

**Harvard University
Computer Science 121**

Problem Set 0

Due Monday, September 14, 2009 at 1:20 PM.

Late problem sets may be turned in until Wednesday, September 16, 2009 at 1:20 PM with a 20% penalty.

Please hand in Parts A and B separately; each part must be stapled.

All problem sets should be dropped off in the CS 121 box in the basement of Maxwell Dworkin.

See syllabus for collaboration policy.

Note: This is our standard prologue. Problem set 0 counts 0 points, but you should complete it, and we will grade it and provide a solution set for your edification.

PART A (Graded by Jesse)

PROBLEM 1 (1+1+1+1 points)

Describe the following sets using formal set notation:

- (A) A is the set containing the empty set.
- (B) B is the set containing the empty string.
- (C) C is the set containing all non-negative, integral powers of 2.
- (D) D is the set containing all strings over $\Sigma = \{a, b\}$ whose length is a non-negative, integral power of 2.

PROBLEM 2 (2+2+2+2 points)

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers. For each of the following functions, $f : \mathbb{N} \rightarrow \mathbb{N}$, state whether f is (i) one-to-one, (ii) onto, and/or (iii) bijective. Briefly justify each of your answers.

- (A) $f(x) = x^2$
- (B) $f(x) = x \bmod 3$
- (C) $f(x) = x!$
- (D) $f(x) = \begin{cases} x + 1 & \text{if } x \text{ is even} \\ x - 1 & \text{if } x \text{ is odd} \end{cases}$

PROBLEM 3 (2+4+4 points)

Consider the binary relation \sim on sets defined by $A \sim B$ if and only if there exists a function $f : A \rightarrow B$ such that f is bijective.

- (A) Give examples of two finite sets A, B such that $A \sim B$, as well as sets C, D such that $C \not\sim D$.
- (B) Prove that \sim is an equivalence relation. (Your proof should work for infinite sets as well as

finite ones. Indeed, this relation is how one defines what it means for two infinite sets to have the same cardinality. Later in the course, we will see that not all infinite sets have the same cardinality, i.e. the infinite sets yield multiple equivalence classes under this relation.)

(C) Now consider the relation \lesssim defined by $A \lesssim B$ if there exists a one-to-one function $f : A \rightarrow B$. Is \lesssim reflexive? symmetric? transitive? Justify your answers.

PART B (Graded by Victor)

PROBLEM 4 (4 points)

Joe the painter has 2009 cans of paint. Show that at least one of the following statements is true about Joe's paint collection:

1. Among the cans, there are at least 10 of them with the same color.
2. Among the cans, there are at least 224 different colors of paint.

(Hint: prove by contradiction)

PROBLEM 5 (6+6 points)

Define the Fibonacci numbers as follows:

$$F_0 = 0$$

$$F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2} \text{ for all } n > 1$$

Prove the following statements by induction:

(A) For $n \geq 2$, F_n equals the number of strings of length $n - 2$ over alphabet $\Sigma = \{a, b\}$ that do not contain two consecutive a 's.

(B)

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

PROBLEM 6 (Challenge!! 3 points)

Note: On every problem set we will provide a challenge problem, generally significantly more difficult than the other problems in the set, but worth only a few points. It is recommended that you attempt these problems, but only after completing the rest of the assignment.

Let \sim and \lesssim be the relations defined as in Problem 3. Prove that $A \sim B$ if and only if $A \lesssim B$ and $B \lesssim A$. (Hint: For the if direction, consider the (possibly infinite) directed graph with vertex set $(A \times \{a\}) \cup (B \times \{b\})$ and edges $\{((x, a), (f(x), b)) : x \in A\} \cup \{((x, b), (g(x), a)) : x \in B\}$ where $f : A \rightarrow B$ and $g : B \rightarrow A$ are one-to-one functions. Characterize the connected components of this graph, and construct the bijection separately for each connected component.