may be superposed. Dr Toomre pointed out to us that properties of the general solution are best studied by use of Green's functions. Any solution may be thought of as made up of

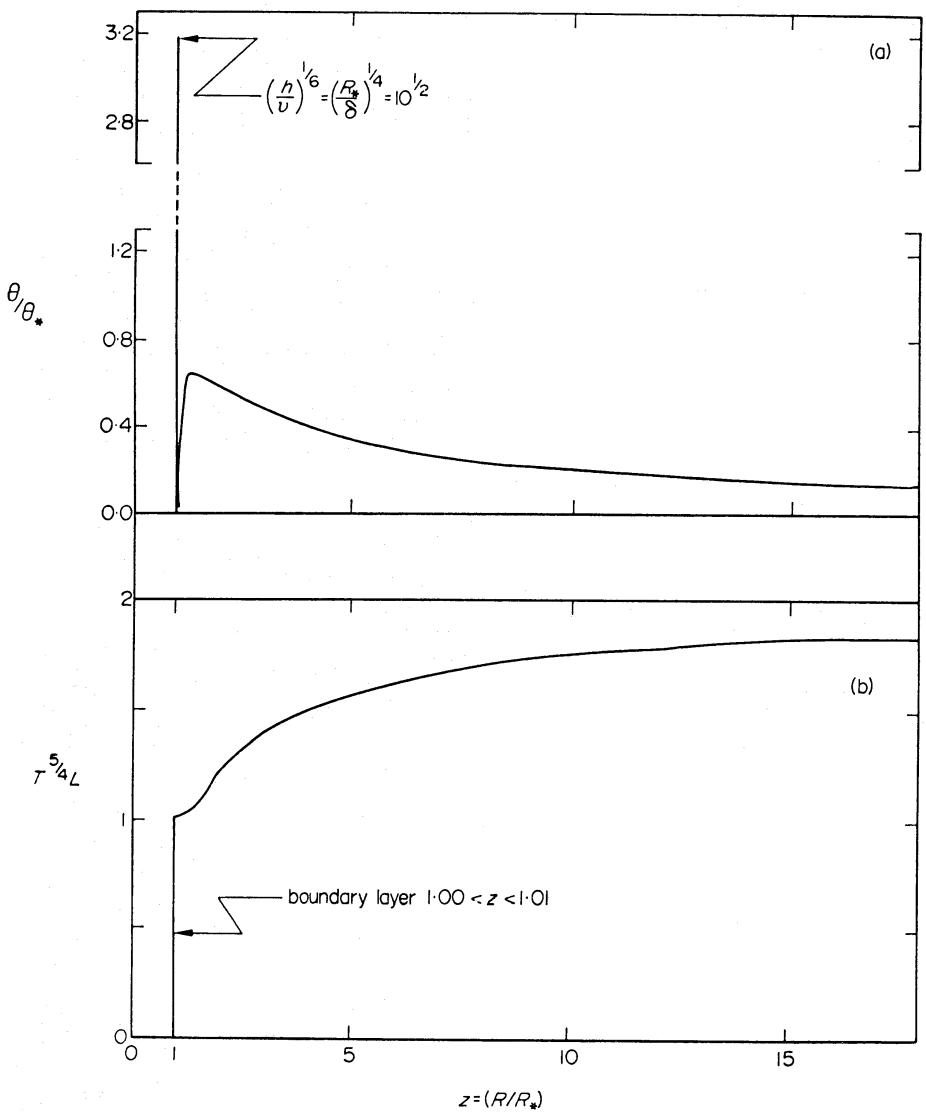

Fig. 3. (*a*) *Black body temperature, θ, in the disc as a function of radius measured in terms of the central stars radius. Both diagrams are drawn for a boundary layer corresponding to* $\nu = 10^{-3}\, h_*$. θ_* *is the reference temperature defined by* $\sigma\theta_*^4 4\pi R_*^2 = \frac{1}{2}GM\dot{M}/R_*$. *For the similarity solutions with no central couple* $\dot{M}$ *varies with time like* $T^{-5/4}$ *and so* $\theta_* \propto T^{-5/16}$. (*b*) *The luminosity produced by the disc and boundary layer within radius R. The unit of* $T^{5/4}\,L$ *is the initial luminosity of each of the disc and the boundary layer. Both diagrams are drawn with the maximum boundary layer luminosity* (*that which is achieved for a very slowly rotating central star*).

elementary solutions whose initial density distributions are of the form

$$\sigma(R_1, R, 0) = (2\pi R_1)^{-1}\,\delta(R-R_1)$$

where δ is Dirac's δ function. The normalization has been chosen to make the total mass unity. We now find how such elementary solutions behave with time. The general solution is a superposition of these with positive coefficients. The couple corresponding to our initial σ is

$$g(h_1, h, 0) = 2R\nu A\left(\frac{dh}{dR}\right)\delta(h-h_1) = 4l^2\kappa^{-2}h_1^{2-1/l}\delta(h-h_1) = 2l\kappa^{-2}x_1^{2l-1}\delta(x-x_1)$$

$g(h_1, h, t)$ follows on using equation (17) to determine $A(k)$ and (16) for g

$$A(k) = 2l\kappa^{-2}x_1{}^l k^{1-l} J_l(kx_1)$$

so from (16)

$$g = 2l\kappa^{-2}x_1{}^l x^l \int_0^\infty k \exp - (k^2\kappa^{-2}t)\, J_l(kx_1)\, J_l(kx)\, dk.$$

The integral may be found in Tables of Hankel transforms and so

$$g = 2l\kappa^{-2}\frac{(x_1x)^l}{x_1{}^2T_*}\exp - \left(\frac{(x_1-x)^2}{2T_*x_1{}^2}\right)\mathscr{I}_l\left(\frac{x/x_1}{T_*}\right) \tag{26}$$

where $T_* = 2\kappa^{-2}t/x_1{}^2$ and $\mathscr{I}_\nu(z) = (\exp - z)\, I_\nu(z)$ and $I_\nu(z)$ is the Bessel function of imaginary argument. If the actual initial distribution is $g(h_1, 0)$ then at later times

$$g(h, t) = \int_0^\infty g(x^{2l}, 0)\frac{x_1{}^{-1-l}x^l}{T_*}\exp - \left[\frac{(x_1-x)^2}{2T_*x_1{}^2}\right]\mathscr{I}_l\left(\frac{x/x_1}{T_*}\right) dx_1 \tag{27}$$

where $x = h^{1/2l}$ as previously.

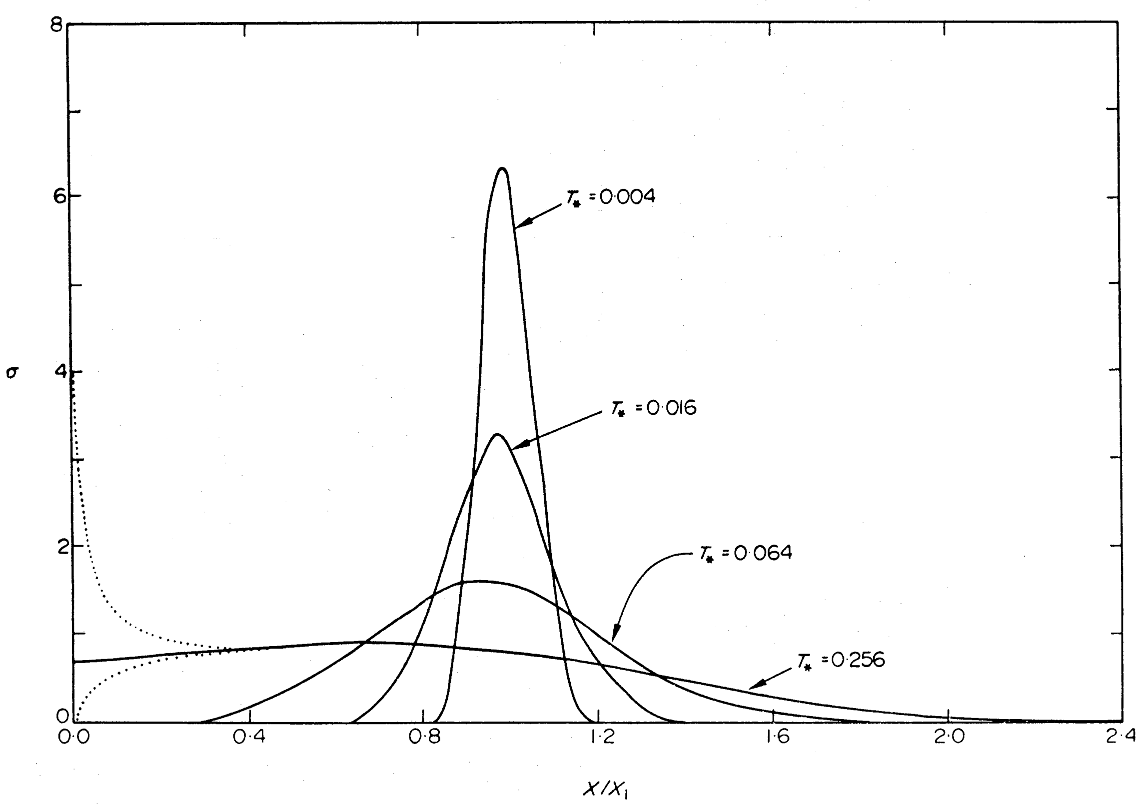

FIG. 4. *The surface density distribution with radius at four times for the δ function initial distribution of the disc. The lower dotted modification corresponds to a central star whose radius is* 10^{-2} *in these units, while the upper modification corresponds to the solution in which a no-central-flux boundary condition is imposed, corresponding to the throwing off of the disc by a strong magnetosphere.*

The function $\mathscr{I}_l(z)$ has the form

$$\mathscr{I}_l(z) = e^{-z}(\tfrac{1}{2}z)^l \sum_{k=0}^{\infty} (\tfrac{1}{4}z^2)^k/[k!(l+k)!]$$

hence for

$$\begin{Bmatrix} x/x_1 \ll T_* \\ 1 \ll T_* \end{Bmatrix}$$

the elementary solution has the form

$$g = (2)^{1-l}\, l\kappa^{-2} x_1^{-1} \left(\frac{x}{x_1}\right)^{2l} T_*^{-(l+1)} \exp - \left(\frac{x^2}{2T_* x_1^2}\right)$$

which has precisely the form of our special solution. Hence after sufficient time each elementary solution approaches the form of our special solution. Since those in turn have steady state form near the origin we see that all solutions attain that form there.

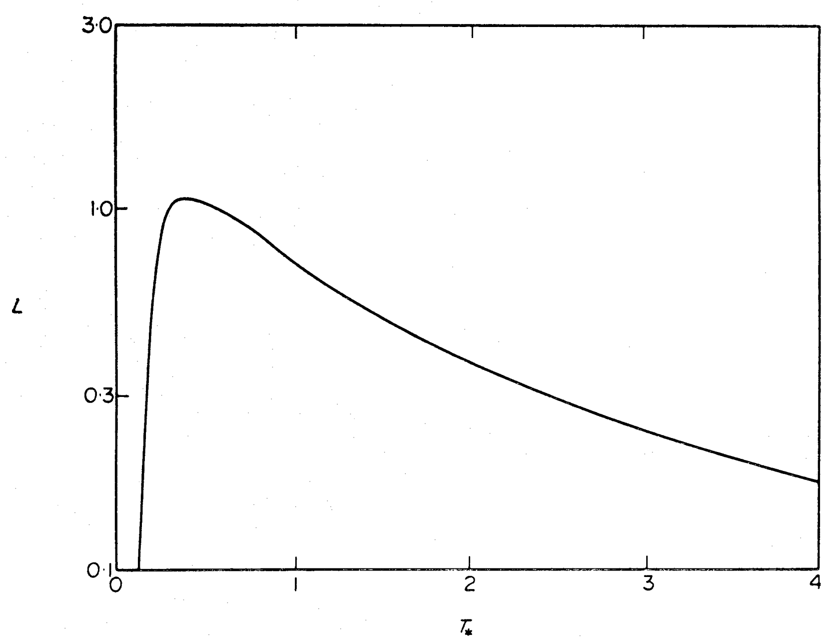

FIG. 5. *The luminosity of the disc as a function of time for the δ function initial condition in the no central couple case. The boundary layer contributes an equal luminosity. Compare the rise and fall with the luminosity curves of U Gem variables for which the unit of T is about 0·5 day. For long times the solution approaches the similarity solution of Figure 1 with* $L \propto T^{-5/4}$.

2.5 *Solution with no central flux*

In our discussion of X-ray sources we shall find it important to consider the situation in which the corotating magnetosphere is sufficiently strong to resist the influx of matter. The relevant solutions are those with no flux of matter onto the central body. Such solutions are approximately those with $\partial g/\partial h = 0$ at $h = 0$. The solutions of equation (15) with this behaviour are the J_{-l} solutions, the J_l solutions must be omitted (we have $0 < l < 1$). The $B(k)$ may be found because equation (17) remains true when J_l is replaced by J_{-l} and $A(k')$ by $B(k')$.

Analogously to our special solution (18) we deduce the similarity solutions with no central flux to be

$$\left.\begin{aligned} g &= g_1 T^{l-1} \exp\left[-\frac{ax^2}{T}\right] = g_1 T^{l-1} \exp\left[-\frac{ah^{1/l}}{T}\right] \\ F &= -g'(h) = g_1 a h^{1/l-1} T^{l-2} \exp\left[-\frac{ah^{1/l}}{T}\right] \\ \sigma &= \frac{g}{(4\pi R^2 \nu A)} \\ u_R &= \frac{F}{(2\pi R\sigma)} = -2R\nu A \frac{\partial \log g}{\partial h} = 2R\nu A \frac{ah^{1/l-1}}{lT} \end{aligned}\right\} \quad (28)$$

where $A = -\frac{1}{2}R(d\Omega/dR)$ and g_1 is a constant.

(Note that other special solutions may be obtained by differentiating with respect to ' a ' but we shall not discuss them as they are not similarity solutions.) Returning to the solution above and using equation (19) we have the total angular momentum within R

$$H = g_1 \frac{\kappa^2}{(4l^2)} \int_0^x T^{l-1} \exp\left(-\frac{ax^2}{T}\right) 2lx\, dx$$

$$= \kappa^2 g_1 \frac{T^l}{(4la)} \int_0^{y(R)} \exp(-y)\, dy$$

from which one checks that $\dot{H} = g_0$, also the total mass between R_1 and R_2 is

$$M_{12} = \frac{\kappa^2 g_1}{4l} a^{l-1} \int_{y_1}^{y_2} y^{-l} \exp(-y)\, dy$$

which is constant if R_1 and R_2 are chosen to change so that y_1 and y_2 are both constant. Note $y = ax^2/T = ah^{1/l}/T$. This demonstrates the similarity form of the solution.

The convected energy flux is of course $g\Omega$, while the energy dissipated per unit area is

$$D = (4\pi R^2)^{-1} g_1 T^{l-1} A \exp\left(-\frac{ah^{1/l}}{T}\right), \quad (29)$$

and the effective temperature in each ring is $(\frac{1}{2}D/\sigma_0)^{1/4}$ where σ_0 is Stefan's constant.

It is also worth recording here the corresponding Green's function solutions for the no central flux case. They are for unit mass in the disc

$$g = 2l\kappa^{-2} \left(\frac{x}{x_1}\right)^l T_*^{-1} \exp\left[-\frac{\frac{1}{2}(x_1 - x)^2}{(T_* x_1^2)}\right] \mathscr{I}_{-l}\left[\frac{x}{(x_1 T_*)}\right] \quad (30)$$

where we use the notation defined under the analogous solution (26). Taking the limit of small x we find $g_0 = 2^{l+1} l\kappa^{-2} (x_1^2 T_*)^{l-1} [\Gamma(1-l)]^{-1} \exp -\frac{1}{2}T_*^{-1}$. This couple starts small and rises to a maximum where $T_* = 1/2(l-1)$ when it achieves the value

$$g_{\max} = 4l\kappa^{-2} \left[\frac{x_1^2}{(l+1)}\right]^{l-1} [\Gamma(1-l)]^{-1} \exp -(l-1).$$

3. APPLICATIONS

3.1 *Galaxies*

The main difficulty in applying the above considerations to galaxies is the calculation of the frictional mechanism. In application to the Galaxy molecular viscosity in the gas is negligibly weak so any evolution of this type will be due to turbulent eddy viscosity or to some form of magnetic friction. The theory of both these is still in a lamentable state so only the crudest of estimates can be made. Some check is obtained from observations of the velocities of interstellar clouds and of the galactic magnetic field in the solar neighbourhood.

(i) *Eddy viscosity.* Although there is considerable random motion of cosmic clouds, both in the solar neighbourhood and elsewhere, it is not at present clear what determines the magnitudes of these motions and whether they are at all related to the rate of shearing caused by the differential rotation. It seems most likely that these random motions are associated with star formation and as we have no clear idea of that subject we cannot write down equations that determine the amount of random motion as a function of distance from the galactic centre. However, it is possible to obtain an idea of this velocity dispersion from the 21 cm line observations.

It is likely that galaxies suffer some weak large scale gravitational instabilities but it is not clear that these can lead to the small scale motions that are observed. Such could be the case through the spiral-shocks discussed by Roberts (**5**).

The formula for the kinematic viscosity may be deduced from that given by Jeans (**6**) and we find

$$\nu = \tfrac{1}{3}\bar{c}l_1$$

where l_1 is the mean free path of a cosmic cloud (assumed to be independent of its velocity) and $\bar{c}$ is the mean speed of a cloud. For cosmic clouds collisions will be inelastic so it is reasonable to take l_1 to be independent of c. For clouds filling a fraction f of the volume the mean free path is approximately a/f where a_1 is the radius of a cloud. In the neighbourhood of the Sun we find typical values of the order of $a_1 \sim 10$ pc, $f \sim \frac{1}{10}$ and for the velocity dispersion in one direction 5 km s^{-1}. This should be approximately $(\pi/8)^{1/2}\,\bar{c}$ so $\bar{c} \sim 8$ km s^{-1}. Thus

$$\nu = \tfrac{1}{3}\times 8\times 100 = 250 \text{ km s}^{-1}\text{ pc} = 8\times 10^{25}\text{ cm}^2\text{ s}^{-1}.$$

In solution (18) for σ we see that significant evolution has taken place when $T = 2$ that is $t = \frac{1}{4}\kappa^2/a$.

If to fix ideas we take a constant kinematic viscosity then approximating the galaxy's rotation by V = constant that is $l = \frac{1}{2}$, we find $a^{-1/2}$ is the initial angular momentum scale and the time for things to happen is $\frac{1}{4}a^{-1}V_0^{-2}\nu^{-1}$.

$$\text{For} \quad (a^{1/2}V_0)^{-1} = 1 \text{ kpc} \quad \text{we have} \quad t = 10^9 \text{ yr.}$$

$$\text{For} \quad (a^{1/2}V_0)^{-1} = 10 \text{ kpc} \quad \text{we have} \quad t = 10^{11} \text{ yr.}$$

Thus the effective viscosity of interstellar clouds has important effects on small scale distributions close to the galactic centre but probably does not grossly affect things close to the Sun. It is not unlikely that $\bar{c}$ actually increases towards the galactic centre as was deduced indirectly by Schmidt (**7**) and directly for the expanding arm by Rougoor (**8**). Their data are fitted within the errors by $\bar{c} \propto R^{-1}$ this implies $\nu \propto R^{-1} \propto h^{-1}$ and $l = \frac{1}{3}$. If we write $R_1 = a^{-1/3}/V_0$ so that R_1 is the initial length scale of the density distribution then the condition of significant evolution $T > 2$, $t > \kappa^2/4a$ gives $t > 4 \times 10^7 R_1^3$ yr where R_1 is in kpc. For $t = 10^{10}$ yr this condition is satisfied for

$$R_1 \leqslant 6 \text{ kpc}.$$

We deduce that frictional evolution of the distribution of the interstellar gas, although negligible in the outer parts of the Galaxy under present conditions, is likely to be important in the inner parts. At 100 pc from the centre it might be important on the 10^4 to 10^5 yr time scale. This conclusion would be seriously modified if the whole central region rotated uniformly, but the observational evidence of Rougoor and Oort does not show uniform rotation. If Becklin and Neugebauer's results at 2·2 μm are interpreted as a stellar density distribution then within 10 pc of the centre (**9**) the circular velocity is only weakly dependent on r, varying like $r^{0.2}$ approximately, assuming that there is no black hole at the centre.

Although on first inclination one might assume that both the angular velocity and the surface densities approach some uniform values at the centre, we show in Appendix II that such a situation is actually unstable in the presence of viscosity. We have earlier demonstrated that this secular instability is also realized in the absence of viscosity, provided that there is a magnetic field. We deduce that the angular velocity is unlikely to become uniform at the centre so that something closer to Mestel's law $V = V_0$ is likely there. Under these circumstances viscosity will be important in forming a central mass.

When a galaxy is first formed there will be much more gas about and both the scale and the speed of non-circular motions must be much larger than those we find here today, since it is unlikely that galaxies are made in any carefully balanced equilibrium. As a result the effective viscosity will be greater yielding a shorter time scale than those we have calculated and the surface density of gas will be larger leading to the generation of larger fluxes and a larger central mass. We have suggested elsewhere that quasars and galactic nuclei arise in this way (**10**).

(ii) *Magnetic friction.* With the magnetic field frozen into the interstellar gas almost any shearing motion will increase the magnetic energy. The shearing will then be opposed by magnetic forces which will depend on the detailed configuration of the magnetic field. Only when the direction of shearing is along the magnetic field lines, can this be avoided. Observed fields are not so directed, so we must consider what will happen. Presumably the field will be stretched, and therefore amplified, until it is able to react back strongly enough to break out of the configuration that leads to its continual amplification. Thereafter a rather chaotic cycle of amplification followed by breaking out might well continue. We have not found a satisfactory way of analysing such a situation with any accuracy, although rough estimates were given in (**3**) and (**10**). Likewise we have not been able to find suitable small scale experiments to investigate such processes.

3.2 *Discs around stars*

A number of interesting questions may be investigated using our basic solutions:

(a) What is the behaviour of the material left around a star after it has formed. Here solution (25) is relevant, provided that any magnetic field of the star is dominated by the weight of material left around it. The disc spreads to accommodate the angular momentum but the inner parts of the disc lose angular momentum and fall onto the star.

(b) A ring of material with specific angular momentum h_1 is deposited about a white dwarf or neutron star by its companion in a binary system. What is the time dependence of the ensuing radiation? There are two cases according as the material finds its way down onto the star or is expelled by a magnetic torque. The first case corresponds to solution (26) and the second to (30).

(c) If the magnetic field is so strong that it dominates over the material, then all the material within the corotation point will fall on the star on the timescale of free fall while the remainder will be expelled following the solutions with no central flux. The no central flux solutions (28) and (30) are appropriate to the expulsion phase. If however the magnetic dominance ever fails at the corotation point, then material will again get into the region where the magnetic field slows it downward this will lead to further accretion into the star.

To apply any of our basic solutions we must have a value for the kinematic viscosity ν. As is usually the case in astrophysics ordinary molecular viscosity is inadequate to effect any substantial evolution over the age of the stars. However, the Reynolds numbers $\Omega_* R_*^2/\nu$ which are involved are very large when ν is the molecular viscosity. This suggests that pure smooth Keplerian flow is likely to be disturbed by turbulent eddies. We suggest that in most fluid dynamical problems of this type these eddies are fed with energy until the eddy viscosity reaches that value at which the effective Reynolds number is reduced to the critical one for the onset of turbulence. Further increase in the eddies would increase the viscosity so much that the tendency to create further large scale eddies would be cut off. On the other hand any lesser value will lead to a flow which has a tendency to create further turbulence. Thus we take $\Omega_* R_*^2/\nu_{\text{eddy}} = \mathscr{R}_c$ the critical Reynolds number.

From experience in other fluid dynamical problems it is reasonable to take $\mathscr{R}_c \simeq 10^3$. This gives the eddy viscosity $\nu = 10^{-3}\, h_*$. We shall hereafter work with this value of ν but we shall take care to mention how changes in the adopted value of ν will affect the results.

3.3 *T Tauri and flare stars*

Well-known problems connected with the shedding of angular momentum during star formation, and the formation of planets lead one to expect that stars form with excess angular momentum which may well be initially stored in discs around them. Before tackling the problem of the evolution of such discs it is convenient to gather together the disc formulae appropriate to solution (25) in a form convenient for application.

Formulae:

Let R_1 be the radius corresponding to the standard deviation of the gaussian density distributions at $T = 1$, the initial time. Then in equations 18′ and 25 we have

$$a = \tfrac{1}{2}(GMR_1)^{-2},$$

and assuming $R_1 \gg R_*$ equation (21) gives the mass of the disc★

$$M_D = \tfrac{2}{3} R_1^2 \nu^{-1} C T^{-1/4} = M_1 T^{-1/4}$$

★ A more accurate formula is derived by integrating $2\pi \int \sigma R\, dR$ from R_* to ∞.

$$M_D = \tfrac{2}{3} R_1^2 \nu^{-1} C T^{-1/4} [1 - 1{\cdot}23\, Y^{1/2} + \tfrac{1}{3} Y^2 - \tfrac{1}{14} Y^4 + \tfrac{1}{66} Y^6 - \ldots]$$

where

$$Y^2 = \frac{R_*^2}{2 R_1^2 T}.$$

if we write M_1 for the initial disc mass. We have

$$C = \tfrac{3}{2}\nu M_1 R_1^{-2}, \quad T = 12(GM)^2 \nu a t + 1 = 6\nu t R_1^2 + 1$$

the flux onto the central star is

$$(-F)_* = \dot{M} = -\dot{M}_D = \tfrac{3}{2} M_1 R_1^{-2} \nu T^{-5/4} \rightarrow \frac{M_1}{4.6^{1/4}} \nu^{-1/4} R_1^{1/2} t^{-5/4} \quad \text{for } T \gg 1.$$

From equations (1) and (25) the density outside the boundary layer is given by

$$\sigma = \frac{M_1 T^{-5/4}}{2\pi R_1^2}\left(1 - \left(\frac{R_*}{R}\right)^{1/2}\right) \exp\left[-\frac{1}{2T}\left(\frac{R}{R_1}\right)^2\right],$$

and the inward velocity by

$$u_R = -\frac{3}{2}\frac{\nu}{R}\left\{\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]^{-1} - 2\frac{R^2}{R_1^2 T}\right\}; \quad R_* \ll R_1.$$

The energy liberated in the boundary layer is $\tfrac{1}{2}(GM\dot{M}/R_*)[1 - \Omega_0^2 R_*^3/GM]$ per unit time. For a central star rotating at significantly less than break up speed the square bracket is close to unity, and for the sake of definiteness we consider this case. Then the power generated in the boundary layer is $\tfrac{1}{2}GM\dot{M}/R_*$ and provided $R_1 \gg R_*$ the power generated in the rest of the disc

$$\int_{R_*}^{\infty} D 2\pi R dR$$

is the same. In fact D is the same as for the steady disc case except for $R \gtrsim R_1$. Since little power comes from those regions we may write

$$D = \frac{3}{4\pi}\frac{GM\dot{M}}{R^3}\left(1 - \left(\frac{R_*}{R}\right)^{1/2}\right).$$

This dissipated energy heats the disc which then radiates on both sides with an effective temperature $\theta(R)$ equal to $((D/2\sigma_0)^{1/4})$ where σ_0 is Stephan's constant. Thus

$$\frac{\theta(R)}{\theta_*} = 3^{1/4}\left[1 - \left(\frac{R}{R_*}\right)^{-1/2}\right]^{1/4}\left(\frac{R}{R_*}\right)^{-3/4} \quad \text{(see Fig. 3)}$$

where the reference temperature

$$\theta_* = \left(\frac{GM\dot{M}}{8\pi R_*^2 \sigma_0}\right)^{1/4} = \left(\frac{G\rho_* \dot{M}}{6\sigma_0}\right)^{1/4} = \left(\frac{L_D}{4\pi R_*^2 \sigma_0}\right)^{1/4}.$$

L_D is the total luminosity of the disc excluding the boundary layer. When the optical depth through the disc is greater than unity the actual temperature of the radiation is close to $\theta(R)$. Notice that when the disc radiates the same amount of light as the central star then $L_D = L_*$ so θ_* is then the effective temperature of the star. In this case the disc is significantly cooler than the star, but, as we should expect, the boundary layer's temperature θ_b is hotter. The greatest temperature in the main disc is $0{\cdot}595\ \theta_*$ at $R = \frac{49}{36} R_*$. To calculate its spectrum we take each part of the disc to radiate like a black body of temperature $\theta(R)$. The spectral distribution per factor θ in frequency is then

$$\lambda F_\lambda = 4\pi^2 \hbar c^2 \lambda^{-4} \int_{R_*}^{\infty} \frac{4\pi R\, dR}{\exp\left[\dfrac{2\pi\hbar c}{k\lambda\theta(R)}\right] - 1}.$$

Writing the integral in terms of $X = R/R_*$ and $\lambda_* = (2\pi\hbar c/3^{1/4}k\theta_*)$ we obtain*

$$\lambda F_\lambda = \left\{8\pi^2 R_*{}^2 c^2 \frac{3k^4\theta_*{}^4}{(2\pi\hbar)^3}\right\}\left(\frac{\lambda_*}{\lambda}\right)^4 \int_1^\infty \frac{X\,dX}{\exp\left[\frac{\lambda_*}{\lambda}(1-X^{-1/2})^{-1/4}X^{3/4}\right]-1}.$$

Using the full expression for Stephan's constant and the definition of θ_* the curly bracket is related to the total flux by {bracket}

$$\{\ \ \} = 45\pi^{-4}L_D = 45\pi^{-4}\left(\frac{\frac{1}{2}GM\dot{M}}{R_*}\right).$$

In the above λ is the wavelength of light emitted: in terms of frequency distribution $\lambda F_\lambda = \nu F_\nu$ and $\nu/\nu_* = \lambda_*/\lambda$ where here the ν is a light frequency and not a kinematic viscosity. The integral was computed as a function of λ_*/λ and the total spectrum is plotted as Fig. 6. Notice that the peak in λF_λ comes at the peak of a black body of temperature $10^{-1/2}\,\theta_*$. Notice that θ_* is set by the accretion rate to the power of one-quarter $\sigma_0\theta_*{}^4 = \frac{1}{6}G\rho_*\dot{M}$. Since for $T \gg 1$,

$$\dot{M} = \tfrac{1}{4}6^{-1/4}M_1\nu^{-1/4}R_1{}^{1/2}t^{-5/4}$$

neither luminosity nor θ_* are strongly dependent on precise values of ν and even R_1 only enters through its square root.

A low estimate of the boundary layer temperature, θ_b, is obtained by assuming that it behaves as a black body of constant temperature and thickness δ. We then have

$$\sigma_0\theta_b{}^4 4\pi R_*\delta = \tfrac{1}{2}GM\dot{M}R_*{}^{-1}\left[1 = \left(\frac{\Omega_0}{\Omega_*}\right)^2\right].$$

Using the value of δ from Appendix 1 equation (A6)

$$\frac{\theta_b}{\theta_*} = \left(\frac{R_*}{\delta}\right)^{1/4}\left[1-\left(\frac{\Omega_0}{\Omega_*}\right)^2\right]^{1/4} = \left(\frac{h_*}{\nu}\right)^{1/6}\left[1-\left(\frac{\Omega_0}{\Omega_*}\right)^2\right]^{1/4} \sim 10^{1/2}$$

where, at the last step, we have assumed that $\Omega_0{}^2 \ll \Omega_*{}^2$ and $\nu = 10^{-3}h_*$. In practice the boundary layer is not usually optically thick and the temperatures are hotter than this as a result.

Inhomogeneities in the disc. Although viscosity smears out density differences radially, nevertheless a turbulent disc will have density clumps within it which form and decay with the turbulence. A lump entering the boundary layer or atmospheric 'entry' zone yields its kinetic energy in a short time and so produces a flare. The shortest time scale involved in such a flare depends on the rate at which the lump burns up its translational energy as it enters the atmosphere. We find this time scale t_f to be approximately $t_f = 6\Omega_*{}^{-1}(b/R_*)^{1/2}$ where b is the scale height of the atmosphere and Ω_* is the angular velocity of the star-grazing orbit. Using $b = k\theta_0/(g_* m_H)$ and $g_* = GM/R^2$ we obtain $t_f = 6(k\theta_0/m_H)^{1/2}g_*{}^{-1}$ where m_H is the mass of the hydrogen atom and θ_0 is the star's effective temperature. We note that this time scale is inversely proportional to the star's surface gravity.

* The integral gets its major contribution in the range below $X = (\lambda/\lambda_*)^{4/3}$. For large λ the integral is proportional to the square of this effective cut-off and is thus proportional to $(\lambda/\lambda_*)^{8/3}$. Thus for small frequency ν: $\nu F_\nu = \lambda F_\lambda \propto \nu^{4/3}$ while for large ν the behaviour is similar to Plank's law. The approximation $F_\nu \propto \nu^{4/3}/[\exp(h\nu/kT)-1]$ gives a rough fit to the computed values which are plotted in Fig. 6 (left-hand hump).

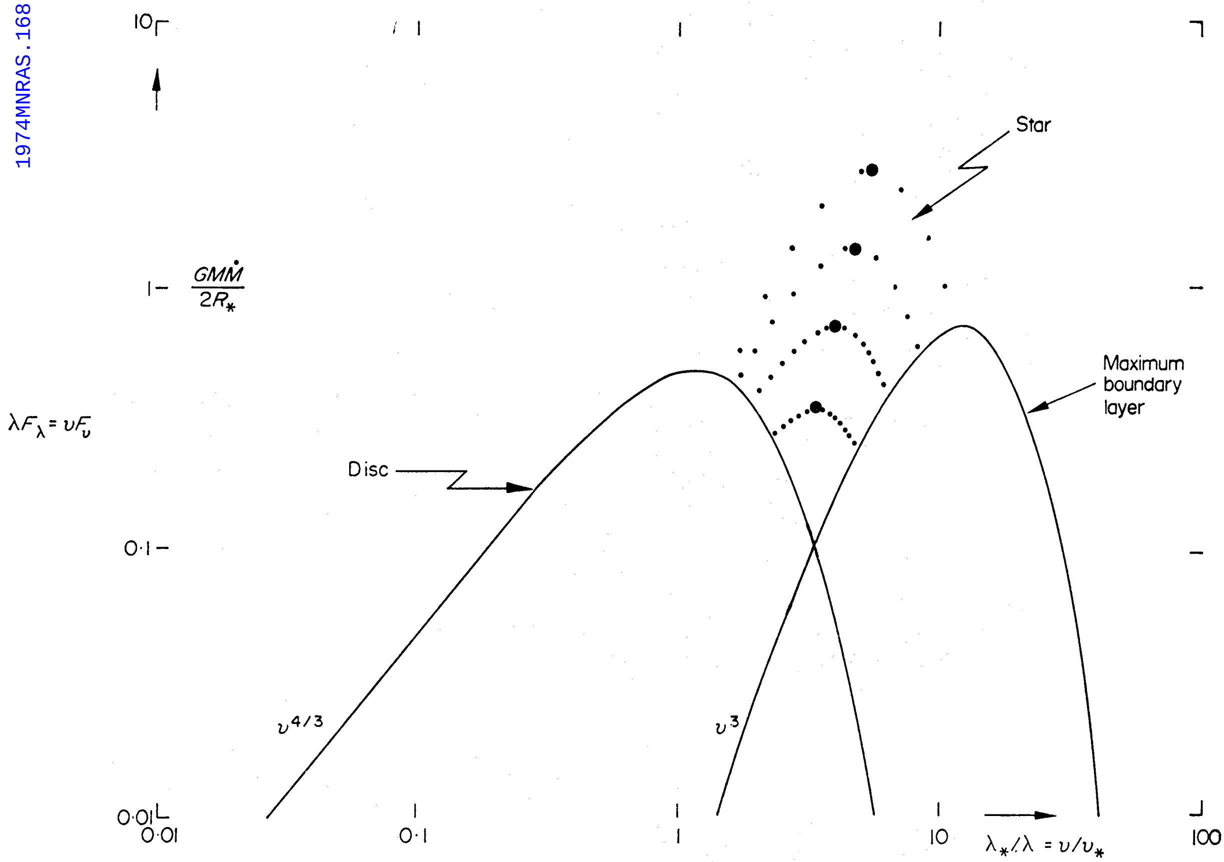

FIG. 6. *The full spectrum of the disc around a slowly rotating star; if the star rotates near break up speed then the boundary layer contribution is greatly reduced. The unit of luminosity varies with time like $T^{-5/4}$ and the unit of frequency, ν_*, like $T^{-5/16}$. Dotted curves show possible positions of the spectrum of the central star. If we assume that it is constant then in our shrinking units of luminosity it will grow with time from the smallest curve giving $\frac{1}{2}$ of the disc luminosity to the next, equal-luminosity case, and on to the 2 × and 4 × positions in which it dominates the spectrum.*

Applications. To fix ideas imagine that each main sequence star when formed has a disc with $R_1 = 20\, R_*$ and a mass $M_1 = \frac{1}{2}M$. Our formulae deduced in the assumption that $R_1 \gg R_*$ is then good initially to an accuracy of better than 20 per cent and they improve in accuracy because the disc expands as time passes. If we take $\nu = 10^{-3} h_*$ we have from equation (18′)

$$T = 1{\cdot}5 \times 10^{-5}\, \Omega_* t + 1 = 1{\cdot}5 \times 10^{-5}\, \Omega_*(t+t_1)$$

where

$$t_1 = \tfrac{2}{3} \times 10^{-5}\, \Omega_*^{-1} = 10^9 \left(\frac{R_*}{R_\odot}\right)^3 \left(\frac{M_\odot}{M_*}\right) \mathrm{s} \sim 30\ \mathrm{yr}.$$

The mass flux from equation (21) is

$$\dot{M} = \tfrac{3}{16}\, 10^{-5}\, \Omega_* M T^{-5/4} = 2{\cdot}0\, M \Omega_*^{-1/4} (t+t_1)^{-5/4}$$

and the luminosities of each of the disc and the boundary layer are

$$L = \frac{\frac{1}{2} G M \dot{M}}{R_*} = 1{\cdot}0\, G M^2 R_*^{-1} \Omega_*^{-1/4} (t+t_1)^{-5/4}.$$

For the young Sun at 10^4 yr old this gives a disc luminosity of 10^{35} erg s^{-1} at a temperature of about $10^{-1/2}\,\theta_* = 4100$ K plus a boundary radiating a similar amount at 41 000 K. After $1{\cdot}4\times10^5$ yr the boundary layer and the disc still radiate with one solar luminosity the one is a factor $10^{1/2}$ hotter than the Sun the other a factor $10^{1/2}$ cooler. For the first 30 yr the disc about the Sun is shining at some 10^{38} erg s^{-1}—a value that encroaches on the Eddington limit at which radiation pressure balances gravity. The temperature of the disc would then be 23 000 K but it is unlikely that our model is accurate in that regime.

TABLE I

Disc luminosity and disc temperature as a function of age and the star's mass

Only discs to the left of the continuous line are brighter than the central main sequence star. Discs to the right of the dotted line are not optically thick. The temperatures will be higher in reality through the optical thickness effect and because of the radiation from the central star.

$M/M_\odot$	Age years		10^4	10^5	10^6	10^7	10^8	10^9
5	L	erg s^{-1}	$9{\cdot}6.10^{35}$	$5{\cdot}5.10^{34}$	$3{\cdot}1.10^{33}$	$1{\cdot}7.10^{32}$	$9{\cdot}6.10^{30}$	$5{\cdot}5.10^{29}$
	$10^{-1/2}\,\theta_*$	K	3600	1800	860	420	210	102
1	L	erg s^{-1}	1.10^{35}	$5{\cdot}6.10^{33}$	$3{\cdot}2.10^{33}$	$1{\cdot}8.10^{31}$	1.10^{30}	$5{\cdot}6.10^{28}$
	$10^{-1/2}\,\theta_*$	K	4100	2000	980	480	240	116
0·4	L	erg s^{-1}	$2{\cdot}8.10^{34}$	$1{\cdot}5.10^{33}$	$8{\cdot}8.10^{31}$	$5{\cdot}0.10^{30}$	$2{\cdot}8.10^{29}$	$1{\cdot}5.10^{28}$
	$10^{-1/2}\,\theta_*$	K	4600	2300	1100	540	270	130
0·1	L	erg s^{-1}	$3{\cdot}9.10^{33}$	$2{\cdot}2.10^{32}$	$1{\cdot}3.10^{31}$	$7{\cdot}0.10^{29}$	$3{\cdot}9.10^{28}$	$2{\cdot}2.10^{27}$
	$10^{-1/2}\,\theta_*$	K	4900	2400	1200	580	280	140

When the boundary layer, the star, and the main disc are all of equal luminosity, the boundary layer will always be at least $10^{1/2}$ times hotter than the star and the main disc about $10^{1/2}$ times cooler. The spectrum of the disc is flatter than a black body towards the red; at low frequency, ν, it varies as $F\nu \propto \nu^{1/3}$ rather than ν^2. The boundary layer contribution will be mainly in the ultraviolet and due to the inherent suddenness of the entry of orbital material into an atmosphere the boundary contribution may well be variable. These variations in the ultraviolet may be of short duration down to

$$t_f = 6\left(\frac{k\theta_0}{M_H}\right)^{1/2}\left(\frac{R_*^2}{GM}\right) = 2{\cdot}5.\frac{\theta_0}{\theta_\odot}\left(\frac{R_*}{R_\odot}\right)^2\left(\frac{M_\odot}{M}\right)\text{ min}$$

while the main disc in the infrared and optical should show longer term variations on the time scales down to $t_l = 3400\,\Omega_*^{-1} = 5{\cdot}4\times10^6\,(R_*/R_\odot)^{3/2}(M_\odot/M)^{1/2}$ s $\sim$ 2 months. In the above θ_0 is the star's own effective temperature. T Tauri stars have large infrared (**12**) and rapidly variable ultraviolet excesses (**13**). They are clearly young and associated with stars of solar mass and below in their formation stages. They have strong and broad emission lines (**14**) and if these widths are interpreted as a rotation then they agree with a total width of

$$2\Omega_* R_* \sin i \sim 875 \sin i \text{ km s}^{-1},$$

the orbital velocity spread. While T Tauri stars occur in clusters of up to 10^6 yr age range, the older clusters have flare stars of progressively later spectral types (**15**). Clusters of 10^8 yr old have M-type flare stars, while younger clusters have K-type flare stars as well (**16**). Kunkel finds that the energy contained in flares of the local

UV Ceti flare stars is less than 1 per cent of the stellar output (**17**), and that for all flare stars the flare time scales are inversely proportional to stellar gravity as found in the last section (**18**). This increases our confidence that flares are the result of agglomeration of matter in old discs entering the stellar atmosphere. It is probable that a fraction of the braking of such material will be magnetic and so it is not unreasonable that Lovell (**19**) should have found 1 per cent of the flare energy to be emitted in the radio region. Flare stars are associated with young clusters and it is probable that as old T associations age they become associations with flare stars. We suggest that these phenomena occur as follows:

1. When a disc is brighter than its parent the star is called a T Tauri type star. These stars are typically somewhat under a solar mass but, being brighter and redder than their solitary equivalent, lie above the main sequence. Exceptionally when the disc is more than 1000 times brighter than the star around which it lies then it may be to the left of the main sequence. Such stars are known and we predict that they evolve to the red and down in the H–R diagram following $L \propto \theta^4$, Table I.

2. In the later stages such discs become dimmer than their parent stars and the density in them drops. The disc breaks up into condensations which are slowly accreted onto the star causing the flares. Table II. Notice from Table II that disc effects last less long for bright stars although related phenomena occur (**20**).

Table II

Disc variation times and ages as a function of stellar mass

$M/M_\odot$	t_f min	t_e months	$t_=$ years	t_i years	$t_{1/2\%}$ years
5	12	5·1	$5{\cdot}6\times10^3$	61	4×10^5
1	2·5	2·0	$1{\cdot}4\times10^5$	30	1×10^7
0·4	1	1·5	$8{\cdot}8\times10^5$	18	$6{\cdot}3\times10^7$
0·1	0·25	0·48	$1{\cdot}4\times10^7$	5·8	1×10^9

t_f flare time scale,
t_e time scale for traversal of energy producing region by a material element,
$t_=$ time until disc, boundary layer and main sequence star will all have the same luminosity,
t_i initial evolution time scale,
$t_{1/2\%}$ the time scale for which ½ per cent of the main sequence stellar luminosity can come from the boundary layer. Kunkel estimates flaring activity gives < 1 per cent of the average stellar luminosity in flare stars.

3.4 *Dwarf novae and binary X-ray sources*

Dwarf novae are binary systems which are thought to be composed of a bright white dwarf and a less luminous (often invisible) late-type dwarf which is filling its Roche lobe (**21**). The systems have typically masses of 1 $M_\odot$ and binary periods of a few hours to a day. The prototype is U Geminorum. Every 100 days or so, U Geminorum suddenly increases in brightness by $\Delta m_V \sim 5^m$ in a few hours and then returns to its normal brightness after a few days (**22**). It is probable (Bath) that these regular outbursts are caused by periodic dumping of mass (about $10^{-9} M_\odot$) onto the white dwarf by its companion (**23**). The transferred mass has, in general, too much angular momentum to fall directly onto the white dwarf, and initially forms a ring which evolves according to solution (26). The physical implications of this solution will be discussed elsewhere and for the time being we merely draw attention to the similarity between the decay of $L(t)$ in Fig. 5 and the decay of a typical U Gem outburst.

Similar considerations apply to the binary systems that contain compact X-ray sources (**24**)–(**29**). In particular in the system Her X1/HZ Her if we attribute the gradual decay of the 10-day 'on' state of the 35^d cycle to an abrupt ($\leqslant 1$ day) cut off of the mass transfer rate for HZ Her, then the decay rate predicts a viscosity $\nu \sim 10^{-3}\, h_*$ in good agreement with our assumption throughout.

3.5 *Magnetospheric dominance*

A rough estimate of the couple that a magnetosphere exerts when it is invaded down to a radius R_i ($\geqslant R_\Omega$, the corotation radius) is obtained by assuming that every line of force that would have crossed the equatorial plane beyond R_i is dragged backwards by the disc so that it meets the equator at an angle of 45°. This gives a torque of

$$g_i = \int_{Ri}^{\infty} 2\frac{B_*^2}{4\pi}\left(\frac{R_*}{R}\right)^6 2\pi R^2\, dR = \tfrac{1}{3}B_*^2 R_*^6 R_i^{-3}$$

where B_* is the equatorial magnetic field of the star. The maximum possible couple occurs when $R_i = R_\Omega$ the corotation radius given by $GM/R_\Omega^3 = \Omega_0^2$ that is $g_{\max} = \frac{1}{3}B_*^2R_*^6\Omega_0^2/(GM)$. The steady constant couple solution will be given by $g = g_i$, $R \geqslant R_i$ and so the approximate similarity solution by

$$g = g_i \exp\left(\frac{-ah^{1/l}}{T}\right),$$

$R \geqslant R_i$ with $g_i = g_{i1}T^{l-1}$ that is $R_i \propto T^{(1-l)/3} \propto T^{1/4}$. This is the approximate modification of solution (28). It is interesting to note that the couple on the star falls off like $T^{-3/4}$ so the star will slow down with $\Omega_0 = K - LT^{1/4}$ where K and L are constants. As $R_\Omega \propto \Omega_0^{-2}$ we see that eventually the condition $R_i \geqslant R_\Omega$ will be violated and after that a flux of material will fall through the magnetosphere and onto the star. The basic solution changes at this point to one in which the star continues to slow down but accumulates material in such a way that R_i is kept equal to R_Ω. Almost all the material will end on the star which will rotate very slowly but a small disc will exist from the now-large-R_Ω outwards, carrying all the angular momentum. In a binary system this conclusion will be modified since some material ends up on the companion.

In solution (30) the central couple rises to a maximum and then decays. If the central star has a magnetic field it will only manage to expel all the disc if the maximum couple called for by solution (30) can be met by a magnetic couple $\leqslant \frac{1}{3}B_*^2R_*^6\Omega_0^2/(GM)$. If that is not the case the magnetosphere will be invaded deeper than the corotation point and the inward flux will only be halted if the disc dies so that the magnetosphere can reassert itself at the corotation point. It is clear that there are interesting phenomena to investigate here.

ACKNOWLEDGMENTS

Our thanks are due to Professor L. Mestel under whose supervision this work began (**1**), to Drs A. Toomre, M. E. Dixon, D. O. Gough, M. J. Rees for discussions and to Mrs Stewart for computing an integral.

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APPENDIX I

THE STEADY VISCOUS BOUNDARY LAYER

The equations of motion read

$$\sigma\mathbf{u}.\nabla \boldsymbol{u} = -\nabla p + \operatorname{div}(2\eta \boldsymbol{e}) + \nabla([\zeta - \tfrac{2}{3}\eta)\operatorname{div}\mathbf{u}] + \sigma\nabla\psi;$$

considering a cold gas for simplicity and therefore omitting the pressure term and writing down the components, we have

$$\sigma\left[\frac{\partial}{\partial R}\left(\frac{u_R{}^2}{2}\right) - \frac{u_\phi{}^2}{2R}\right] = 2\frac{\partial}{\partial R}\left(\eta\frac{\partial u_R}{\partial R}\right) + 2\eta\frac{\partial}{\partial R}\left(\frac{u_R}{R}\right) + \frac{\partial}{\partial R}\left[(\zeta - \tfrac{2}{3}\eta)\frac{1}{R}\frac{\partial}{\partial R}(Ru_R)\right] - \sigma\frac{GM}{R^2} \qquad \text{(A1)}$$

$$\sigma u_R\frac{\partial}{\partial R}(Ru_\phi) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(\eta R^3\frac{\partial\Omega}{\partial R}\right). \qquad \text{(A2)}$$

In the steady state the continuity equation yields constant flux $F = 2\pi R\sigma u_R$; so we may re-write the above equation $F\,dh/dR = -\partial g/\partial R$, where g is the couple

$g = -\nu\sigma R^3\, d\Omega/dR$ and $\nu\sigma = \eta$. Integrating we get $g = (-F)(h-H)$, where H is the value of Ru_ϕ at which g and $d\Omega/dR$ vanish. Writing the full expressions for g and F into this last equation and solving for u_R we have

$$u_R = \frac{\nu \dfrac{d\Omega}{dR}}{\Omega - HR^{-2}} = \frac{\nu \dfrac{d\Omega}{dR}}{\Omega - \Omega_{\max}} \tag{A3}$$

where $\Omega = u_\phi/R = h/R^2$.

We now substitute expressions (A3) for u_R and solve equation (A1) for Ω using the constant flux condition to eliminate σ. If we use the dimensionless variables v, ω and x defined by

(A) $R = R_* + x\delta$ where δ is the thickness of the boundary layer and x is of order unity within the layer.

(B) $$u_R = \left(\frac{\nu}{\delta}\right) v \quad \text{where} \quad v = \frac{\dfrac{d\Omega}{dx}}{\Omega - \Omega_{\max}} = \frac{\dfrac{d\omega}{dx}}{\omega - \omega_{\max}} \tag{A4}$$

(C) $$\omega = \Omega \left(\frac{GM}{R_*^3}\right)^{-1/2}, \quad \omega < 1$$

then the resulting equation has the form

$$\frac{d}{dx}\left(\frac{v^2}{2}\right) + 2\frac{d}{dx}\left(\frac{1}{R}\frac{d(vR)}{dx}\right) - \frac{d}{dx}\left[\left(n - \tfrac{2}{3}\right)\frac{1}{R}\frac{d}{dx}(Rv)\right] + \left[2\frac{dv}{dx} + \left(n - \tfrac{2}{3}\right)\frac{1}{R}\frac{d}{dx}(Rv)\right]\frac{\partial}{\partial x}(\log Rv)$$

$$= -\left[\frac{\delta^3}{\nu^2}\left(\frac{GM}{R_*^2} - \Omega_0^2 R_*\right)\right]\left(\frac{R_*}{R}\right)^2\left(\frac{1 - \omega^2\left(\dfrac{R}{R_*}\right)^3}{1 - \omega_*^2}\right) \tag{A5}$$

where $n = \zeta/\eta$, ν has been assumed constant and ω_* is the value of ω when $R = R_*$. Notice that the final bracket is a dimensionless measure of the imbalance of centrifugal force and gravity which varies from unity at the star to zero as $R \to \infty$. By construction each major bracket in the above equation is dimensionless. Further by our assumption that x is of order unity all these brackets save possibly

$$\left[\frac{\delta^3}{\nu^2}\left(\frac{GM}{R_*^2} - \Omega_0^2 R_*\right)\right]$$

are made up from expressions of order unity. Thus we must choose our as yet indefinitely defined δ so that this can indeed be true. We achieve this by taking the bracket to be unity. The boundary layer thickness is then

$$\delta = \nu^{2/3}\left(\frac{GM}{R_*^2} - \Omega_0^2 R_*\right)^{-1/3}. \tag{A6}$$

The details of the boundary layer behaviour can only be found by solving equations (A4) and (A5) simultaneously, but as we only need the above estimate of boundary layer thickness we shall not do this here.

It is of interest to discuss how the situation with a slowly rotating star and a boundary layer changes as the star is taken to have a more rapid rotation. Notice that as centrifugal-force-gravity balance is approached at the star's equator so the thickness of the boundary layer δ becomes larger. Eventually, just before centrifugal force gravity balance is achieved, ω will cease to have a maximum within the boundary layer and our argument for taking $g_c = 0$ will then cease to hold. Finally for a star in exact centrifugal force gravity balance no boundary layer occurs and the disc solution (18) is valid right up to the star's surface. Whenever the viscous couple does not vanish there is the energy flow $g\Omega$ into the viscous disc. In the special case of centrifugal force gravity balance this flux of energy is double the total power that becomes available as the material descends from orbit to orbit down to the stellar surface.

If we imagine a rigid star rotating so fast the centrifugal force is stronger than gravity at the equator then there is no valid solution of the type we have discussed, because the vorticity in the boundary layer is oppositely directed to the vorticity just outside. As a result any boundary layer will be unstable. Further one may argue that if any fluid is available at the stellar surface it will be flung off so that the flux is outwards rather than inwards. In the absence of such material there is no steady state in the absence of pressure and even with pressure there will be a highly turbulent region outside the rigid star that will extend out to the radius at which h is somewhat greater than $\Omega_0 R_*^2$.

We shall not investigate this unphysical case further.

The condition that the boundary be so thick that Ω no longer has a maximum within it, is approximately the condition that just outside the boundary layer the angular velocity shall be less than that for corotation with the star. That is $GM/(R_*+\delta)^3 < \Omega_0^2$, which may be written

$$\frac{\delta}{R_*} > \left(\frac{\Omega_*}{\Omega_0}\right)^{2/3} - 1,$$

where Ω_0 is the star's spin and $\Omega_*^2 \equiv GM/R_*^3$.

Using expression (A6) for δ this gives

$$\left(\frac{\nu}{h_*}\right)^{2/3} > \left(\left(\frac{\Omega_*}{\Omega_0}\right)^{2/3} - 1\right)\left(1 - \left(\frac{\Omega_0}{\Omega_*}\right)^2\right)^{1/3}$$

where as before $h_* = \Omega_* R_*^2$.

If further we assume $\nu \ll h_*$ then Ω_0/Ω_* is close to 1 and the condition above reduces to $(\Omega_0/\Omega_*)^2 > 1 - \Delta$ where $(\nu/h_*)^{2/3} = \frac{1}{3}\Delta^{4/3}$ that is $\Delta = 3^{3/4}(\nu/h_*)^{1/2}$.

In summary then we have four cases:

(1) The common case Centrifugal Force < gravity and a thin boundary layer $(\Omega_0/\Omega_*)^2 < 1 - \Delta$ that is $\delta/R_* < \frac{1}{3}\Delta = 3^{-1/4}(\nu/h_*)^{1/2}$.

(2) The thick boundary layer case $\Omega_*^2(1-\Delta) < \Omega_0^2 < \Omega_*^2$.

(3) $\Omega_0 = \Omega_*$—when there is no boundary layer, and the viscous solution continues to the star. The couple on the star provides two-thirds of the energy dissipated in the disc in this case.

(4) Centrifugal force over-balances gravity; This case is unphysical and there is no steady solution. Even in the presence of a flux source on the star the boundary layer would be unstable.

This boundary layer treatment is correct for a constant viscosity, but when applied to our turbulent viscosity problem it leads to inward velocities of order

ν/δ. These are greater than the velocities of the turbulent elements so no elements are travelling fast enough to communicate the couple outwards against the inward flow. This is an inadequacy of treating the turbulent viscosity as a constant.

If near the star we modify our earlier argument for $h_*/\nu = \mathscr{R}_c \sim 10^3$ by taking the distance to the stellar surface multiplied by the velocity relative to that surface to define the Reynolds number we then have

$$\frac{(\Omega R-\Omega_0 R_*)(R-R_*)}{\nu} = \mathscr{R}_c \sim 10^3$$

which gives

$$\nu = \mathscr{R}_c^{-1} h_* \left[\left(\frac{R_*}{R}\right)^{1/2} - \left(\frac{\Omega_0}{\Omega_*}\right)\right]\left(\frac{R}{R_*}-1\right)$$

and decreases the viscosity near the star. Since in this region we have steady state form we can use our steady state formulae which we derived without any assumptions about the viscosity. These give the same energy generation in the disc as before and

$$u_R = -\tfrac{3}{2}\nu R^{-1}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]^{-1}$$
$$= -\tfrac{3}{2}\mathscr{R}_c^{-1}\frac{h_*}{R}\left[\left(\frac{R_*}{R}\right)^{1/2}-\frac{\Omega_0}{\Omega_*}\right]\frac{R}{R_*}\left[1+\left(\frac{R_*}{R}\right)^{1/2}\right], \quad R>R_*+\delta.$$

Thus within the reduced value of the viscosity these velocities are no longer faster than the turbulent elements. The reduced and variable viscosity leads to higher densities in the inner parts of the disc so that the same couple can be carried. There are modifications to the boundary layer equations but once again they lead to the conclusion that $[\delta^3 \nu - {}^2R_*(\Omega_*{}^2-\Omega_0{}^2)]_{R_*+\delta}$ must be of order unity. But now ν itself contains a term (R/R_*-1) of order δ/R_*; we thus obtain using our new viscosity $\delta/R_* \simeq \mathscr{R}_c^{-2}$ in place of $\mathscr{R}_c^{-2/3}$ and for ordinary stars these new boundary layers are so thin that they will be less than the scale height of the atmosphere into which the disc is entering. In addition the temperature of the gas in the boundary layer swells the disc to a thickness greater than the width of the boundary layer.

In conclusion the turbulent disc has a region of modified viscosity whose extent is of order R_*. This does not effect the energy generation as a function of radius but does lead to greater densities and smaller inward velocities taking in the same flux. Although the energy generated in the boundary layer is the same the layer is formally much thinner. However, it is thinner than the disc thickness and so the temperature of the boundary layer is now set by the disc thickness and the scale height of the stellar atmosphere. The high velocities of entry into the stellar atmosphere keep the boundary layer hot so that any emission from this region will be predominantly in the ultraviolet.

APPENDIX II

STABILITY OF THE UNIFORMLY ROTATING SHEET WHEN FRICTION IS PRESENT

Earlier work on non-viscous self-gravitating slabs demonstrates that the thickness of the slab is unimportant but that the effects of pressure within the plane of the sheet are essential to a proper discussion of stability. For this reason we shall neglect the thickness but take into account the lateral pressure. To do this we assume relative perturbations in surface pressure and surface density are proportional

$$\frac{\Delta \int p \, dz}{\int \rho_0 \, dz} = \gamma \frac{\Delta \int \rho \, dz}{\int p_0 \, dz}$$

when p_0 and ρ_0 are unperturbed pressures and densities and the integrations are performed right through the thickness of the disc. For a uniform unperturbed disc we have on denoting perturbed quantities at a given position by a suffix 1.

$$\int p_1 \, dz = \gamma \frac{\int p_0 \, dz}{\int \rho_0 \, dz} \sigma_1 = c^2 \sigma, \tag{B1}$$

where c is a constant velocity ' of sound ' and $\sigma_1 (= \int \rho_1 \, dz)$ is the perturbation in surface density. We use (A1) to eliminate pressure in favour of surface density in the equation of motion. Fourier transforming the usual perturbed equations of motion for a uniformly rotating sheet at equilibrium we have on the disc

$$i\omega \mathbf{u} + 2\Omega \times \mathbf{u} = -i\mathbf{k}c^2 \frac{\sigma_1}{\sigma_{00}} + i\mathbf{k}\psi_1 - \nu k^2 \mathbf{u} + (\zeta + \tfrac{1}{3}\nu)\, \mathbf{k}(\mathbf{k}.\mathbf{u}). \tag{B2}$$

Acceleration Coriolis Surface Perturbed Viscous terms
Pressure gravity

where ν = kinematic viscosity. $\zeta\rho$ = bulk viscosity.

Here Ω is the angular velocity of the unperturbed sheet. The perturbed quantities have been Fourier analysed in the form $\sigma = \sigma_{00} + \sigma_1 \exp(i\omega t + ikR)$ where $R = (x, y, 0)$. $\mathbf{u}$ is the perturbed velocity relative to axes that rotated with angular velocity Ω and ψ_1 is the perturbed gravitational potential. The latter satisfies Poisson's equation

$$\nabla^2 \psi_1 = \frac{\partial^2 \psi_1}{\partial z^2} - k^2 \psi_1 = -4\pi G \sigma_1 \, \delta(z)$$

where the appropriate solution is

$$\psi_1 = \frac{2\pi G \sigma_1}{|k|} \exp - |k|\,|z| \tag{B3}$$

which reduces to $\psi_1 = 2\pi G \sigma_1 / |k|$ on $z = 0$ where the disc lies.

Equations (B2) and (B3) must be supplemented by the continuity equation which reads

$$i\omega \sigma_1 + ik.u\sigma_{00} = 0 \tag{B4}$$

$\mathbf{u}$ and ψ may be eliminated from equations (B2–B4) by taking first the equation (B2) $\mathbf{k}$ and then (B2) $\times\, \mathbf{k}$ we find the dispersion relation

$$\left(\frac{\sigma_1}{\sigma_{00}}\right) \{(i\omega)^3 + (i\omega)^2 k^2(\zeta + \tfrac{7}{3}\nu) + i\omega[k^2c^2 - 2\pi G\sigma_{00}|k| + 4\Omega^2 + k^2\nu(\zeta + \tfrac{4}{3}\nu)] + \nu k^2(k^2c^2 - 2\pi G\sigma_{00}|k|)\} = 0. \tag{B5}$$

When $\nu = \zeta = 0$ we find the familiar dispersion relation for the flat sheet:

$$i\omega[-\omega^2+k^2c^2-2\pi G\sigma_{00}|k|+4\Omega^2] = 0. \tag{B6}$$

There are three modes, the sound waves travelling in each direction correspond to the vanishing of the square bracket. We are interested here in sheets that are ordinarily stable so we should have no solution with ω^2 negative. The quadratic in k must be positive to avoid such instabilities. This requires $\pi^2G^2\sigma_{00}^2<4\Omega^2c^2$; that is sufficient rotation and pressure to overcome the self-gravity at all wavelengths. However, beside these stable modes there are neutral modes with $\omega = 0$, all $\mathbf{k}$. These correspond to distortions of the sheet into neighbouring wavy equilibrium flows in which the velocities are balanced by Coriolis forces. When we return to (B5) we shall see that it is these modes that become unstable in the presence of viscosity. They are the flat sheet generalizations of the deformations that make Jacobi ellipsoids out of Maclaurin spheroids.

Return to equation (B5) and denote the expression in curly brackets by $E(i\omega)$. Take modes such that $2\pi G\sigma_{00}|k|<k^2c^2$. Then for all positive $i\omega$, $E(i\omega)$ is positive because every coefficient is positive. Furthermore for small ζ and ν one may show that all the modes are damped by viscosity. However, if we assume instead that $2\pi G\sigma_{00}|k|>k^2c^2$ then $E(0)$ is negative while $E(\infty)$ is positive so there is a real positive root for $i\omega$. Since the variables behave like exp $i\omega t$ this is an instability. Now for wavelengths longer than $\lambda = c^2/(G\sigma_{00})$ this latter assumption is true and $c^2/(G\sigma_{00})$ is the natural length over which a pressure support could hold the body up even in the absence of rotation. For an isotropic pressure it would be the thickness of the sheet. The effect of the viscosity is to change back the condition of stability to exactly what it would have been in the absence of rotation. So viscosity removes the rotational stabilization. But only on the longer time-scale that the viscosity take to act. (Notice also it is not ζ that causes this instability and the presence of ζ alone does *not* change the rotational stabilization.) A uniformly rotating disc can only be stable when it is essentially able to support itself by pressure. For this reason it seems sensible for our discussion of the frictional evolution of non-pressure-supported discs to consider only the rotation laws in which no pressure support is likely to be realistic.