CS152: Programming Languages

Lecture 7 — Lambda Calculus

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Where we are

- ▶ Done: Syntax, semantics, and equivalence
  - ▶ For a language with little more than loops and global variables

- ▶ Now: Didn't IMP leave some things out?
  - ▶ In particular: scope, functions, and data structures
  - ▶ (Not to mention threads, I/O, exceptions, strings, ...)

Time for a new model...

Data + Code

Higher-order functions work well for scope *and* data structures

- Scope: not all memory available to all code
  ```
  let x = 1
  let add3 y =
     let z = 2 in
     x + y + z
  let seven = add3 4
  ```

- Data: Function closures store data. Example: Association "list"

```
let empty = (fun k -> raise Empty)
let cons k v lst = (fun k' -> if k'=k then v else lst k')
let lookup k lst = lst k
```

(Later: Objects do both too)

Adding data structures

Extending IMP with data structures isn't too hard:

$$e ::= c \mid x \mid e + e \mid e * e \mid (e, e) \mid e.1 \mid e.2$$
$$v ::= c \mid (v, v)$$
$$H ::= \cdot \mid H, x \mapsto v$$

$\boxed{H \; ; \; e \Downarrow c}$ all old rules plus:

$$\frac{H \; ; \; e_1 \Downarrow v_1 \quad H \; ; \; e_2 \Downarrow v_2}{H \; ; \; (e_1, e_2) \Downarrow (v_1, v_2)} \quad \frac{H \; ; \; e \Downarrow (v_1, v_2)}{H \; ; \; e.1 \Downarrow v_1} \quad \frac{H \; ; \; e \Downarrow (v_1, v_2)}{H \; ; \; e.2 \Downarrow v_2}$$

Notice:

- We allow pairs of values, not just pairs of integers
- We now have *stuck* programs (e.g., $c.1$)
    - What would C++ do? Scheme? ML? Java? Perl?
    - Division also causes stuckness

What about functions

But adding functions (or objects) does not work well:

$$e ::= \dots \mid \text{fun } x \text{ -> } s$$
$$v ::= \dots \mid \text{fun } x \text{ -> } s$$
$$s ::= \dots \mid e(e)$$

$$\boxed{H \; ; \; e \Downarrow c} \qquad\qquad \boxed{H \; ; \; s \to H' \; ; \; s'}$$

Additions:

$$\frac{}{H \; ; \; \text{fun } x \text{ -> } s \Downarrow \text{fun } x \text{ -> } s} \qquad \frac{H \; ; \; e_1 \Downarrow \text{fun } x \text{ -> } s \quad H \; ; \; e_2 \Downarrow v}{H \; ; \; e_1(e_2) \to H \; ; \; x := v; s}$$

Does this match "the semantics we want" for function calls?

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$$e \quad ::= \quad \dots \mid \text{fun } x \text{ -> } s$$
$$v \quad ::= \quad \dots \mid \text{fun } x \text{ -> } s$$
$$s \quad ::= \quad \dots \mid e(e)$$

$$\frac{}{H \; ; \; \text{fun } x \text{ -> } s \Downarrow \text{fun } x \text{ -> } s} \qquad \frac{H \; ; \; e_1 \Downarrow \text{fun } x \text{ -> } s \quad H \; ; \; e_2 \Downarrow v}{H \; ; \; e_1(e_2) \to H \; ; \; x := v; s}$$

NO: Consider $\text{x} := 1; (\text{fun } x \text{ -> } y := x)(2); \text{ans} := x.$

Scope matters; variable name doesn't. That is:

- Local variables should "be local"
- Choice of local-variable names should have only local ramifications

Another try

$$\frac{H \; ; \; e_1 \Downarrow \text{fun } x \to s \qquad H \; ; \; e_2 \Downarrow v \qquad y \text{ "fresh"}}{H \; ; \; e_1(e_2) \to H \; ; \; y := x; x := v; s; x := y}$$

Another try

$$\frac{H \; ; \; e_1 \Downarrow \texttt{fun } x \texttt{ -> } s \qquad H \; ; \; e_2 \Downarrow v \qquad y \text{ "fresh"}}{H \; ; \; e_1(e_2) \rightarrow H \; ; \; y := x; x := v; s; x := y}$$

- "fresh" isn't very IMP-like but okay (think malloc)

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▶ yuck: the way we want to think about something as fundamental as a call?

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- "fresh" isn't very IMP-like but okay (think malloc)

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- yuck: the way we want to think about something as fundamental as a call?

- NO: wrong model for most functional and OO languages
  - (Even wrong for C if s calls another function that accesses the global variable x)

The wrong model

$$\frac{H \; ; \; e_1 \Downarrow \text{fun } x \text{ -> } s \qquad H \; ; \; e_2 \Downarrow v \qquad y \text{ "fresh"}}{H \; ; \; e_1(e_2) \rightarrow H \; ; \; y := x; x := v; s; x := y}$$

```
f₁ := (fun x -> f₂ := (fun z -> ans := x + z));
f₁(2);
x := 3;
f₂(4)
```

"Should" set ans to 6:

- $f_1(2)$ should assign to f_2 a function that adds 2 to its argument and stores result in ans

"Actually" sets ans to 7:

- $f_2(2)$ assigns to f_2 a function that adds *the current value of* x to its argument

Punch line

Cannot properly model local scope via a global heap of integers.

- ▶ Functions are not syntactic sugar for assignments to globals

So let's build a new model that focuses on this essential concept

- ▶ (can add back IMP features later)

Or just borrow a model from Alonzo Church

And drop mutation, conditionals, integers (!), and loops (!)

The Lambda Calculus

The Lambda Calculus:

$$e ::= \lambda x.\, e \mid x \mid e\; e$$
$$v ::= \lambda x.\, e$$

You *apply* a function by *substituting* the argument for the *bound variable*

- (There is an equivalent *environment* definition not unlike heap-copying; see future homework)

Example Substitutions

$$e ::= \lambda x. \, e \mid x \mid e \, e$$
$$v ::= \lambda x. \, e$$

Substitution is the key operation we were missing:

$$(\lambda x. \, x)(\lambda y. \, y) \rightarrow (\lambda y. \, y)$$

$$(\lambda x. \, \lambda y. \, y \, x)(\lambda z. \, z) \rightarrow (\lambda y. \, y \, \lambda z. \, z)$$

$$(\lambda x. \, x \, x)(\lambda x. \, x \, x) \rightarrow (\lambda x. \, x \, x)(\lambda x. \, x \, x)$$

After substitution, the bound variable is gone, so its "name" was irrelevant. (Good!)

A Programming Language

Given substitution $(e_1[e_2/x] = e_3)$, we can give a semantics:

$$\boxed{e \to e'}$$

$$\frac{e[v/x] = e'}{(\lambda x.\ e)\ v \to e'} \qquad \frac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2} \qquad \frac{e_2 \to e_2'}{v\ e_2 \to v\ e_2'}$$

A small-step, *call-by-value (CBV)*, left-to-right semantics

- ▶ Terminates when the "whole program" is some $\lambda x.\ e$

But (also) gets stuck when there's a *free variable* "at top-level"

- ▶ Won't "cheat" like we did with $H(x)$ in IMP because scope is what we're interested in

This is the "heart" of functional languages like Caml

- ▶ But "real" implementations don't substitute; they do something *equivalent*

Roadmap

- Motivation for a new model (done)

- CBV lambda calculus using substitution (done)

- Notes on concrete syntax

- Simple Lambda encodings (it's Turing complete!)

- Other reduction strategies

- Defining substitution

Concrete-Syntax Notes

We (and Caml) resolve concrete-syntax ambiguities as follows:

1. $\lambda x.\ e_1\ e_2$ is $(\lambda x.\ e_1\ e_2)$, not $(\lambda x.\ e_1)\ e_2$
2. $e_1\ e_2\ e_3$ is $(e_1\ e_2)\ e_3$, not $e_1\ (e_2\ e_3)$
   ▶ Convince yourself application is not associative

More generally:

1. Function bodies extend to an unmatched right parenthesis
   Example: $(\lambda x.\ y(\lambda z.\ z)w)q$

2. Application associates to the left
   Example: $e_1\ e_2\ e_3\ e_4$ is $(((e_1\ e_2)\ e_3)\ e_4)$

- ▶ Like in IMP, assume we really have ASTs
  (with non-leaves labeled λ or "application")
- ▶ Rules may seem strange at first, but it's the most convenient
  concrete syntax
  - ▶ Based on 70 years experience

Lambda Encodings

Fairly crazy: we left out constants, conditionals, primitives, and data structures

In fact, we're *Turing complete* and can *encode* whatever we need (just like assembly language can)

Motivation for encodings:

- ▶ Fun and mind-expanding
- ▶ Shows we aren't oversimplifying the model ("numbers are syntactic sugar")
- ▶ Can show languages are *too expressive* (e.g., unlimited C++ template instantiation)

Encodings are also just "(re)definition via translation"

Encoding booleans

The "Boolean ADT"

- ▶ There are two booleans and one conditional expression.
- ▶ The conditional takes 3 arguments (e.g., via currying). If the first is one boolean it evaluates to the second. If it's the other boolean it evaluates to the third.

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Here is one of an infinite number of encodings:

$$\text{"true"} \quad \lambda x.\ \lambda y.\ x$$
$$\text{"false"} \quad \lambda x.\ \lambda y.\ y$$
$$\text{"if"} \quad \lambda b.\ \lambda t.\ \lambda f.\ b\ t\ f$$

Example: "if" "true" $v_1\ v_2 \rightarrow^*\ v_1$

Evaluation Order Matters

Careful: With CBV we need to "thunk"...

$$\text{``if''} \text{``true''} (\lambda x.\ x) \underbrace{((\lambda x.\ x\ x)(\lambda x.\ x\ x))}_{\text{an infinite loop}}$$

diverges, but

$$\text{``if''} \text{``true''} (\lambda x.\ x) \underbrace{(\lambda z.\ ((\lambda x.\ x\ x)(\lambda x.\ x\ x))\ z))}_{\text{a value that when called diverges}}$$

doesn't.

Encoding Pairs

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$$
\begin{array}{ll}
\text{"mkpair"} & \lambda x.\ \lambda y.\ \lambda z.\ z\ x\ y \\
\text{"fst"} & \lambda p.\ p(\lambda x.\ \lambda y.\ x) \\
\text{"snd"} & \lambda p.\ p(\lambda x.\ \lambda y.\ y)
\end{array}
$$

Example:

$$\text{"snd"}\ (\text{"fst"}\ (\text{"mkpair"}\ (\text{"mkpair"}\ v_1\ v_2)\ v_3)) \rightarrow^* v_2$$

Reusing Lambdas

Is it weird that the encodings of Booleans and pairs both used $\lambda x.\ \lambda y.\ x$ and $\lambda x.\ \lambda y.\ y$ for different purposes?

Is it weird that the same bit-pattern in binary code can represent an int, a float, an instruction, or a pointer?

Von Neumann: Bits can represent (all) code and data

Church (?): Lambdas can represent (all) code and data

Beware the "Turing tarpit"

Encoding Lists

Rather than start from scratch, notice that booleans and pairs are enough to encode lists:

- Empty list is "mkpair" "false" "false"
- Non-empty list is $\lambda h.\ \lambda t.$ "mkpair" "true" ("mkpair" $h\ t$)
- Is-empty is ...
- Head is ...
- Tail is ...

Note:

- Not too far from how lists are implemented
- Taking "tail" ("tail" "empty") will produce some lambda
  - Just like, without page-protection hardware,
    `null->tail->tail` would produce some bit-pattern

Encoding Recursion

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- ▶ Write a function that takes an f and calls it in place of recursion
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- ▸ The details, especially for CBV, are icky; the point is it's possible and you define "fix" only once

- ▸ Not on exam:
  "fix" $\lambda g.\ (\lambda x.\ g\ (\lambda y.\ x\ x\ y))(\lambda x.\ g\ (\lambda y.\ x\ x\ y))$

Encoding Arithmetic Over Natural Numbers

How about arithmetic?

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How I would do it based on what we have so far:

- ▶ Lists of booleans for binary numbers
    - ▶ Zero can be the empty list
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But instead everybody always teaching Church numerals. Why?

- ▶ Tradition? Some sense of professional obligation?
- ▶ Better reason: You don't need fix: Basic arithmetic is often encodable in languages where all programs terminate
- ▶ In any case, we'll show some basics "just for fun"

Church Numerals

$$
\begin{array}{ll}
\text{``0''} & \lambda s.\ \lambda z.\ z \\
\text{``1''} & \lambda s.\ \lambda z.\ s\ z \\
\text{``2''} & \lambda s.\ \lambda z.\ s\ (s\ z) \\
\text{``3''} & \lambda s.\ \lambda z.\ s\ (s\ (s\ z)) \\
& \ldots
\end{array}
$$

- Numbers encoded with two-argument functions
- The "number i" composes the first argument i times, starting with the second argument
  - z stands for "zero" and s for "successor" (think unary)
- The trick is implementing arithmetic by cleverly passing the right arguments for s and z

Church Numerals

"0"	$\lambda s.\ \lambda z.\ z$
"1"	$\lambda s.\ \lambda z.\ s\ z$
"2"	$\lambda s.\ \lambda z.\ s\ (s\ z)$
"3"	$\lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
"successor"	$\lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$

successor: take "a number" and return "a number" that (when called) applies s one more time

Church Numerals

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"2"	$\lambda s.\ \lambda z.\ s\ (s\ z)$
"3"	$\lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
"successor"	$\lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$
"plus"	$\lambda n.\ \lambda m.\ \lambda s.\ \lambda z.\ n\ s\ (m\ s\ z)$

plus: take two "numbers" and return a "number" that uses one number as the zero argument for the other

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"0"	$\lambda s.\ \lambda z.\ z$
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"successor"	$\lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$
"plus"	$\lambda n.\ \lambda m.\ \lambda s.\ \lambda z.\ n\ s\ (m\ s\ z)$
"times"	$\lambda n.\ \lambda m.\ m\ ("plus"\ n)\ "zero"$

times: take two "numbers" m and n and pass to m a function that adds n to its argument (so this will happen m times) and "zero" (where to start the m iterations of addition)

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"1"	$\lambda s.\ \lambda z.\ s\ z$
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"plus"	$\lambda n.\ \lambda m.\ \lambda s.\ \lambda z.\ n\ s\ (m\ s\ z)$
"times"	$\lambda n.\ \lambda m.\ m\ ("plus"\ n)\ "zero"$
"isZero"	$\lambda n.\ (\lambda x.\ "false")\ "true"$

isZero: an easy one, see how the two arguments will lead to the correct answer

Church Numerals

"0"	$\lambda s.\ \lambda z.\ z$
"1"	$\lambda s.\ \lambda z.\ s\ z$
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"plus"	$\lambda n.\ \lambda m.\ \lambda s.\ \lambda z.\ n\ s\ (m\ s\ z)$
"times"	$\lambda n.\ \lambda m.\ m$ ("plus" n) "zero"
"isZero"	$\lambda n.\ (\lambda x.$ "false") "true"
"predecessor"	(with 0 sticky) the hard one; see Wikipedia
"minus"	similar to times with pred instead of plus
"isEqual"	subtract and test for zero

Roadmap

- ▶ Motivation for a new model (done)

- ▶ CBV lambda calculus using substitution (done)

- ▶ Notes on concrete syntax (done)

- ▶ Simple Lambda encodings (it's Turing complete!) (done)

- ▶ Other reduction strategies

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Then start type systems

- ▶ Later take a break from types to consider first-class continuations and related topics