Harvard University
Computer Science 121

Problem Set 6

Due Friday, November 6, 2009 at 1:20 PM.
Late problem sets may be turned in until Monday, November 9, 2009 at 1:20 PM with a 20% penalty.
Please hand in Parts A and B separately; each part should be stapled.
All problem sets should be dropped off in the CS 121 box in the basement of Maxwell Dworkin.
See syllabus for collaboration policy.
WRITE YOUR TF's NAME ON EACH PART OF THE PROBLEM SET

In this problem set, if you are describing a Turing machine, give a high-level rather than a formal description. See Sipser page 157 for guidance.

PART A (Graded by Jesse)

PROBLEM 1 (6+12 points)

(A) Prove that the decidable languages are closed under union.
(B) A *homomorphism* is defined as follows:

> Let Σ and Δ be alphabets. Consider a function h from Σ to Δ^*. Extend h to a function from Σ^* to Δ^* by applying it to each symbol in its input. More formally:
>
> $$\begin{aligned} h(\varepsilon) &= \varepsilon \\ h(w\sigma) &= h(w)h(\sigma), \text{ for any } w \in \Sigma^*, \sigma \in \Sigma \end{aligned}$$
>
> Any function $h : \Sigma^* \to \Delta^*$ defined in this way from a function $h : \Sigma \to \Delta^*$ is called a **homomorphism**.

Note that homomorphisms can "erase": $h(w)$ may be ε, even though w is not.
Prove that the Turing-recognizable languages are closed under *homomorphism*.

PROBLEM 2 (12 points)

Show that every infinite recognizable language has an infinite decidable subset. (Hint: There was a useful fact discussed in lecture on 10/27).

PROBLEM 3 (15 points)

Write a general grammar that generates $\{ww : w \in \{a, b\}^*\}$. Explain in words what each rule in your grammar does.

PART B (Graded by Victor)

PROBLEM 4 (8+10 points)

(A) Let $L = \{\langle D, k\rangle : D \text{ is a DFA that accepts exactly } k \text{ strings, where } k \in \mathbb{N} \cup \{\infty\}\}$. Show that L is decidable. (*Hint:* Show how to find a p such that if D accepts any string of length at least p, then D accepts infinitely many strings).

(B) Let $L = \{\langle M\rangle : M \text{ is a TM that accepts at least one string in } \Sigma^*\}$. Show that L is recognizable.

PROBLEM 5 (10 points)

Define $\textsc{Prefix}(L) = \{x \mid xy \in L \text{ for some } y \in \Sigma^*\}$.

Show that if L is Turing-recognizable, then $\textsc{Prefix}(L)$ is Turing-recognizable.

PROBLEM 6 (Challenge! 3 points)

For any language L, let $d_L(n)$ be the number of strings of length n in L. Prove that if L is regular, then $d_L(n)$ is either $O(n^k)$ for some k or $\Omega(2^n)$.

(See http://en.wikipedia.org/wiki/Big_O_notation#The_family_of_Bachmann.E2.80.93Landau_notations for a definition of Ω)