Amplitude-period relation for El Niño: analytic approximation in the weakly nonlinear regime

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Hi Fiona,

Here is a summary of what I did based on Joe Keller's suggestion. I haven't checked the algebra at all, so I am sure there are errors. Hopefully the general procedure is roughly OK in spite of these errors. If we could add the calculation described here to the work you have done during the summer plus find the 2/3/4 UPOs, we should be able to perhaps write a reasonably nice manuscript on the whole thing…What do you think?

best, Eli

1 Analytic derivation

Start from the delayed oscillator equations of [1] based on the recharge oscillator and two strip approximation of [2].

$$\begin{aligned} h(t) &= ah(t-\tau_1)+bT(t-\tau_2)+cT(t-\tau_3) && (1)\\ \frac{\partial T}{\partial t} &= -\varepsilon T-\frac{\overline{w}}{H_1}\gamma(T-T_{sub}(h)) \\ &\approx -pT+rh-sh^3 && (2)\end{aligned}$$

Substitute a solution of the form

$$\begin{pmatrix} T \\ h \end{pmatrix} = \begin{pmatrix} A\cos(\omega t) \\ B\cos(\omega t+\phi) \end{pmatrix} e^{\lambda t} \qquad (3)$$

Now, assume that the model is neutrally stable ($\lambda = 0$), substitute (3) in (1) and (3) in (2),, multiplying each of the resulting equations by either $\sin(\omega t)$ or $\cos(\omega t)$, and integrate over a full period $\int_0^{2\pi/\omega} dt$. Furthermore, use

$$\cos^3(\theta) = \frac{3}{4}\cos(\theta)+\frac{1}{4}\cos(3\theta) \qquad (4)$$

and neglect the $\cos(3\theta)$ term (equivalent to projecting our solution on the first Fourier term). Using the identities

$$\int_0^{2\pi/\omega} \cos(\omega t+\phi)\cos(\omega t) = \frac{1}{2}\cos(\phi); \qquad \int_0^{2\pi/\omega} \cos(\omega t+\phi)\sin(\omega t) = -\frac{1}{2}\sin(\phi) \tag{5}$$

the result of the above procedure is the four equations

$$\begin{aligned} 0 &= -pA + rB\cos\phi - \frac{3}{4}sB^3 && (6) \\ \omega A &= rB\sin(\phi) && (7) \\ B\cos(\phi) &= aB\cos(\phi-\omega\tau_1) + bA\cos(\omega\tau_2) - cA\cos(\omega\tau_3) && (8) \\ B\sin(\phi) &= aB\sin(\phi-\omega\tau_1) + bA\sin(\omega\tau_2) + cA\sin(\omega\tau_3) && (9) \end{aligned}$$

There are 5 unknown in the problem: $A, B, \lambda, \phi, \omega$. Since we are treating the problem as being quasi-linear, there is an overall scale that cannot be determined. These four equations therefore serve to determine A (or B), ϕ, ω, and the condition that needs to be satisfied for our assumption that $\lambda = 0$. Substituting A from (7) into (6) and then also into (9), we get

$$\begin{aligned} 0 &= \frac{pr}{\omega}\sin\phi + r\cos\phi - \frac{3}{4}sB^2 && (10) \\ 1 &= a\cos(\omega\tau_1) - a\sin(\omega\tau_1)\cot\phi + \frac{br}{\omega}\sin(\omega\tau_2) + \frac{cr}{\omega}\sin(\omega\tau_3) && (11) \end{aligned}$$

Equations (10) and (11) may be used to solve for ω and ϕ. Writing $\Phi \equiv \cos\phi$, we get from (10) a quadratic equation for Φ

$$\Phi^2\{-r^2(\frac{p^2}{\omega^2}+1)\} + \Phi\{\frac{3}{2}srB^2\} + \{\frac{p^2r^2}{\omega^2} - \frac{9}{16}s^2B^4\} = 0 \tag{12}$$

whose solution for Φ may be substituted into (11) to get a single equation for ω and its amplitude dependence. Given that these are all complex transcendental equations, we can simplify in two ways. First, we can assume that $\omega\tau_{1,2,3} \ll 1$ although that is not actually true. . . Alternatively, we can solve for the frequency and phase without the nonlinear correction, ω_0, ϕ_0 by setting the term with $B^2 = 0$ in (10, and then add the nonlinear correction as a small perturbation. For the linear case, (10) gives

$$\cot\phi_0 = \frac{-p}{\omega_0} \tag{13}$$

and substituting this in (11) we find

$$1 = a\cos(\omega_0\tau_1) - a\sin(\omega_0\tau_1)\frac{p}{\omega_0} + \frac{br}{\omega_0}\sin(\omega_0\tau_2) + \frac{cr}{\omega_0}\sin(\omega_0\tau_3) \tag{14}$$

The transcendental equations (13) and (14) may now be used to solve (numerically, as this is a transcendental equation. . . , note that we also expect it to have more than one solution) for the frequency and phase ω_0, ϕ_0 without the effects of the nonlinearity. Next, treating the B^2 term as a small perturbation and writing $\phi = \phi_0 + \phi_1$ and $\omega = \omega_0 + \omega_1$ where ϕ_1, ω_1 are perturbations due to the nonlinear B^2 term, assuming $\phi_1, \omega_1 \ll 1$ and linearizing (10) and (11) we find

$$\phi_1\{\frac{pr}{\omega_0}\cos\phi_0 - r\sin\phi_0\} + \omega_1\{-\frac{pr}{\omega_0^2}\sin\phi_0\} = \frac{3}{4}sB^2 \tag{15}$$

$$\begin{aligned} &\phi_1\{+a\sin(\omega_0\tau_1)/\sin^2\phi_0\} + \omega_1\{-((-(b\omega_0 r\tau_2\cos[\omega_0\tau_2]) - c\omega_0 r\tau_3\cos[\omega_0\tau_3] + \\ &a\omega_0^2\tau_1\cos[\omega_0\tau 1]\cot[\phi_0] + a\omega_0^2\tau_1\sin[\omega_0\tau 1] + br\sin[\omega_0\tau_2] + cr\sin[\omega_0\tau_3])/\omega_0^2)\} = 0 \end{aligned} \tag{16}$$

which is a system of two linear equations for ω_1, ϕ_1, with known coefficients (that is, known given the solution for ω_0, ϕ_0). These two equations may now be solved for the perturbation to the frequency and phase in terms of the amplitude B of the small nonlinearity.

This solution is what we needed(!): it give us $\omega_1(B)$, which is the desired effect of the small nonlinearity on the period of El Niño.

Things that might be interesting to calculate and plot from the above computation:

- ω_0, ϕ_0 as function of some of the model parameters such as $\tau_{1,2,3}, a, b, c, \varepsilon$ etc. Note that there is likely more than one solution for ω_0, ϕ_0 for a given set of model parameters.

- ω_1, ϕ_1 as function of the amplitude B and of all of the same model parameters as above.

Things we probably need to do with the actual numerical model of the delayed oscillator in order to better connect it to the analysis above. The objective is to verify the above analysis using the numerical solution of the model, as follows

- Set the model parameters to the neutrally stable regime $\lambda = 0$: a small initial perturbation will simply cause oscillations that will not grow nor be damped.

- In this regime, try different initial amplitudes, all small. Each such initial perturbation will result in a neutral oscillation with a different amplitude. Note that in the nonlinear regime, any small perturbation grows until the nonlinearity limits further growth and the oscillation is then saturated at some amplitude that is not at all related to the amplitude of the initial perturbation.

- plot the period as function of the amplitude for each of the above experiments. Compare to the prediction of the above theory from the solution for $\omega_1(B)$.

References

[1] E. Galanti and E. Tziperman. On enso's phase locking to the seasonal cycle in the fast sst, fast wave, and mixed mode regimes. *Journal of the Atmospheric Sciences*, in press, 2000.

[2] F.-F. Jin. An equatorial ocean recharge paradigm for enso. part i: conceptual model. *J. Atmos. Sci.*, 54:811–829, 1997.