Harvard University
Computer Science 121

Problem Set 7

Due Friday, November 13, 2009 at 1:20 PM.
Late problem sets may be turned in until Monday, November 16, 2009 at 1:20 PM with a 20% penalty.
Please hand in Parts A and B separately; each part must be stapled.
All problem sets should be dropped off in the CS 121 box in the basement of Maxwell Dworkin.
See syllabus for collaboration policy.

PART A (Graded by Olga)

PROBLEM 1 (6+6+6 points)

For each of the following languages, determine whether it is decidable or undecidable. If it is undecidable, determine whether it recognizable, co-recognizable, or neither. Prove your answers succinctly - any constructions or reductions can be given at a high level, etc. In each case, M is an arbitrary Turing machine.

(A) $\{\langle M \rangle : M \text{ takes more than 121 steps on some input}\}$
(B) $\{\langle M \rangle : M \text{ does not halt on any input}\}$
(C) $\{\langle M \rangle : L(M) \text{ contains a string of length 121}\}$

PROBLEM 2 (8+6+10 points)

(A) Let L_1 be any Turing-recognizable language. Prove that there exists some decidable language L_2 such that $L_1 = \{x : \text{there exists } y \text{ such that } \langle x, y \rangle \in L_2\}$
(Hint: Imagine y gives you some information about an accepting computation on x, if one exists.)

(B) Prove that the class of decidable languages is *not* closed under homomorphisms.

(C) A *nonerasing* homomorphism is one that sends no element of the alphabet to the empty string, ε. Prove that the class of decidable languages *is* closed under nonerasing homomorphisms.

PART B (Graded by David)

PROBLEM 3 (8+8 points)

Let $S = \{\langle M\rangle | M \text{ is a TM and } L(M) = \{\varepsilon\}\}$. By using mapping reductions:
(A) Show that S is not recognizable.
(B) Show that S is not co-recognizable.

PROBLEM 4 (6+8 points)

(A) Prove that a language L is recursive if and only if $L \leq_m a^*b^*$.

(B) Let $L = \{\langle M, w, b\rangle : b \in \{0,1\} \text{ and } b = 1 \text{ iff } M \text{ halts on } w\}$.
Prove that L is neither r.e nor co-r.e.

PROBLEM 5 (10 points)

Two languages H and K are *recursively separable* if there exists some recursive language R such that $H \subseteq R$ and $K \subseteq \overline{R}$.
Prove that the languages $L_1 = \{\langle M\rangle : M \text{ accepts } \langle M\rangle\}$ and $L_2 = \{\langle M\rangle : M \text{ rejects } \langle M\rangle\}$ are not recursively separable.

PROBLEM 6 (Challenge!! 3 points)

Given a particular method of encoding a Turing machine M into a string $\langle M\rangle$, define T_w to be the Turing machine encoded by the string w, or if w is not a proper encoding of any Turing machine, then define T_w to be the Turing machine that immediately accepts on any input. That is,

$$T_w = \begin{cases} M & \text{if } w = \langle M\rangle \text{ for some TM } M \\ \text{The TM that always accepts} & \text{otherwise} \end{cases}$$

Prove that if f is any computable function from $\Sigma^* \to \Sigma^*$, then there exists some string x such that $L(T_x) = L(T_{f(x)})$.