CS 152 Programming Languages

Probabilistic Programming & Probabilistic Programming Languages

Yizhou Zhang
University of Waterloo

What is a Probabilistic Program?

What is a Probabilistic Program?

Drawing samples

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Drawing samples

Conditioning

specifies samples that are good

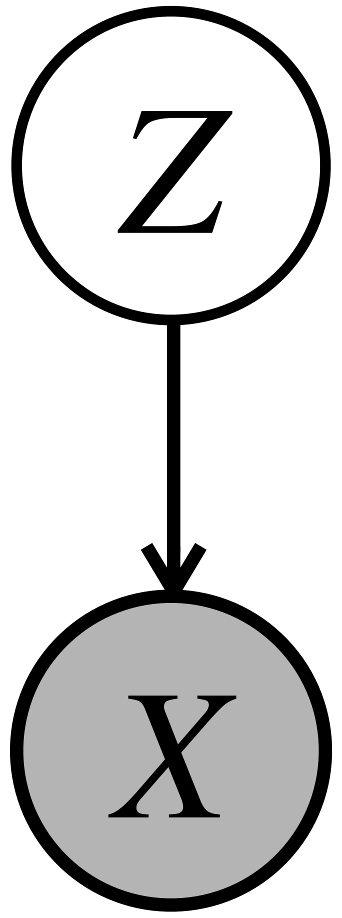
What is a Probabilistic Program?

Drawing samples → Describes a distribution

Conditioning

specifies samples that are good

→ Describes a conditional distribution



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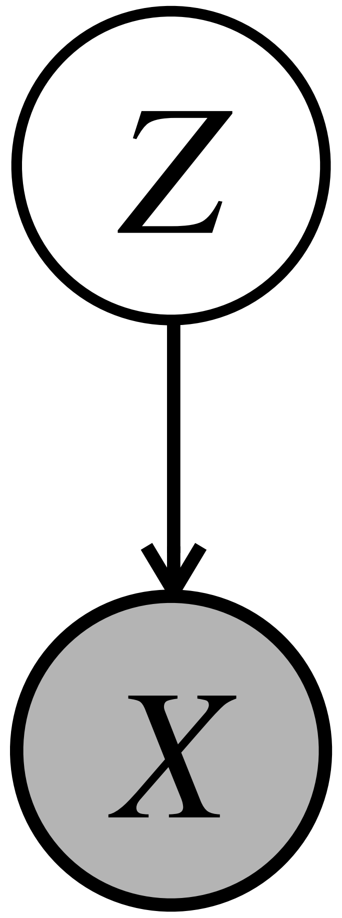
$$p(Z = z \mid X = x)$$



What is a Probabilistic Program?

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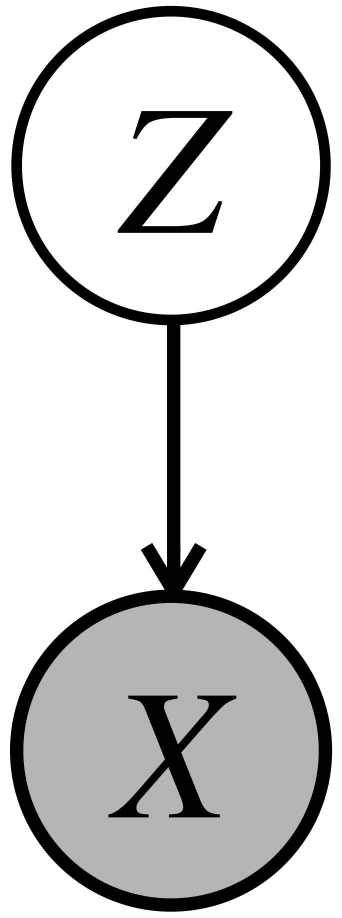
$$p(\underbrace{Z}_{\text{latent}} = z \mid \underbrace{X}_{\text{observed}} = x)$$



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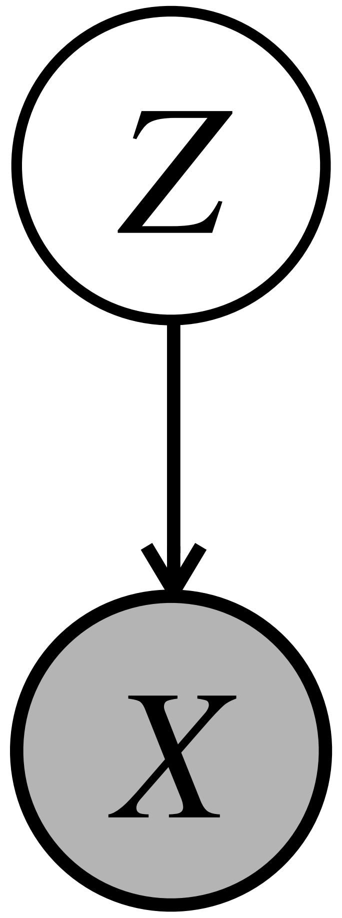
Bayes' Theorem

$$p(\underbrace{Z}_{\text{latent}} = z \mid \underbrace{X}_{\text{observed}} = x) = \frac{p(Z = z, X = x)}{p(X = x)}$$



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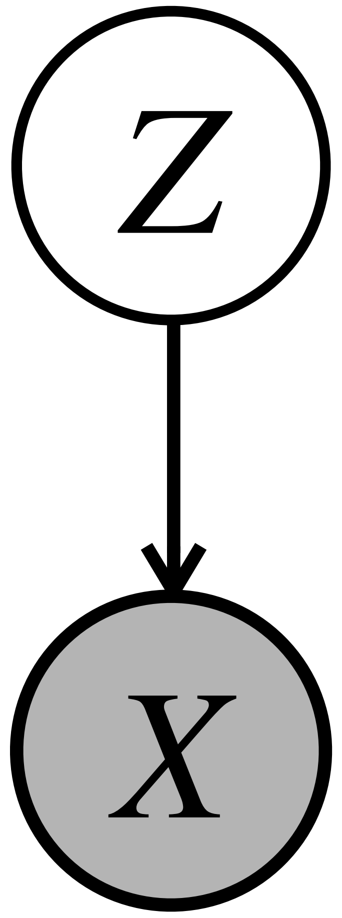
$$\underbrace{p(z \mid x)}_{\text{posterior}} = \frac{\overbrace{p(z, x)}^{\text{joint}}}{\underbrace{p(x)}_{\text{evidence}}}$$



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Bayes' Theorem

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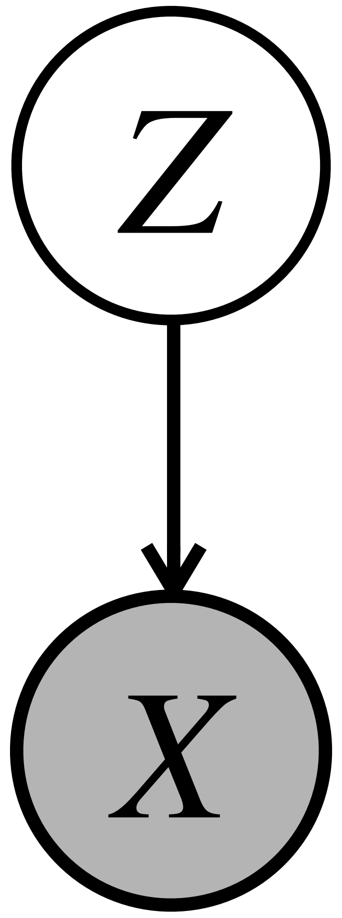

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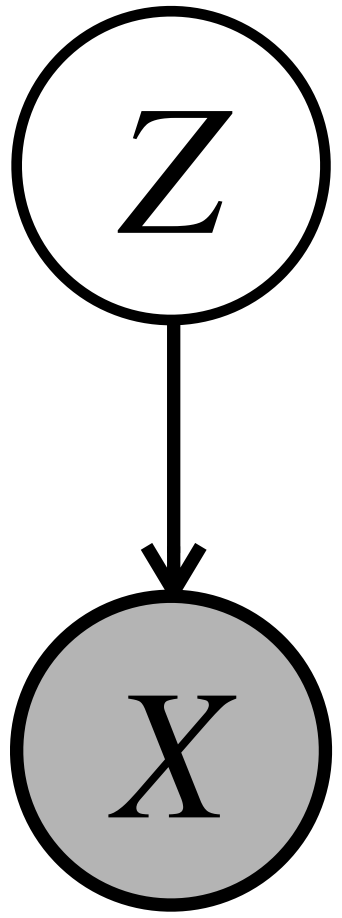
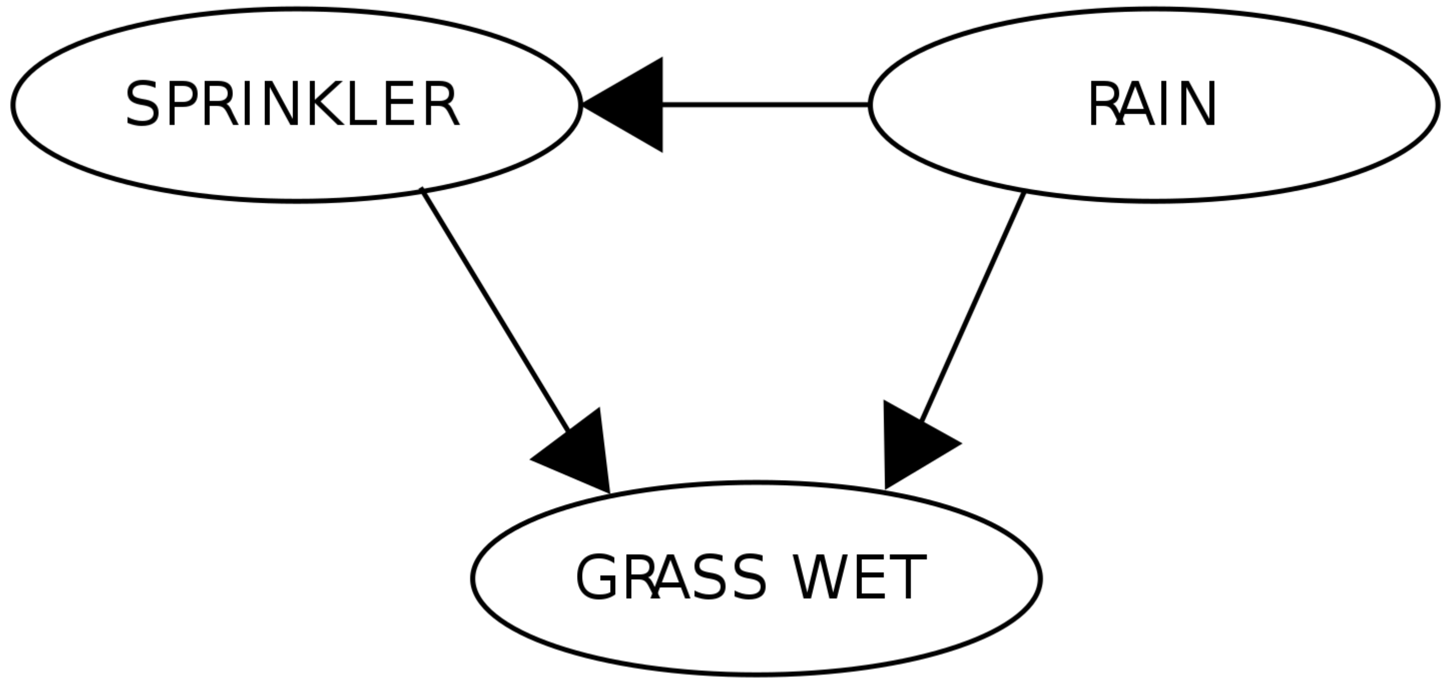


Example: Rain-Sprinkler-Grass

https://en.wikipedia.org/wiki/Bayesian_network#Example

	SPRINKLER	
RAIN	T	F
F	0.4	0.6
T	0.01	0.99

$p(S \mid R)$



RAIN	
T	F
0.2	0.8

$p(R)$

		GRASS WET	
SPRINKLER	RAIN	T	F
F	F	0.0	1.0
F	T	0.8	0.2
T	F	0.9	0.1
T	T	0.99	0.01

$p(W \mid S, R)$

Example: Rain-Sprinkler-Grass

https://en.wikipedia.org/wiki/Bayesian_network#Example

RAIN	SPRINKLER T	SPRINKLER F
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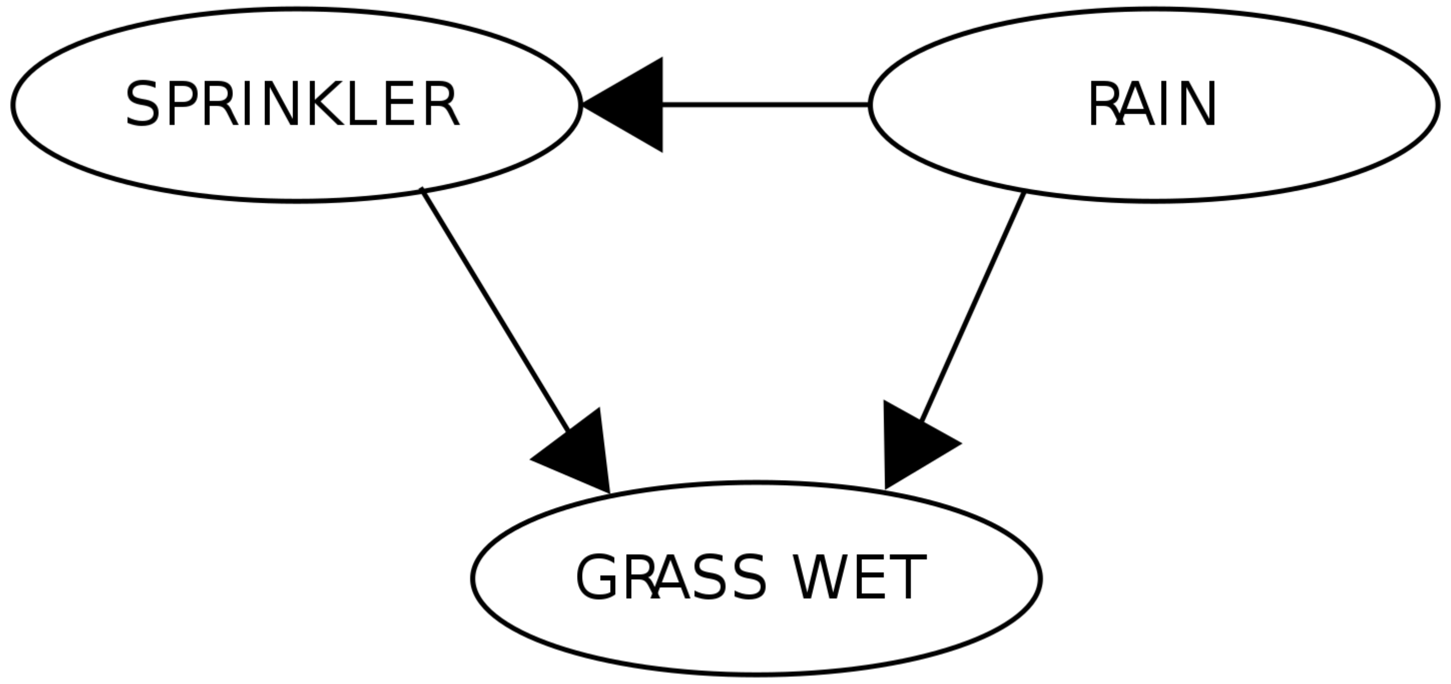
$p(S\,|\,R)$



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$p(R)$

$p(W\,|\,S, R)$

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T	F	0.9	0.1
T	T	0.99	0.01

Aside on notation:

$p(W = \mathsf{T}\,|\,S = \mathsf{T}, R = \mathsf{F})$

probability mass

$p(W\,|\,S = \mathsf{T}, R = \mathsf{F})$

distribution

$p(W\,|\,S, R)$

family of distributions

Example: Rain-Sprinkler-Grass

https://en.wikipedia.org/wiki/Bayesian_network#Example

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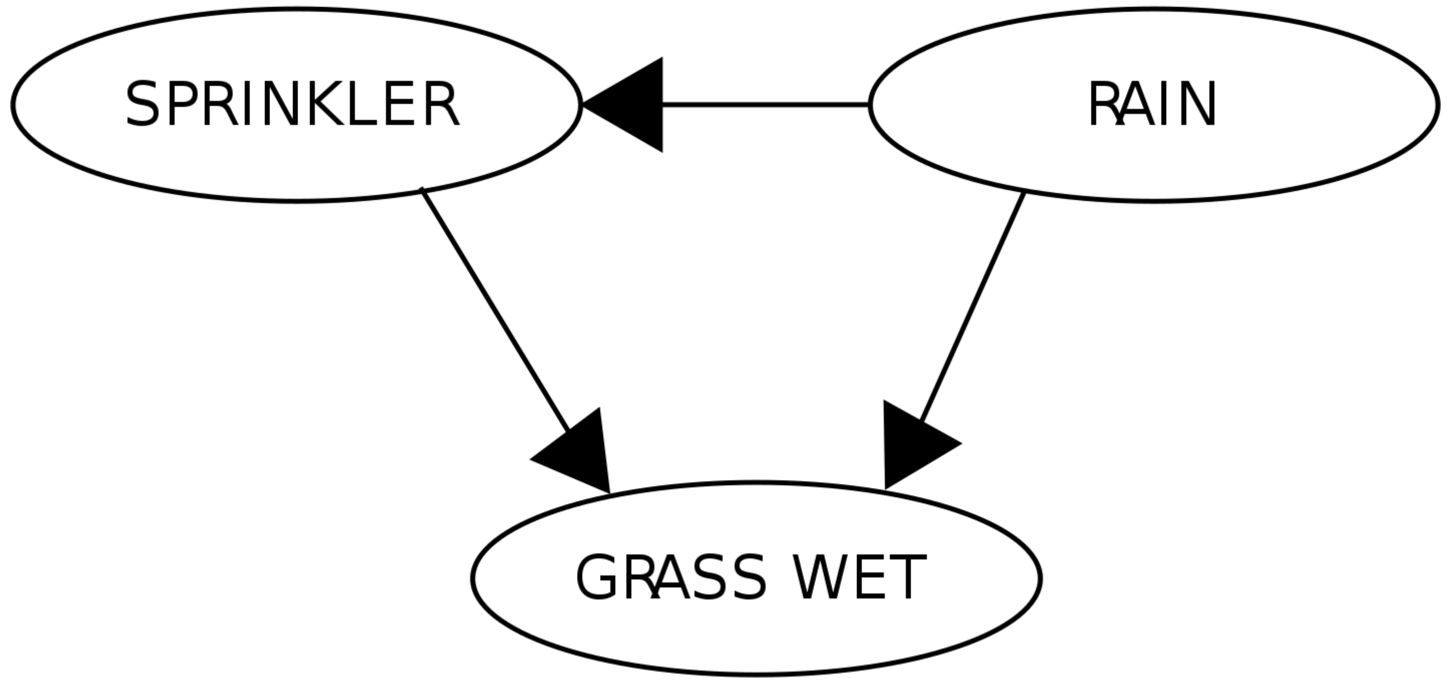

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$p(W \mid S, R)$

Q1: Given that it rained, how likely is that the sprinkler was active?

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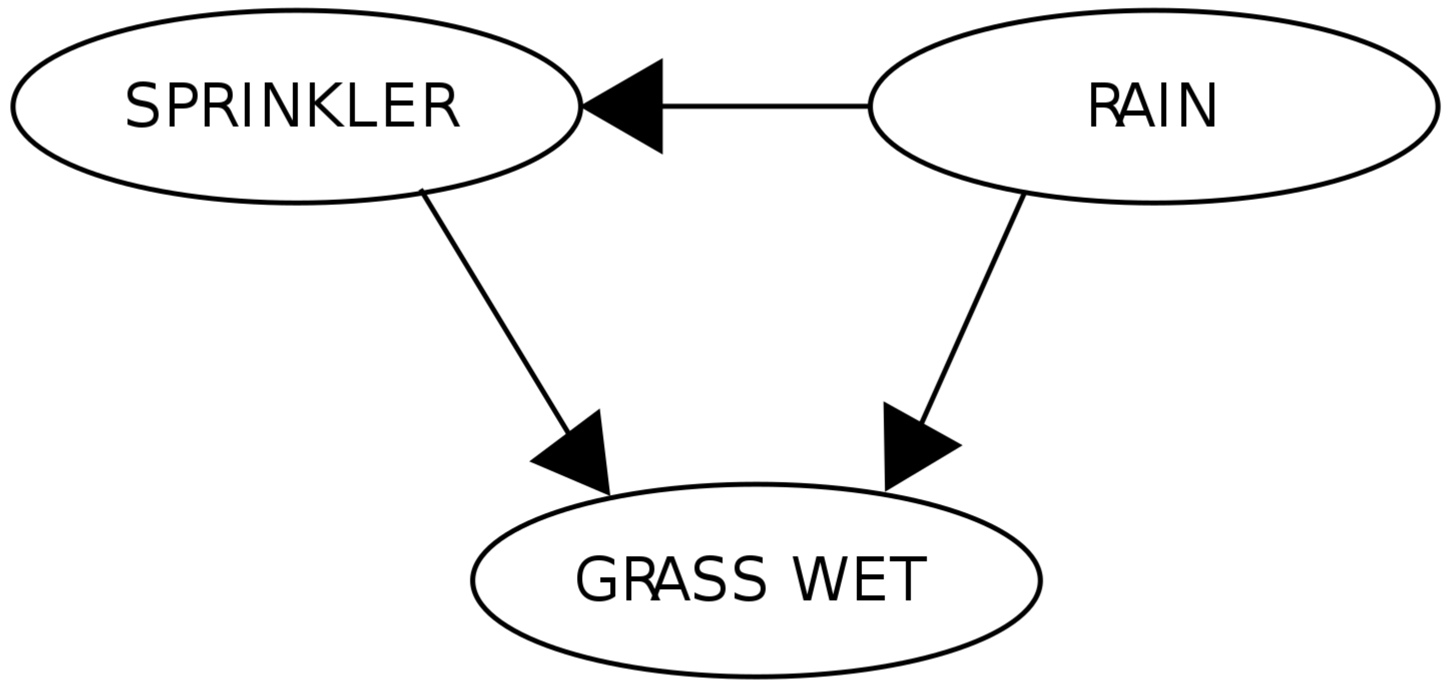

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$p(R)$

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$p(W \mid S, R)$

Q1: Given that it rained, how likely is that the sprinkler was active?

Q2: Given that it rained, how likely is that the grass is wet?

$$p(w \mid R=T)$$
$$= \sum_s p(w, s \mid R=T)$$
$$= \sum_s p(w \mid S=s, R=T)\, p(s \mid R=T)$$
$$= 0.99 \times 0.01 + 0.8 \times 0.99$$
$$= 0.8019$$

Example: Rain-Sprinkler-Grass

https://en.wikipedia.org/wiki/Bayesian_network#Example

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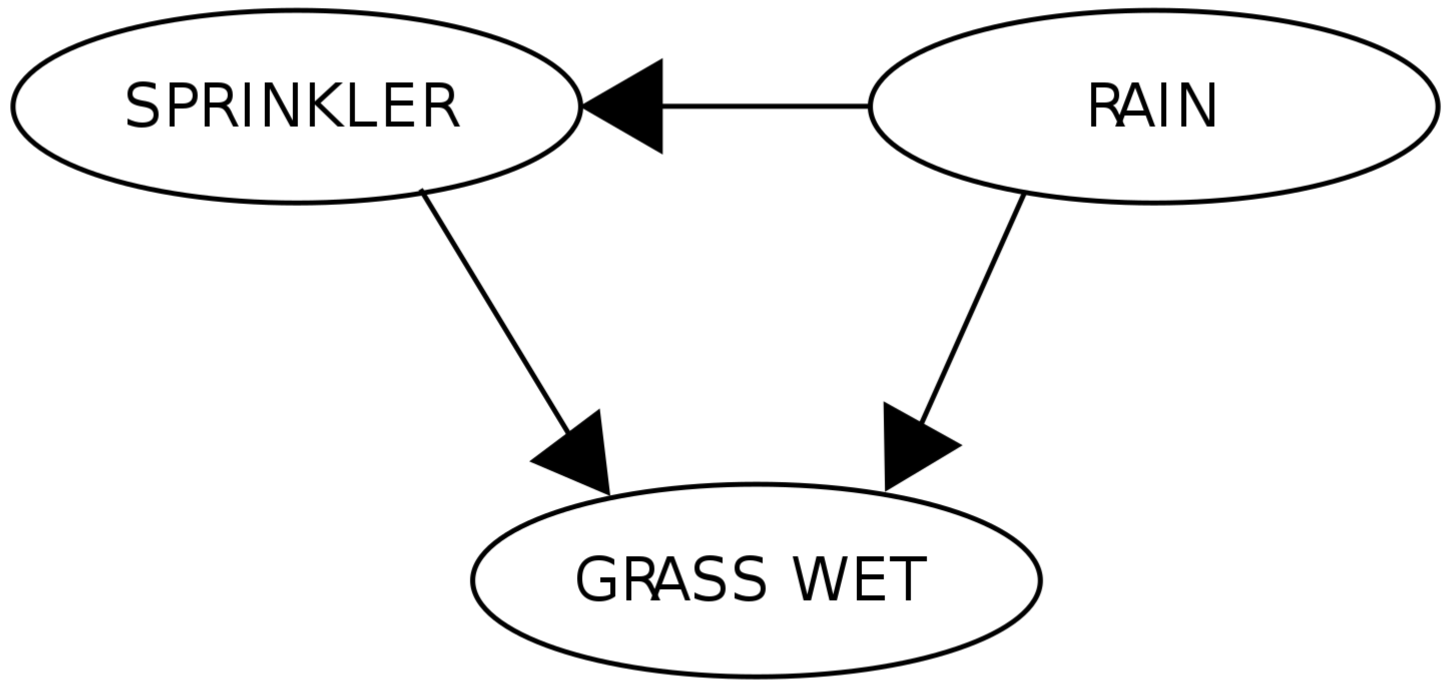

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$p(W \mid S, R)$

Q1: Given that it rained, how likely is that the sprinkler was active?

Q2: Given that it rained, how likely is that the grass is wet?

Q3: Given that grass is wet, how likely is that it rained?

Example: Rain-Sprinkler-Grass

```
var model = function() {
  // Pr(R)
  var r = sample(Bernoulli({p : 0.2}))

  // Pr(S|R=r)
  var s = sample(Bernoulli({p : r ? 0.01 : 0.4}))

  // Pr(W|R=r,S=s)
  var w = sample(Bernoulli({p :
    r ? (s ? 0.99 : 0.8) : (s ? 0.9 : 0.00)
  }))

  // condition model on W being true
  condition(w == true);

  return {R: r, S: s, W: w}
}

// apply Bayesian inference
var R_dist = Infer({
  method: 'enumerate',
  model: function() {
    var result = model()
    return result.R
  }
})
```

Q3: Given that grass is wet, how likely is that it rained?

Example: Rain-Sprinkler-Grass

```
var model = function() {
  // Pr(R)
  var r = sample(Bernoulli({p : 0.2}))

  // Pr(S|R=r)
  var s = sample(Bernoulli({p : r ? 0.01 : 0.4}))

  // Pr(W|R=r,S=s)
  var w = sample(Bernoulli({p :
    r ? (s ? 0.99 : 0.8) : (s ? 0.9 : 0.00)
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  // condition model on W being true
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  return {R: r, S: s, W: w}
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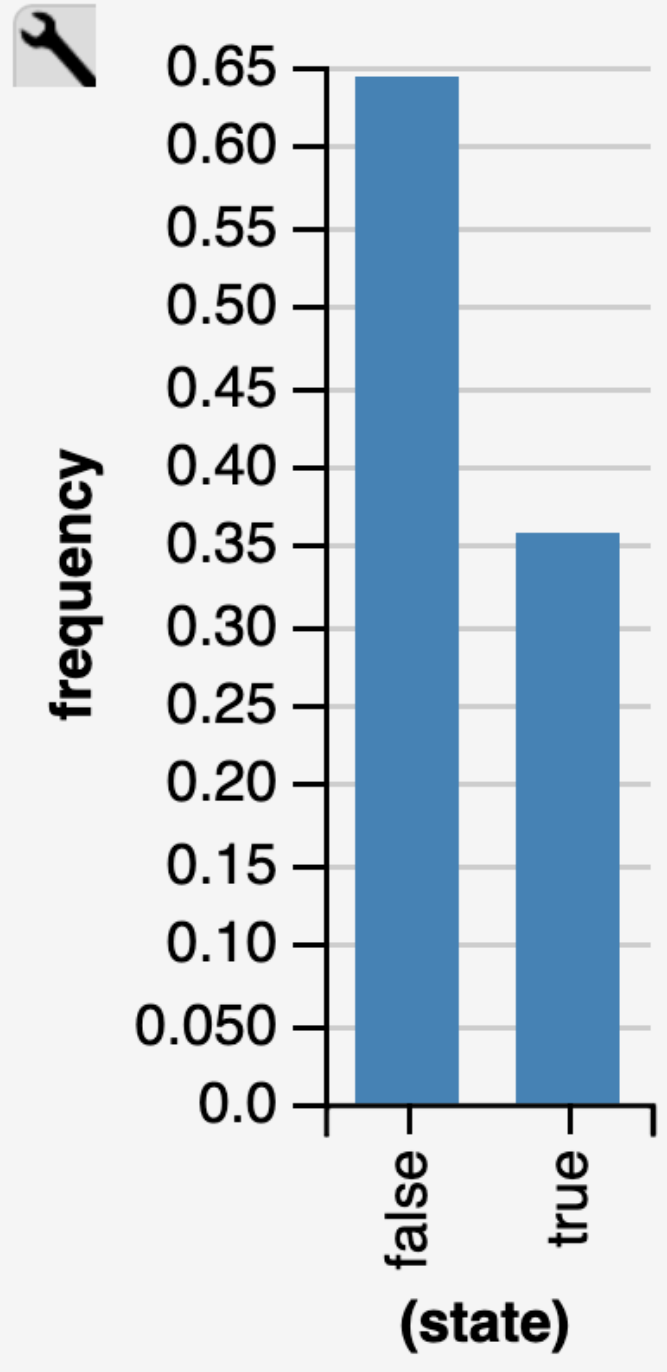


```
Chance of rain is 35.76876756322762%
```

Q3: Given that grass is wet, how likely is that it rained?

Example: Rain-Sprinkler-Grass

https://en.wikipedia.org/wiki/Bayesian_network#Example

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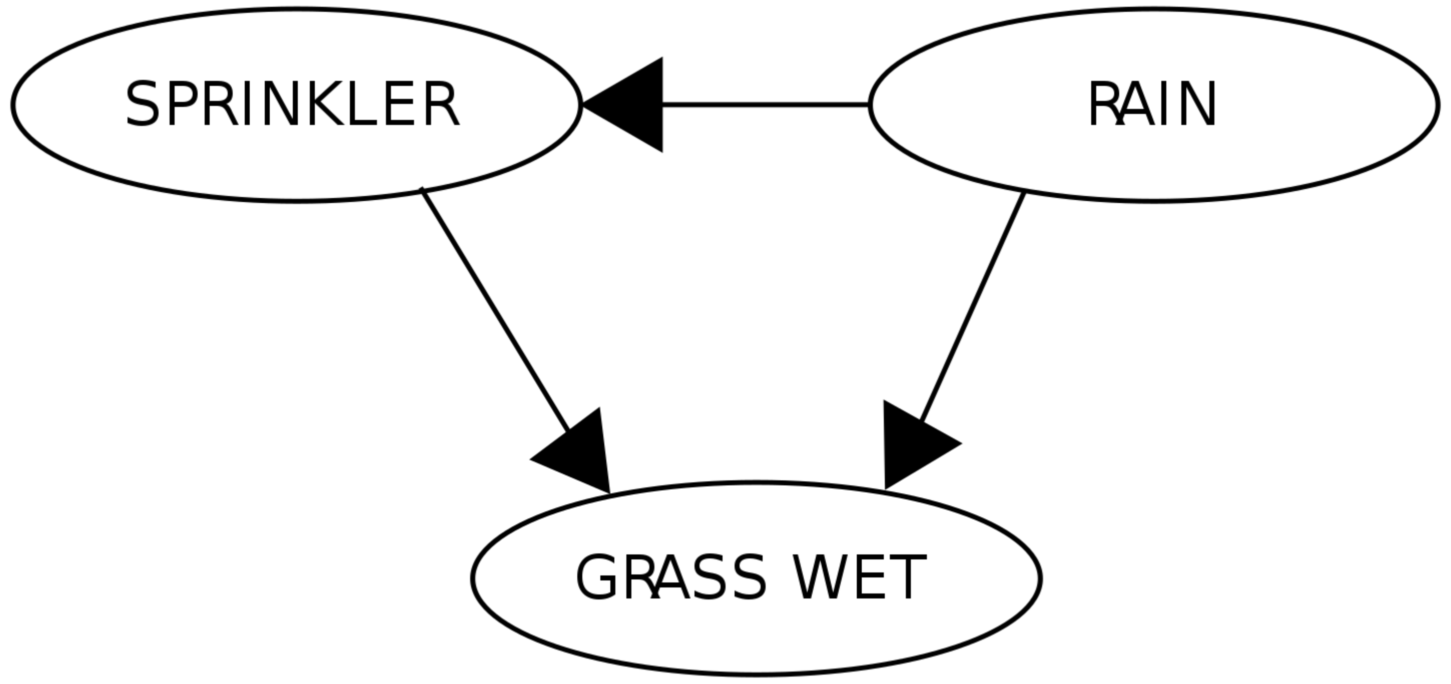
$p(S \mid R)$



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$p(R)$

$p(W \mid S, R)$

SPRINKLER	RAIN	GRASS WET T	GRASS WET F
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Q3: Given that grass is wet, how likely is that it rained?

associational

Example: Rain-Sprinkler-Grass

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Ladder of Causation

	SPRINKLER	
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F	0.4	0.6
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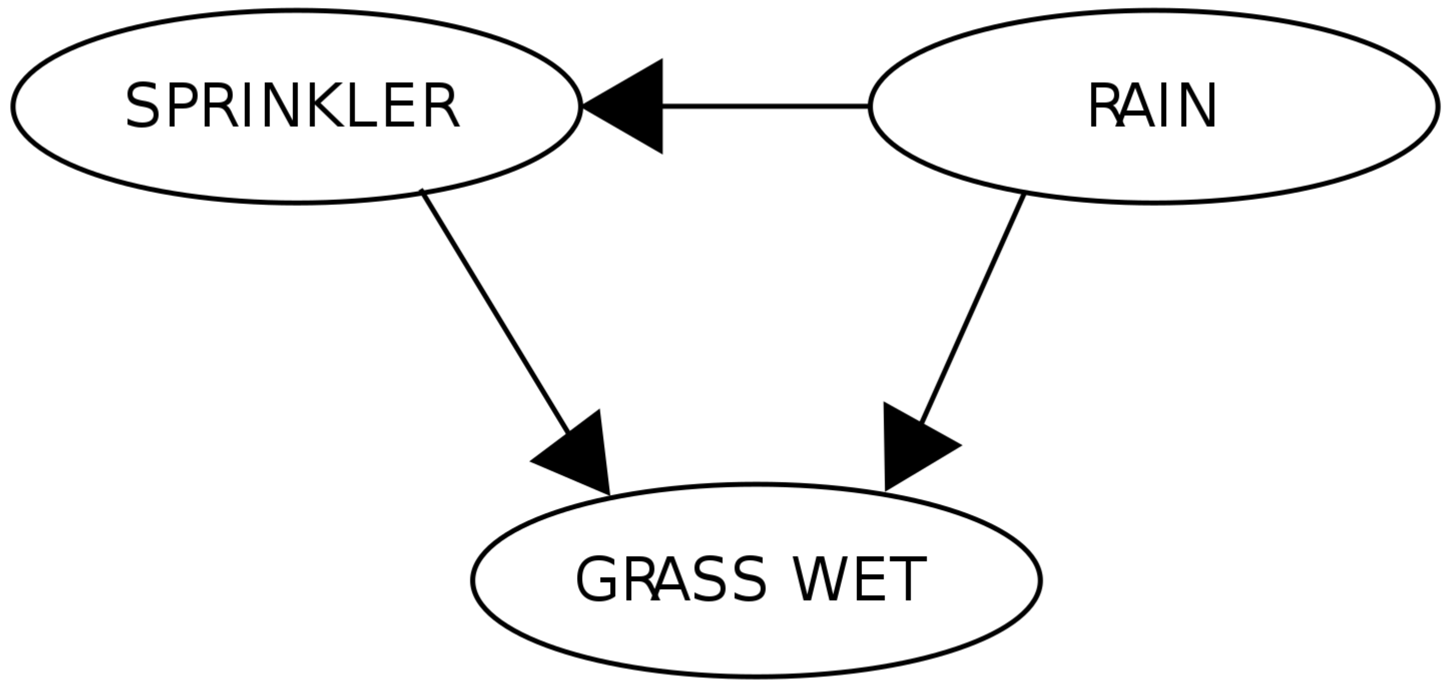
$p(S \mid R)$



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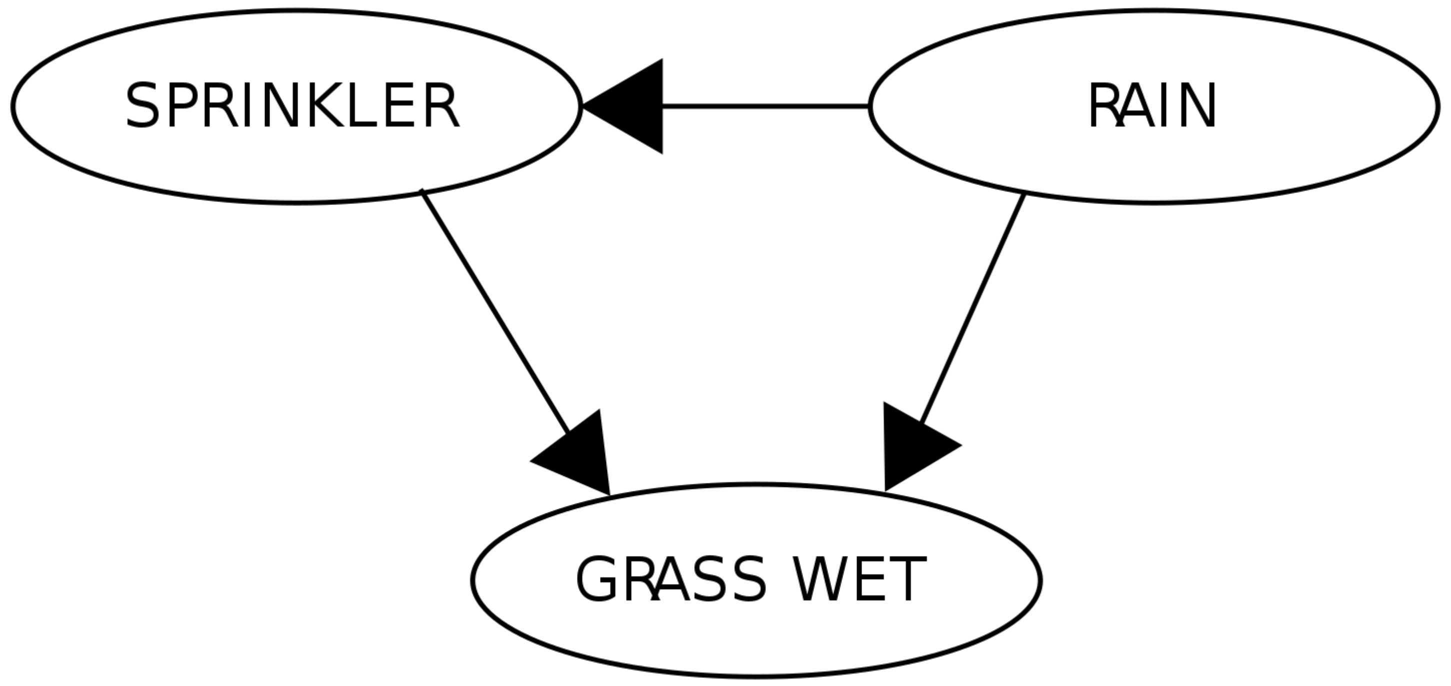

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Ladder of Causation

Q4: If we were to turn on the sprinkler, how likely would the grass be wet?
interventional

Q3: Given that grass is wet, how likely is that it rained?
associational

Example: Rain-Sprinkler-Grass

https://en.wikipedia.org/wiki/Bayesian_network#Example

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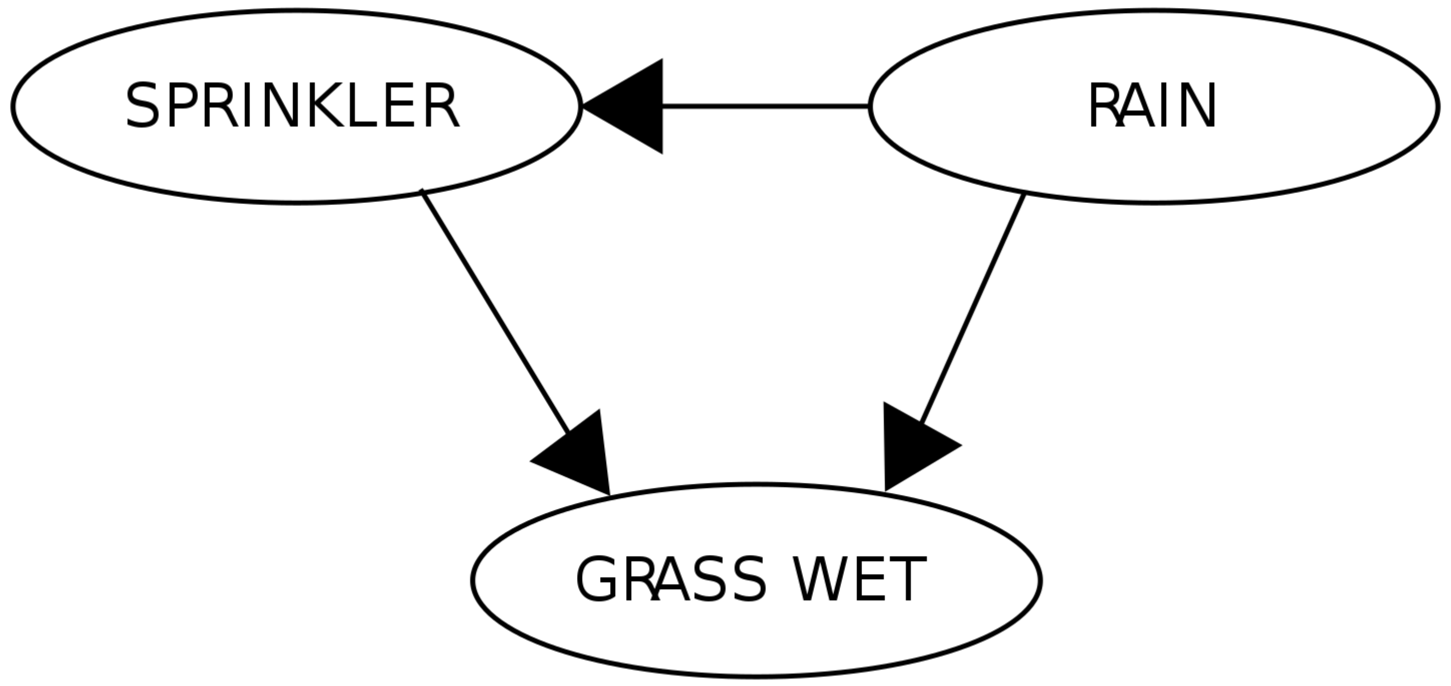

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$p(R)$

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$p(W \mid S, R)$

Ladder of Causation

Q5: Given that the sprinkler is active, had we turned off the sprinkler, how likely would the grass still be wet?
counterfactual

Q4: If we were to turn on the sprinkler, how likely would the grass be wet?
interventional

Q3: Given that grass is wet, how likely is that it rained?
associational

Example: Rain-Sprinkler-Grass

https://en.wikipedia.org/wiki/Bayesian_network#Example

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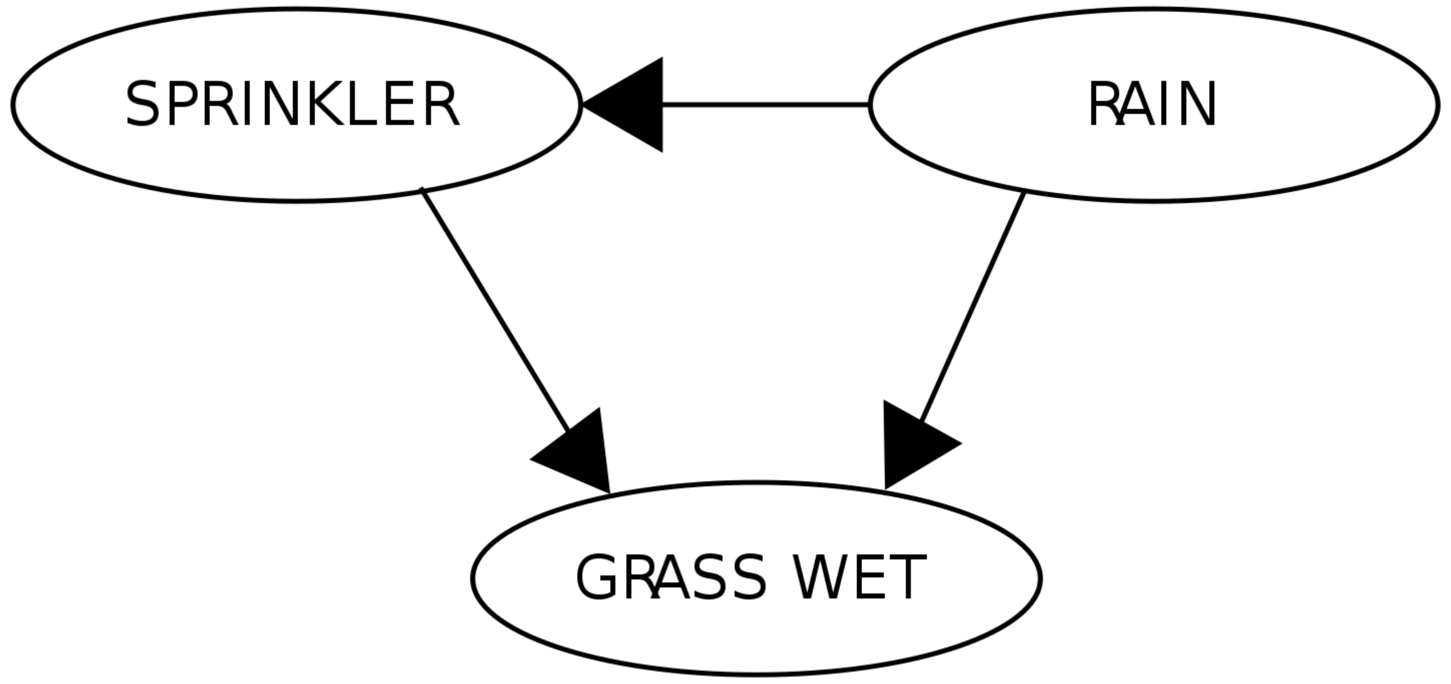
$p(S \mid R)$



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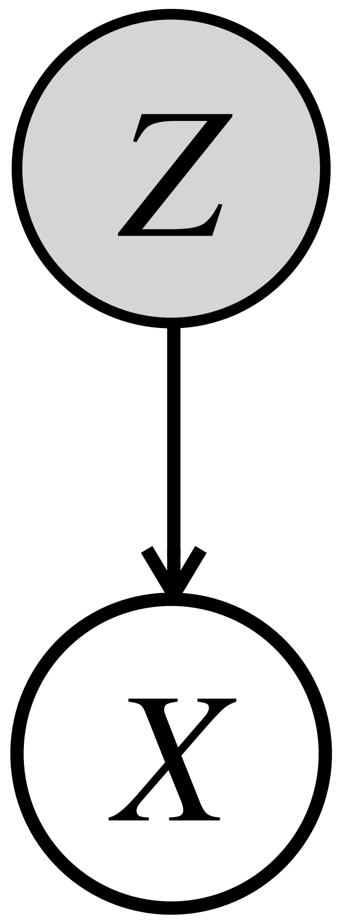
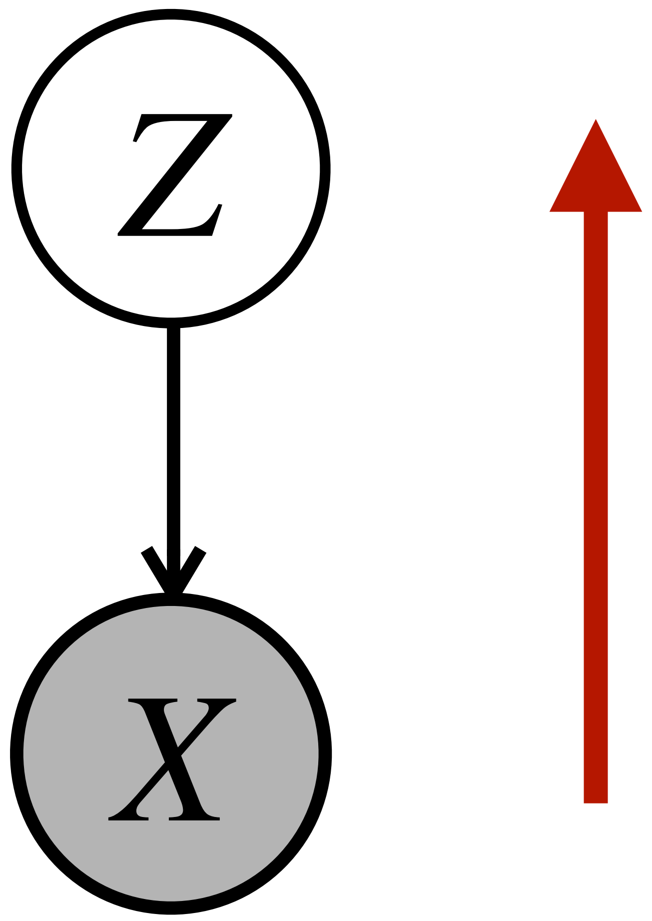
associational

Programming

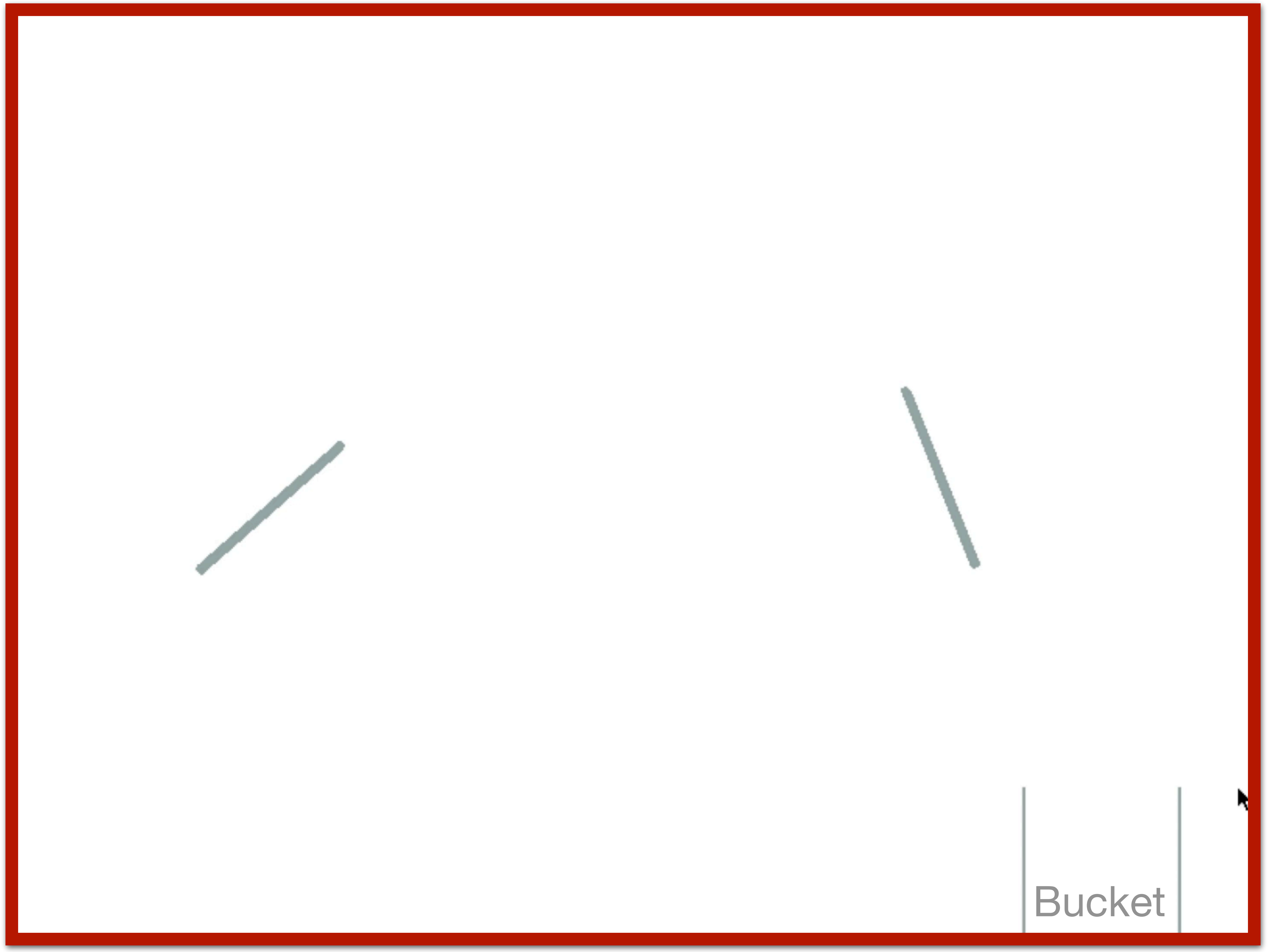


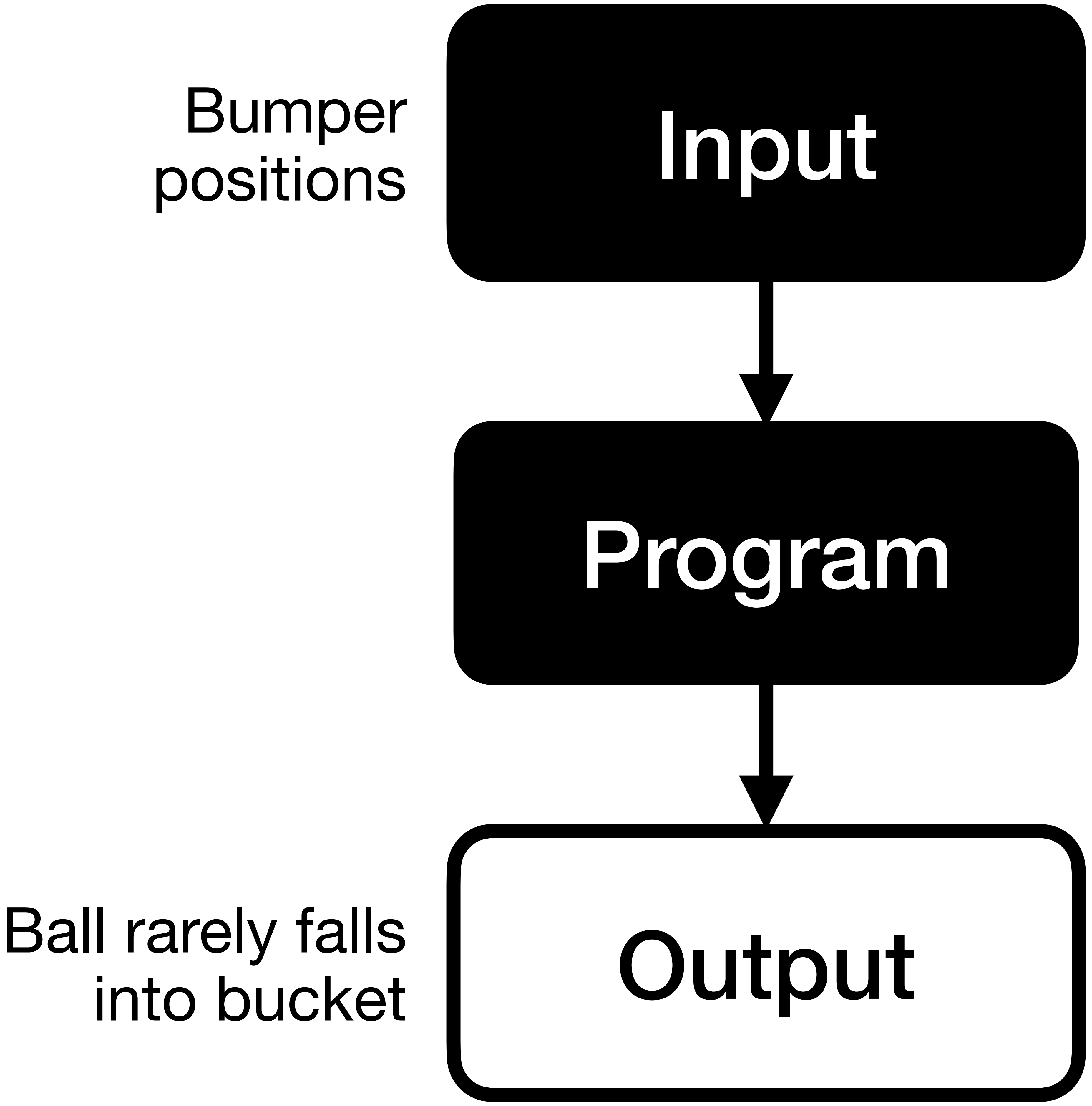
Probabilistic Programming



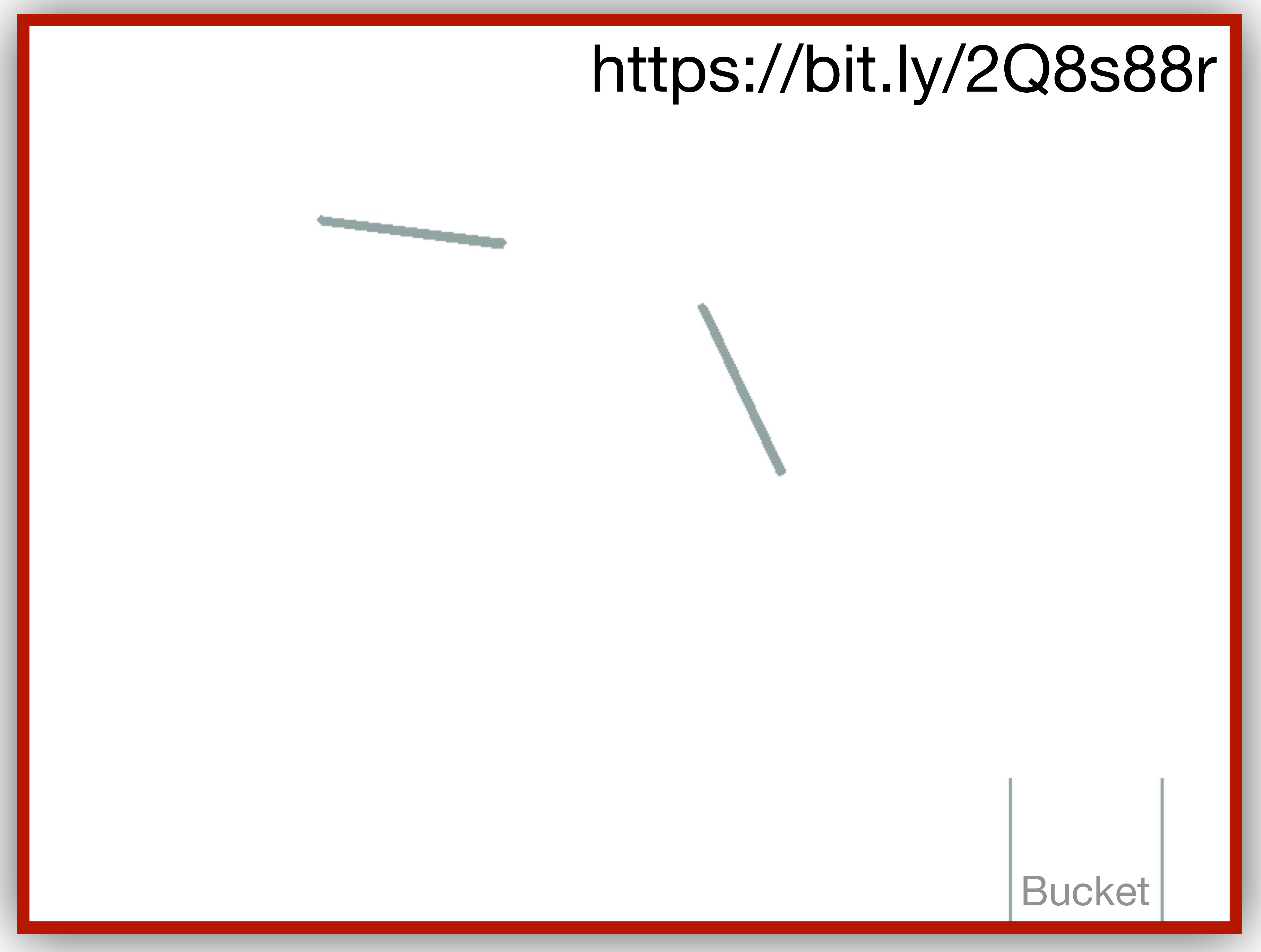
$$p(z\,|\,x) = \frac{p(x\,|\,z)\,p(z)}{p(x)}$$

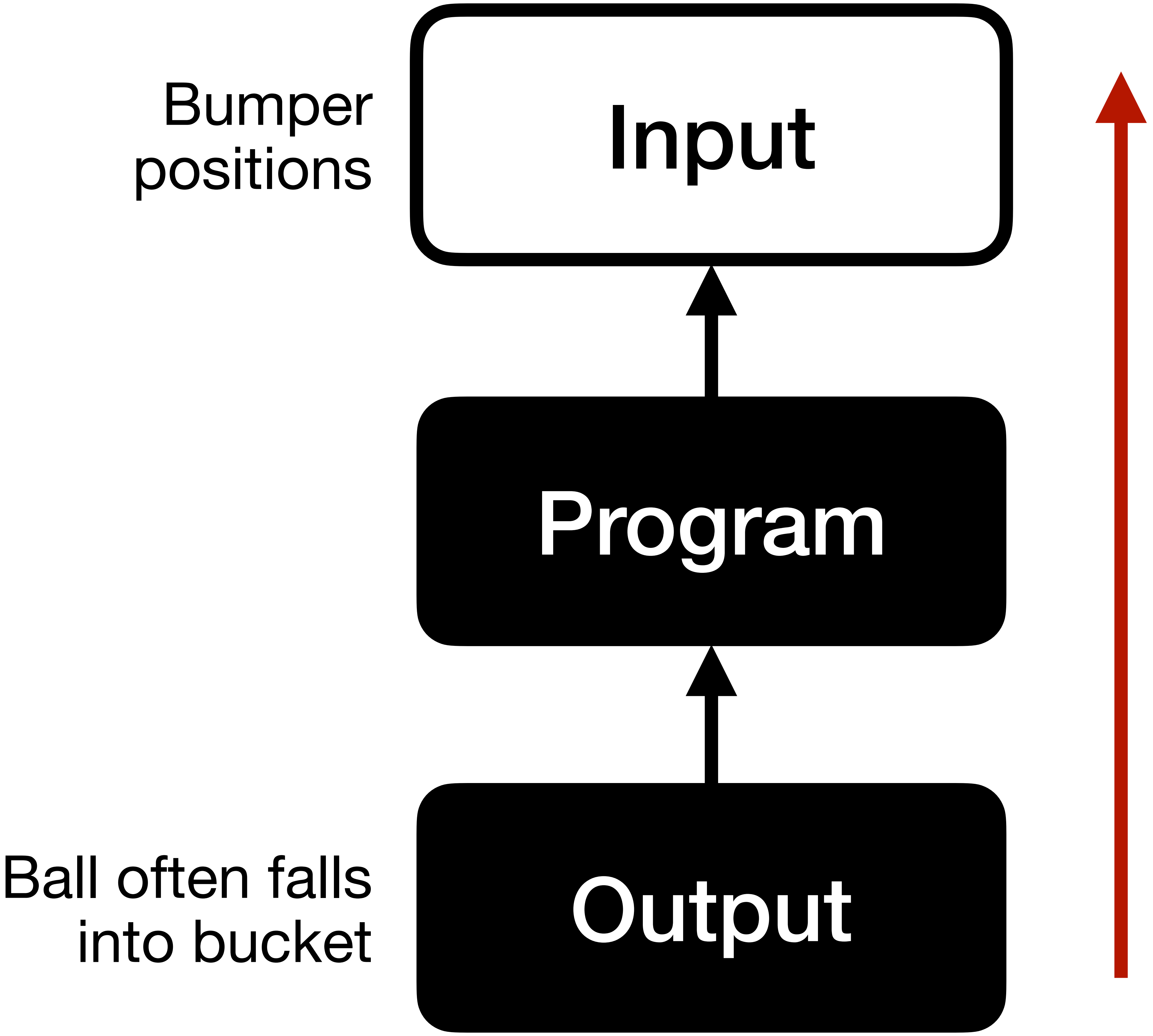
Example: Bouncing Balls



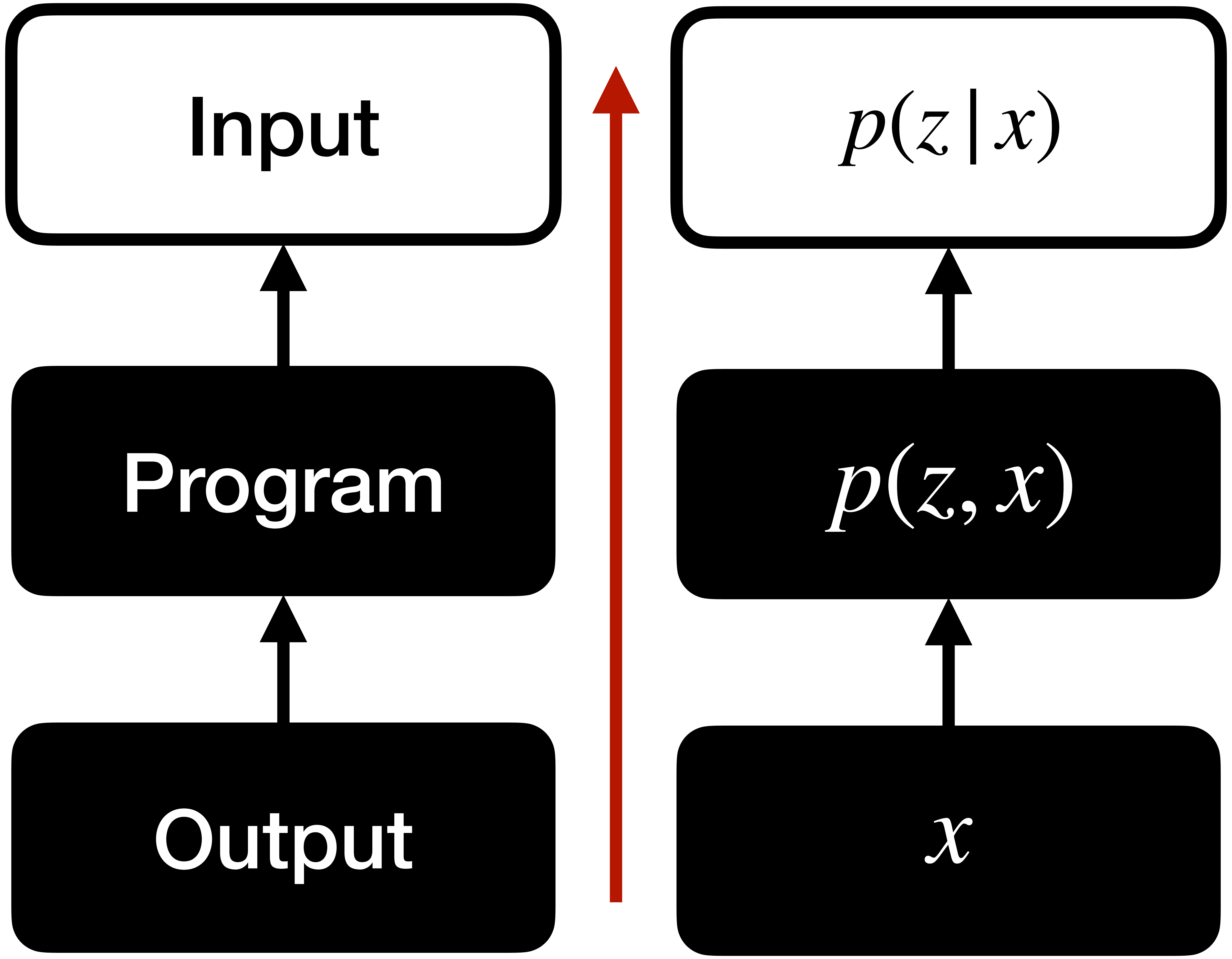


Example: Bouncing Balls into Bucket



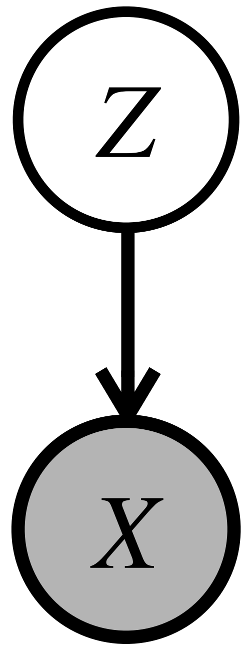


Applications of Probabilistic Programming



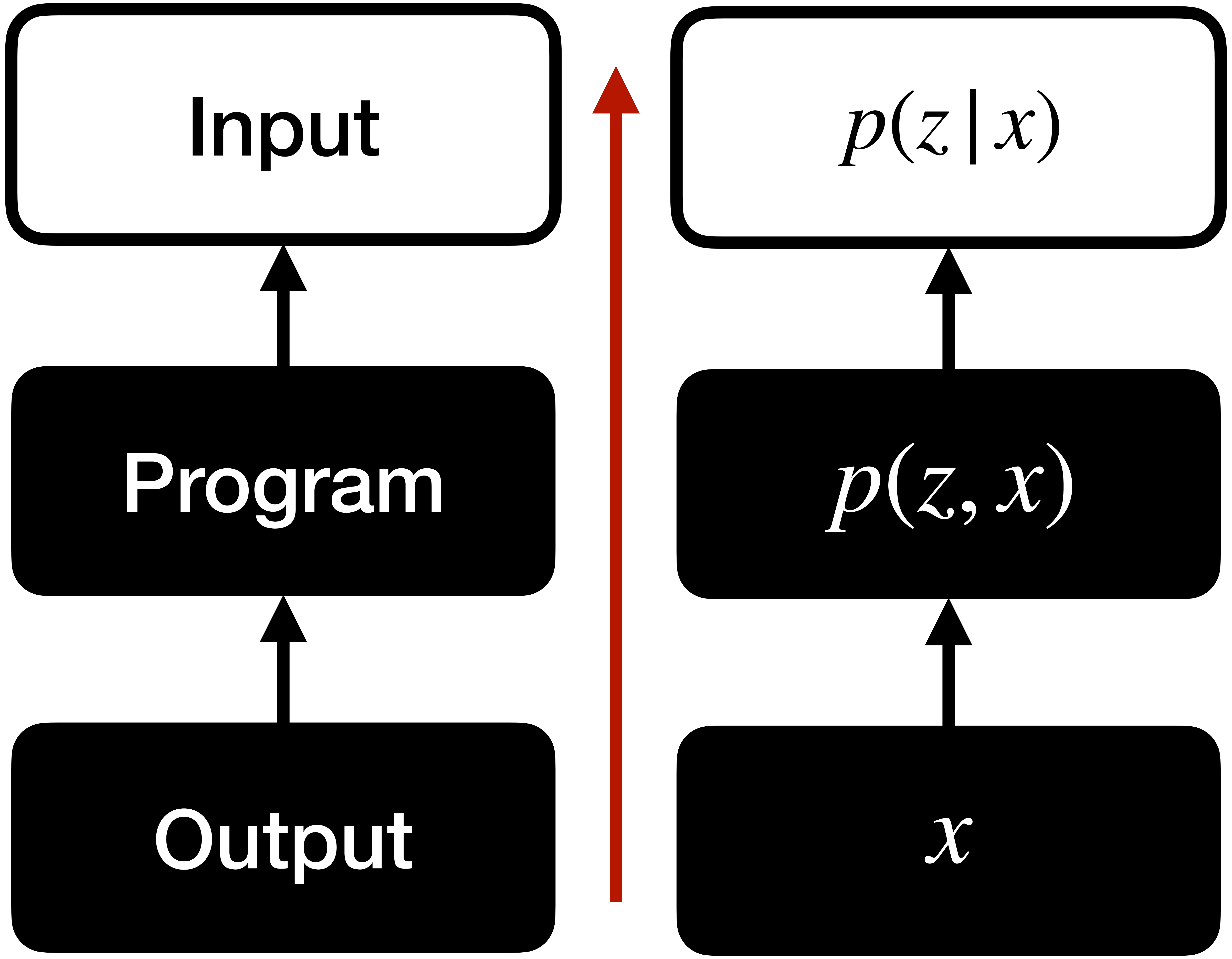
Program source code | Agent's policy/plan | Captcha letters

$$p(z \mid x) = \frac{p(x \mid z)\, p(z)}{p(x)}$$



Input-output examples | Agent's reward | Captcha images

Applications of Probabilistic Programming



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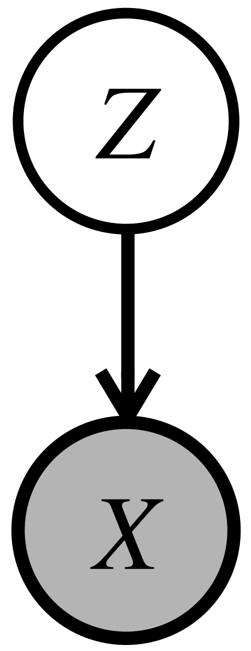
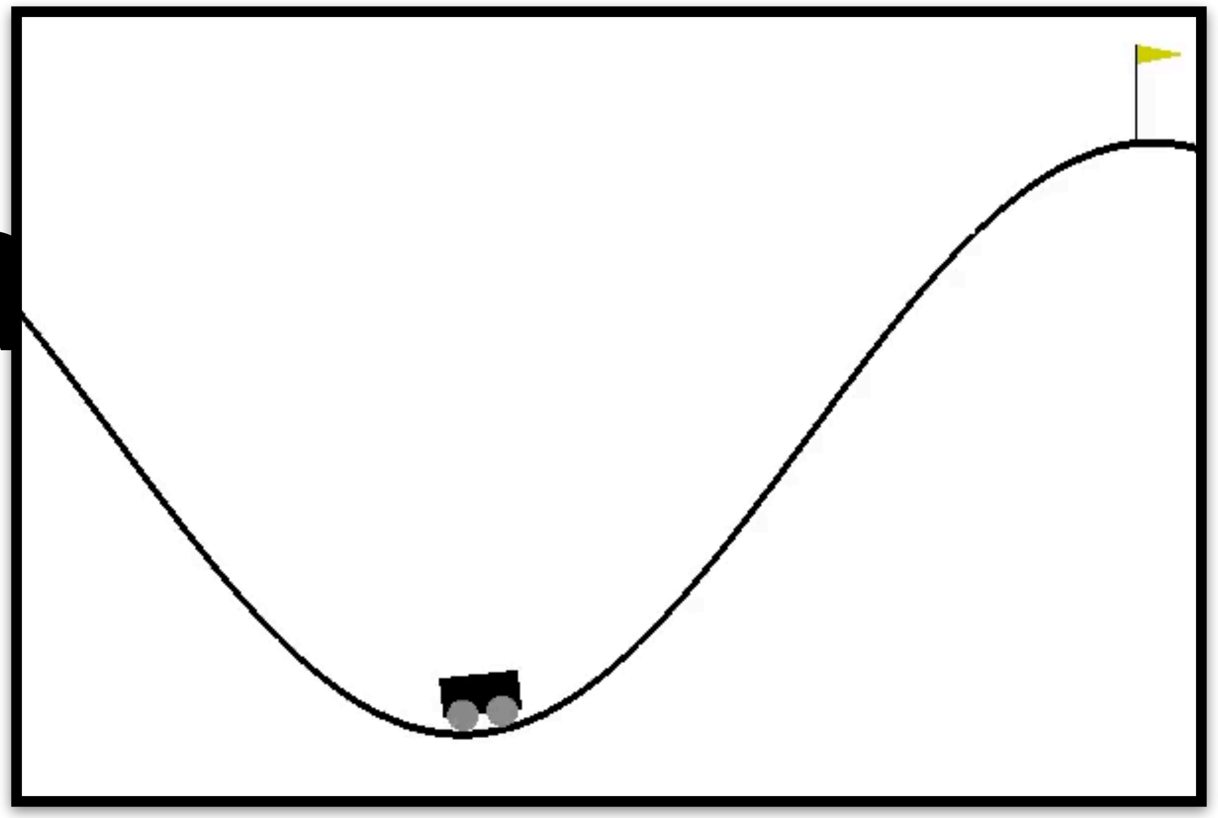


Input-output examples | Agent's reward | Captcha images

Applications of Probabilistic Progra



Input

Program

Output

$p(z \mid x)$

$p(z, x)$

x

Program source code

Agent's policy/plan

Captcha letters

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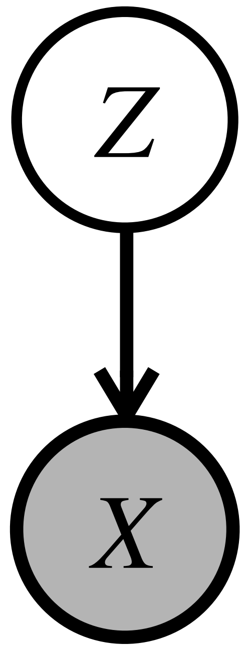
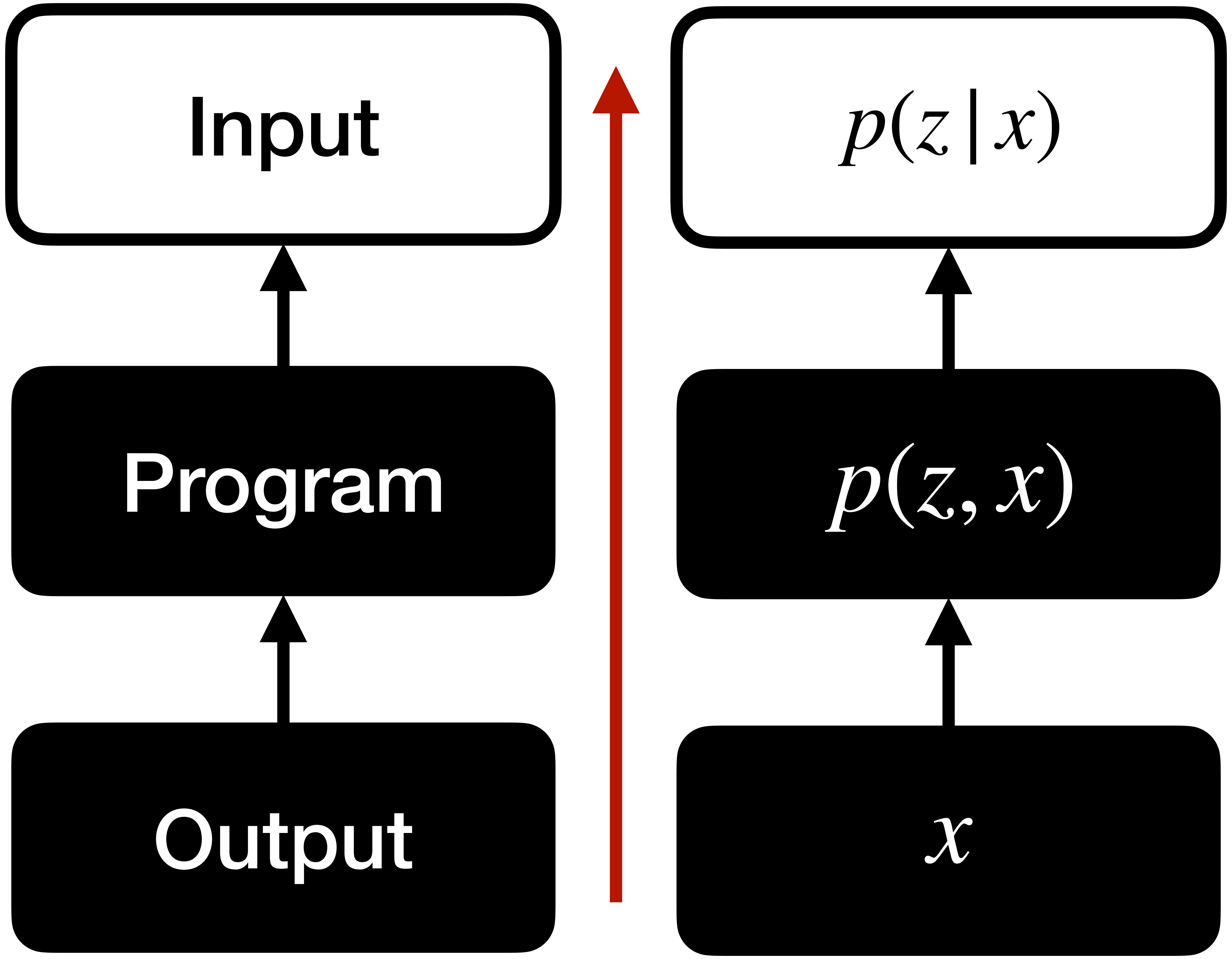


Input-output examples

Agent's reward

Captcha images

Applications of Probabilistic Programming



Program source code	Agent's policy/plan	Captcha letters

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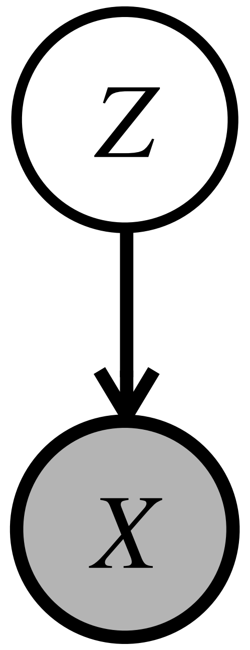


Input-output examples	Agent's reward	Captcha images

PL as an Abstraction Layer

Programs

PL

Interpreter / Compiler

PL as an Abstraction Layer

Programs | expression

PL

Interpreter / Compiler | solving

PPL as an Abstraction Layer

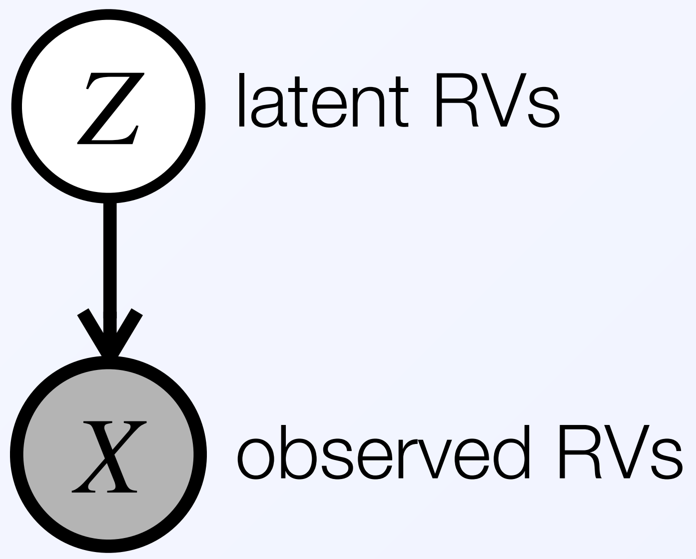
Probabilistic Programs — expression

PPL

Probabilistic Inference — solving

PPL as an Abstraction Layer

Probabilistic Programs



expression

PPL

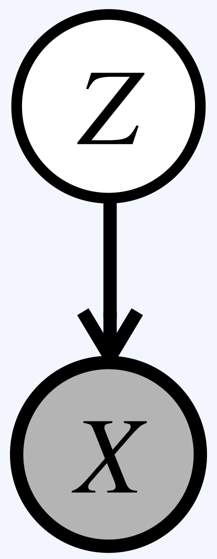
Probabilistic Inference

solving

PPL as an Abstraction Layer

data-generation process
generative model
stochastic simulation
decoders
inductive bias

Probabilistic Programs



latent RVs

observed RVs

expression

PPL

Probabilistic Inference

solving

PPL as an Abstraction Layer

data-generation process
generative model
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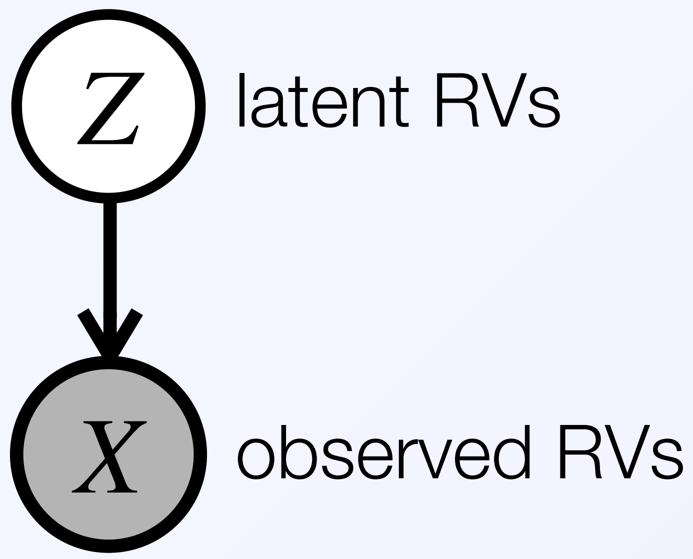
decoders
inductive bias

Probabilistic Programs



expression

PPL

Probabilistic Inference

Interpreters & Compilers

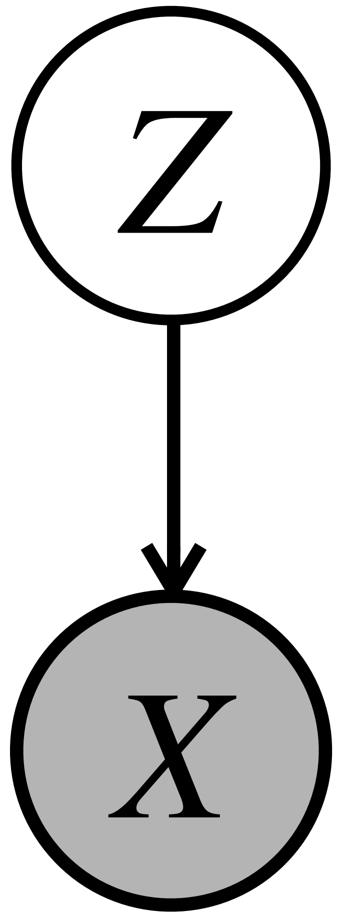
solving

$$p(z \mid x) = \frac{p(x \mid z)\, p(z)}{p(x)}$$

What is hard about Bayesian inference?

Bayes' Theorem

$$\underbrace{p(z \mid x)}_{\text{posterior}} = \frac{\overbrace{p(z, x)}^{\text{joint}}}{\underbrace{p(x)}_{\text{evidence}}} = \frac{\overbrace{p(x \mid z)}^{\text{likelihood}}\,\overbrace{p(z)}^{\text{prior}}}{\underbrace{\boxed{\int p(z, x)\,\mathrm{d}z}}_{\text{marginal likelihood}}}$$

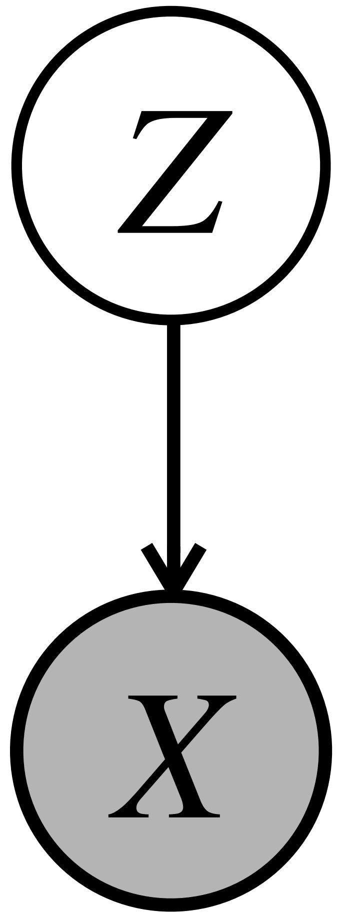


Integration of joint over all execution traces $\Rightarrow$ Enumerating all traces is unrealistic

What is hard about Bayesian inference?

Bayes' Theorem

$$\underbrace{p(z \mid x)}_{\text{posterior}} = \frac{\overbrace{p(z, x)}^{\text{joint}}}{\underbrace{p(x)}_{\text{evidence}}} = \frac{\overbrace{p(x \mid z)}^{\text{likelihood}}\,\overbrace{p(z)}^{\text{prior}}}{\underbrace{\boxed{\int p(z, x)\,\mathrm{d}z}}_{\text{marginal likelihood}}}$$



Integration of joint over all execution traces ⇨ Enumerating all traces is unrealistic

Joint is defined by a program ⇨ Integration rarely has analytical solutions

Have to Approximate or Limit Expressivity

Bayes' Theorem

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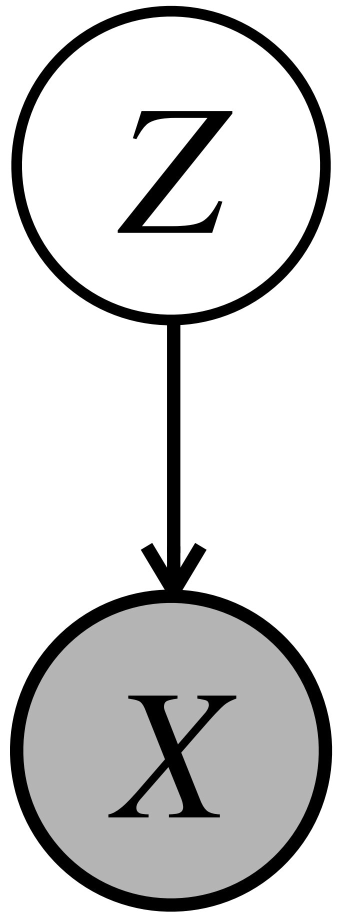

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Have to Approximate or Limit Expressivity

Approximate

Rejection Sampling

Likelihood Weighting

Importance Sampling

Sequential Monte Carlo (SMC)

Markov Chain Monte Carlo (MCMC)

Variational Inference

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Variational Inference

Limit Expressivity

Reduced expressive power

$\Rightarrow$ Improved run-time efficiency

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Sequential Monte Carlo (SMC)

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Variational Inference

Limit Expressivity

Reduced expressive power
⇨ Improved run-time efficiency

Most common restriction to impose:
Ban recursion/unbounded loops
(think of finite graphical models)

Examples: Stan, Infer.NET, …

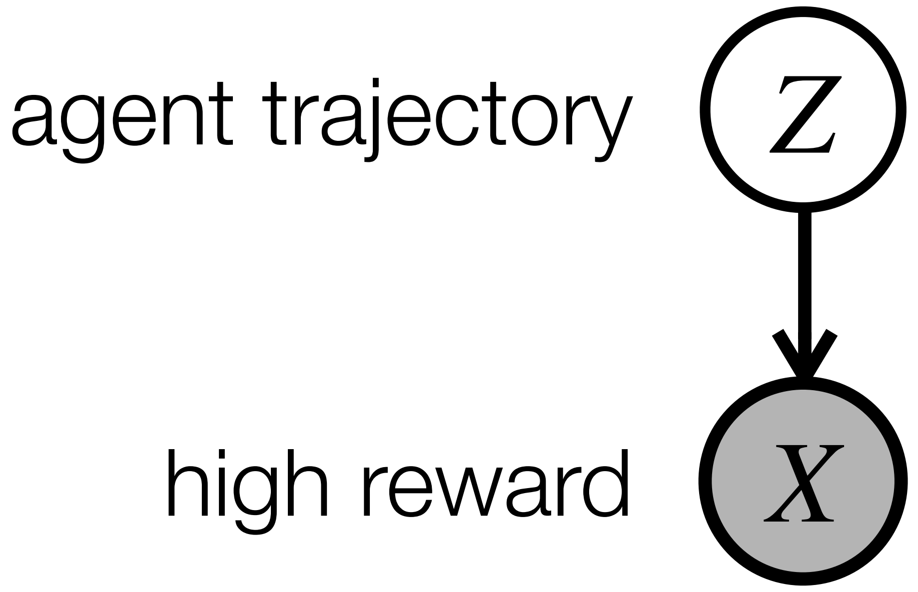
PPL as an Abstraction Layer

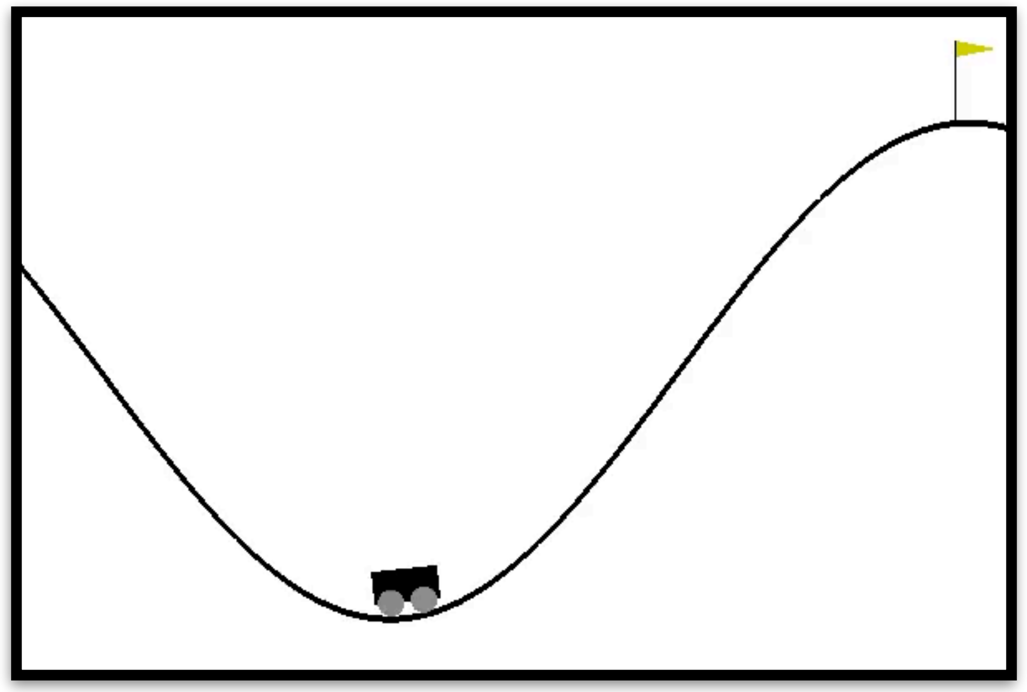
Probabilistic Programs — expression

PPL

Probabilistic Inference — solving

Example: Reinforcement Learning





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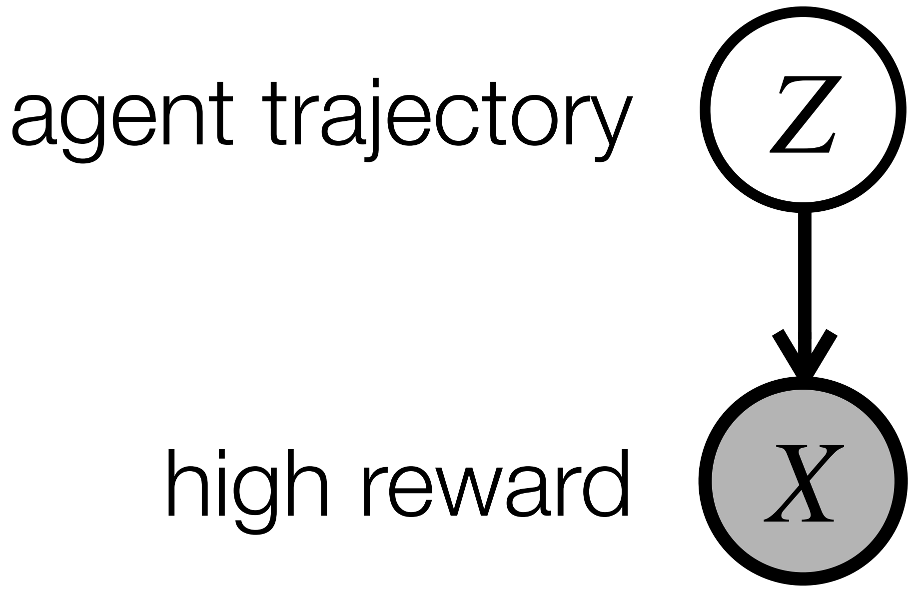

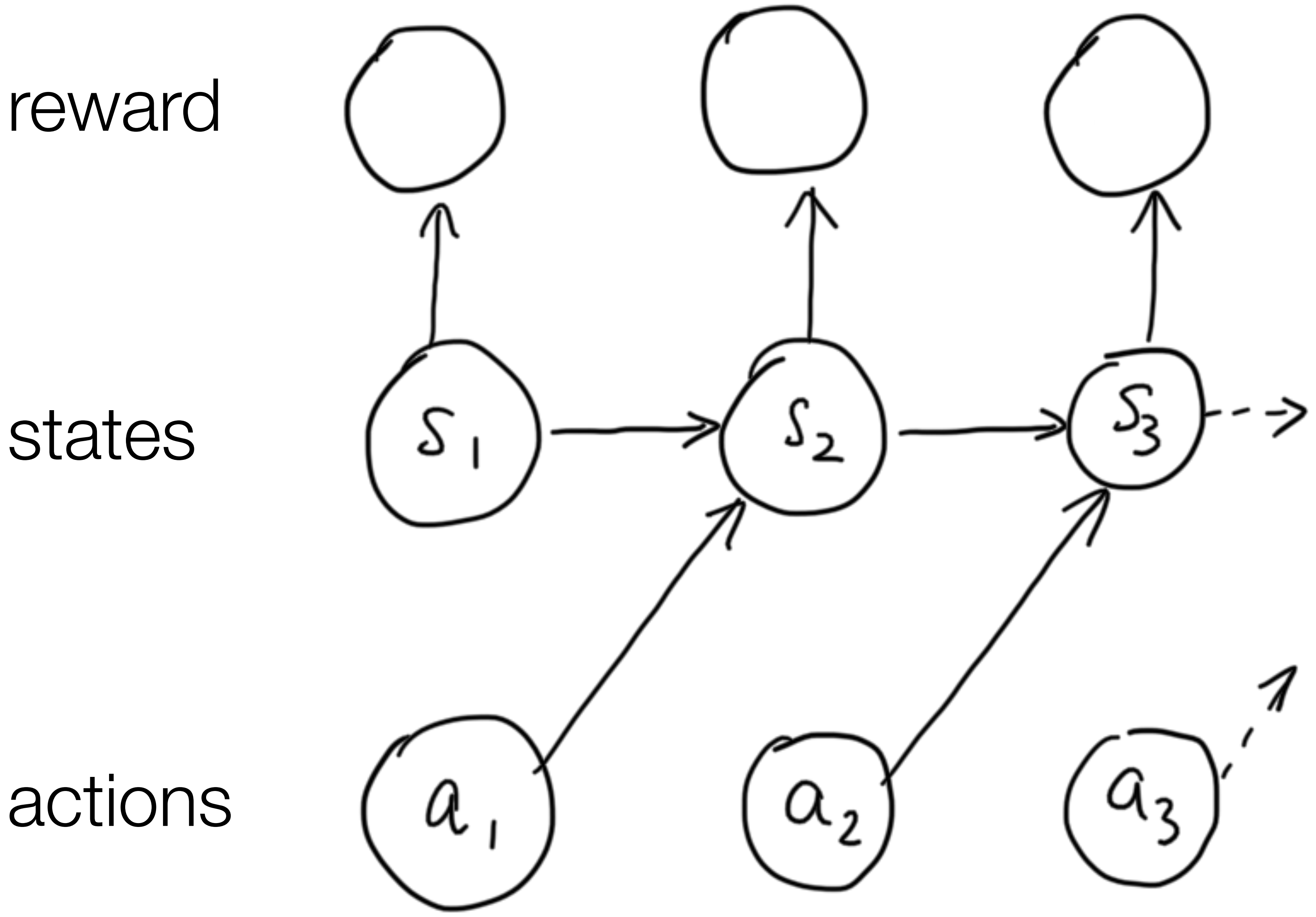

Environment

```
def reward(state) // immediate reward
def transition(state, action) // step the environment
```

Agent

```
def MDP(state): // recursive MDP description
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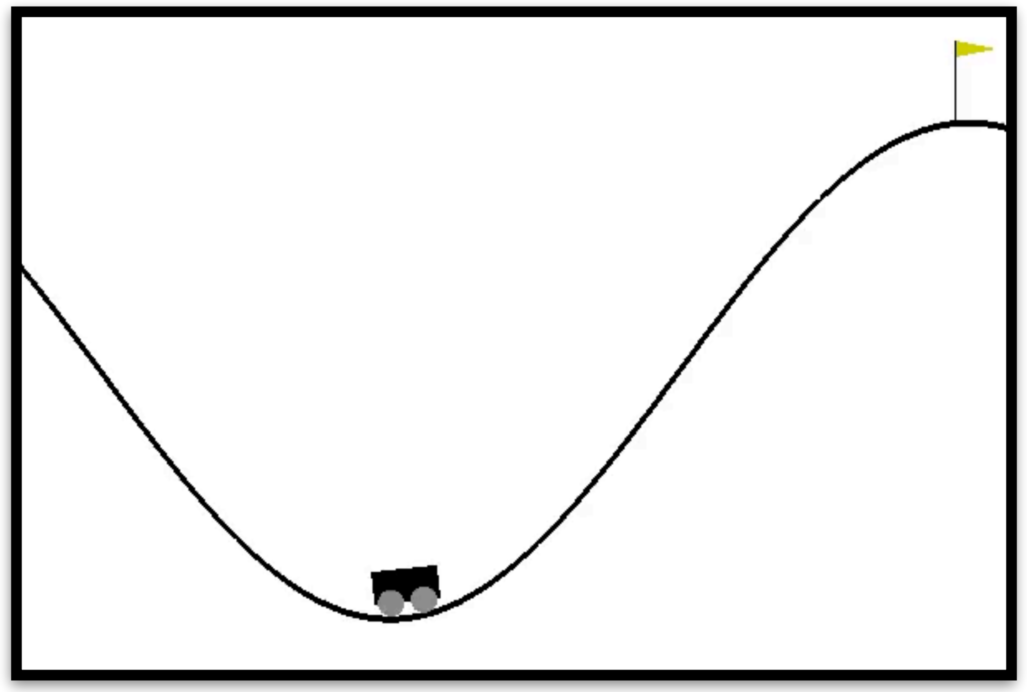

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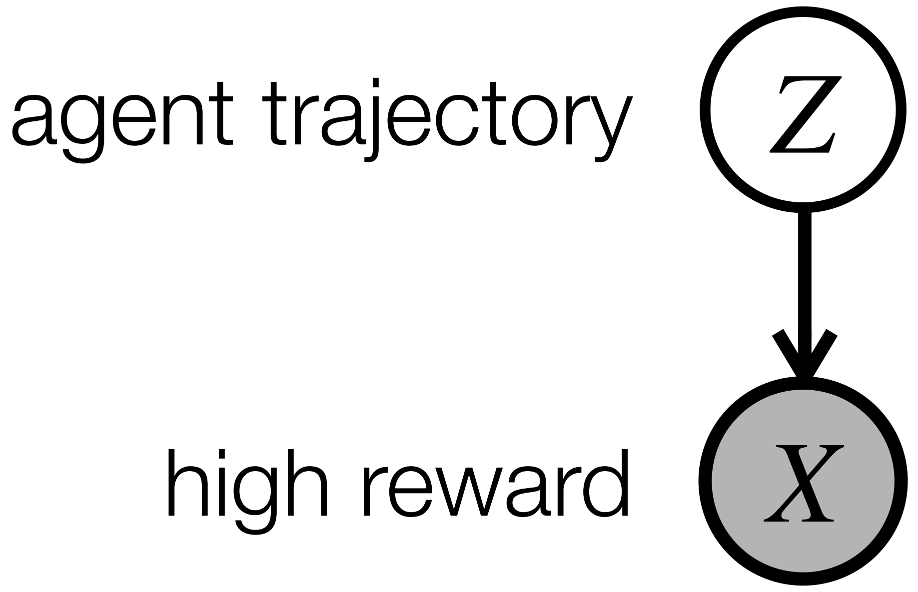

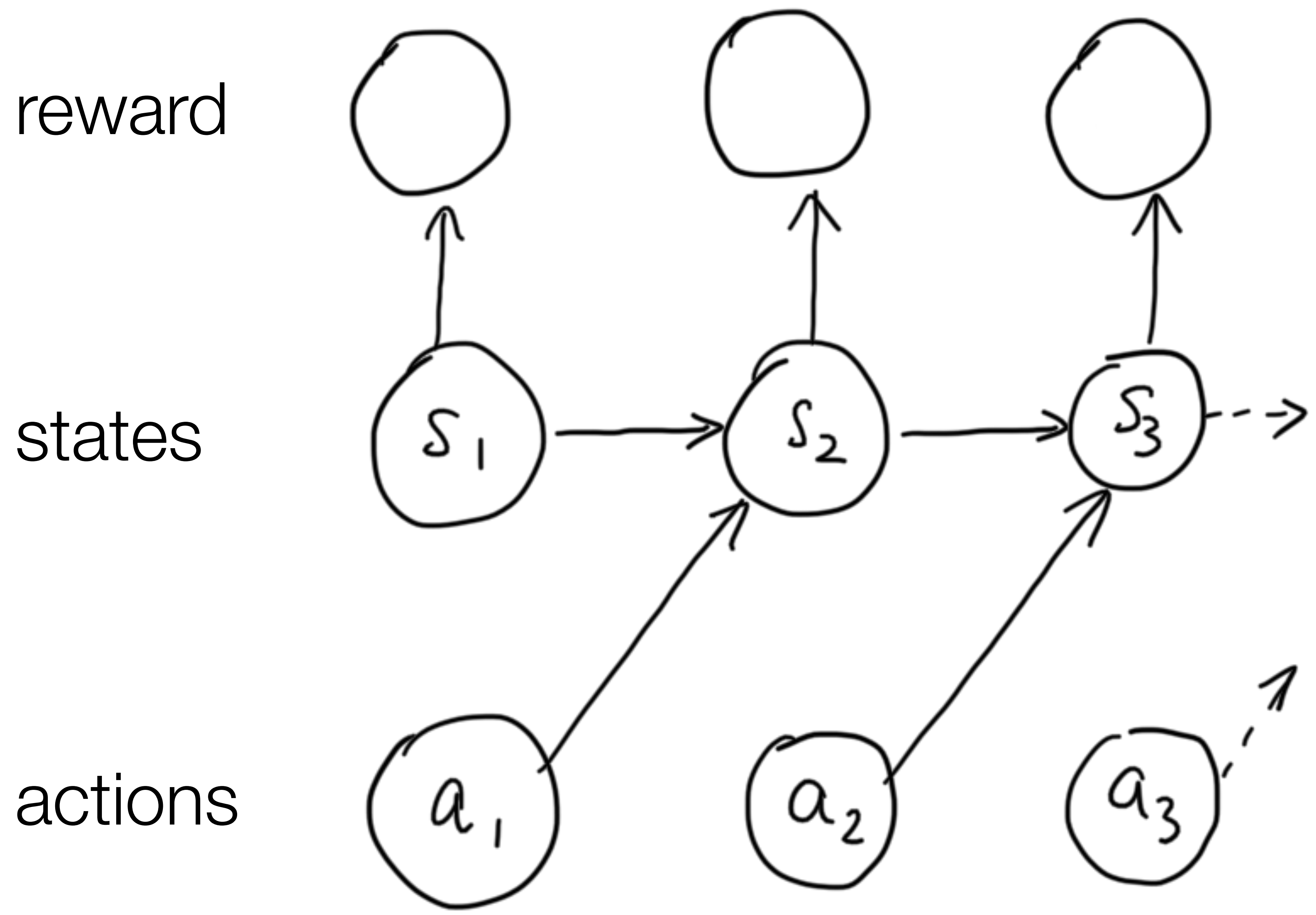

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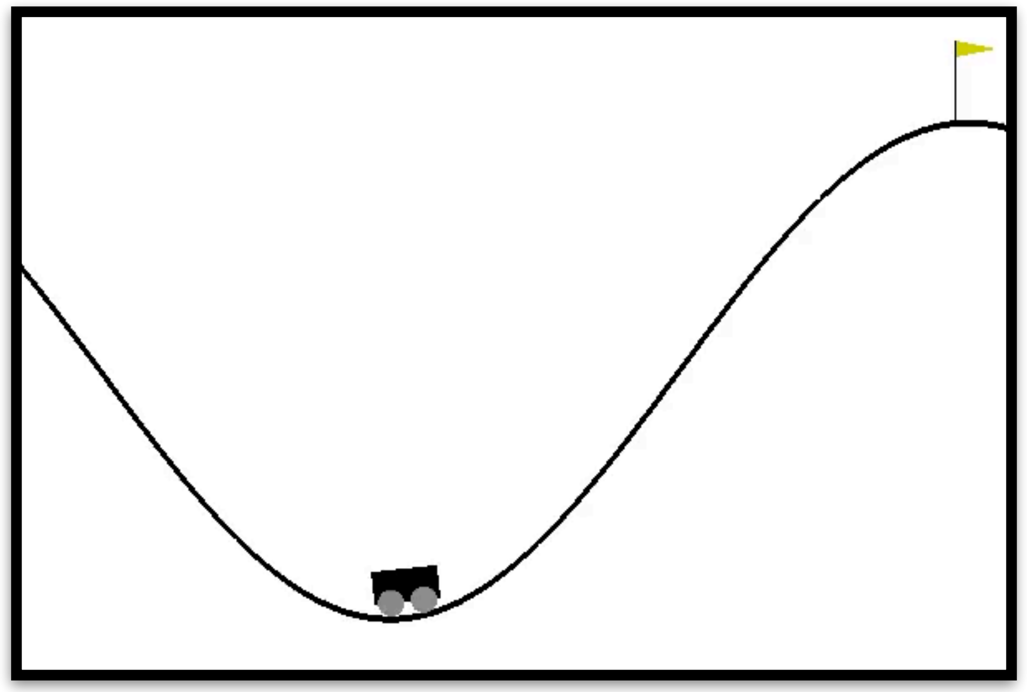

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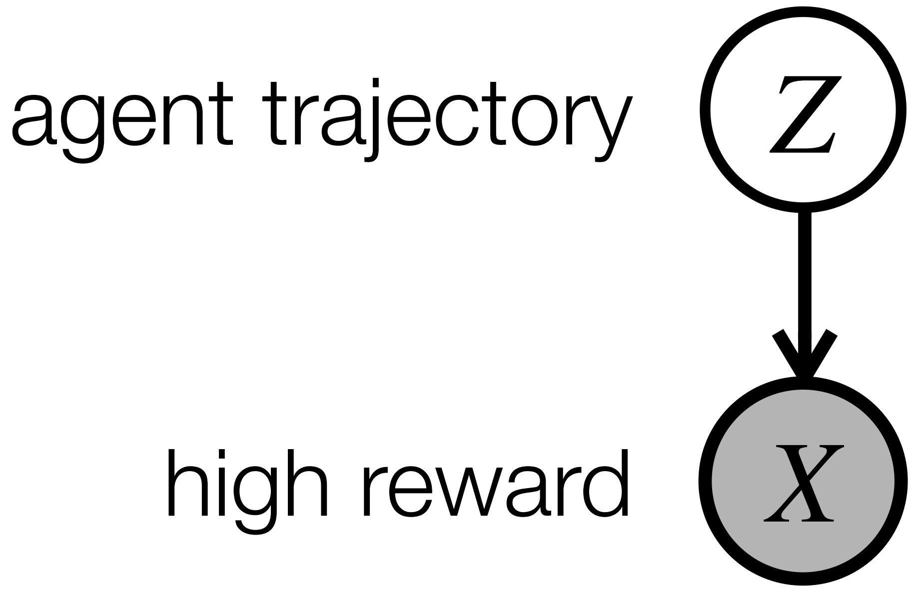

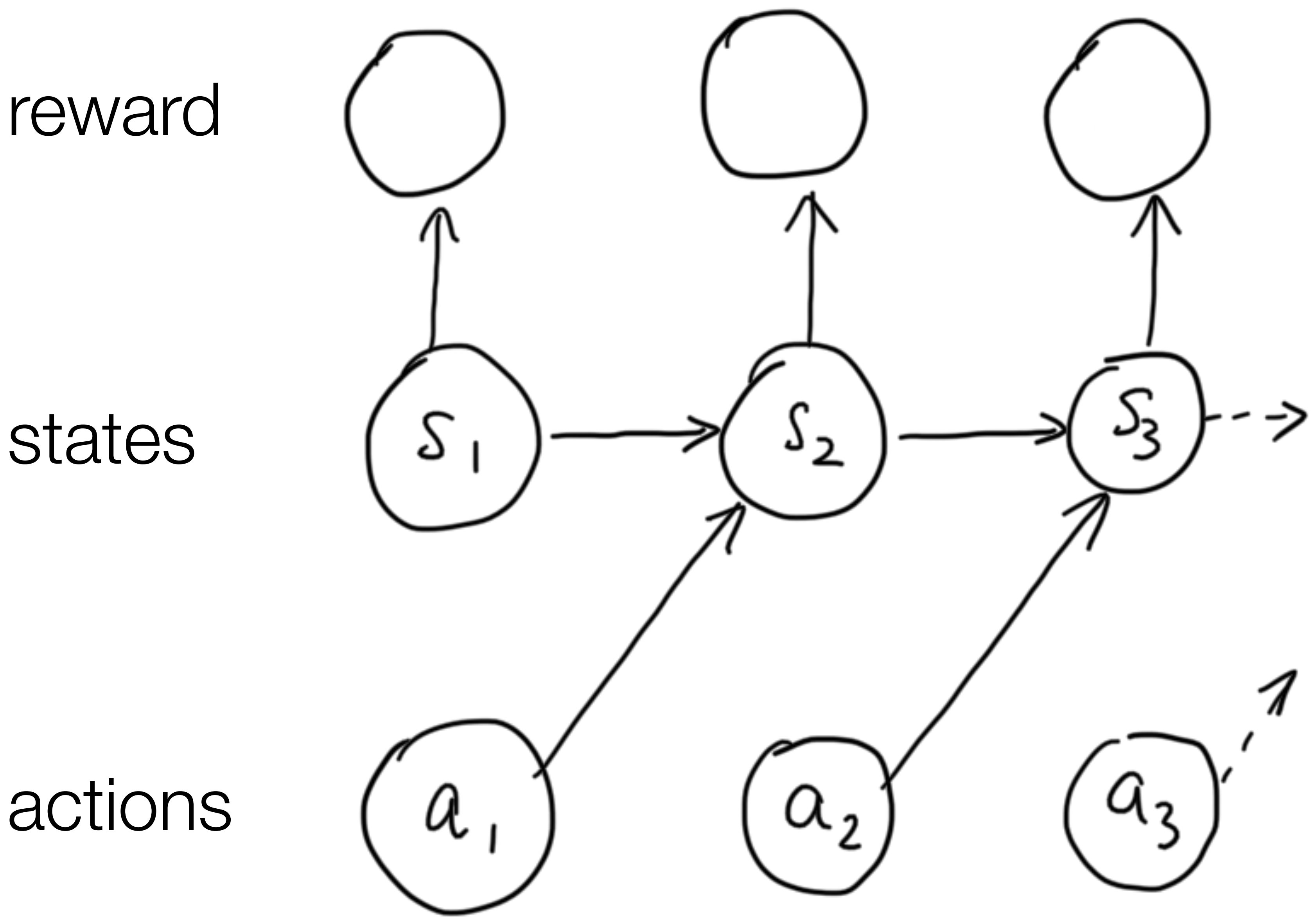

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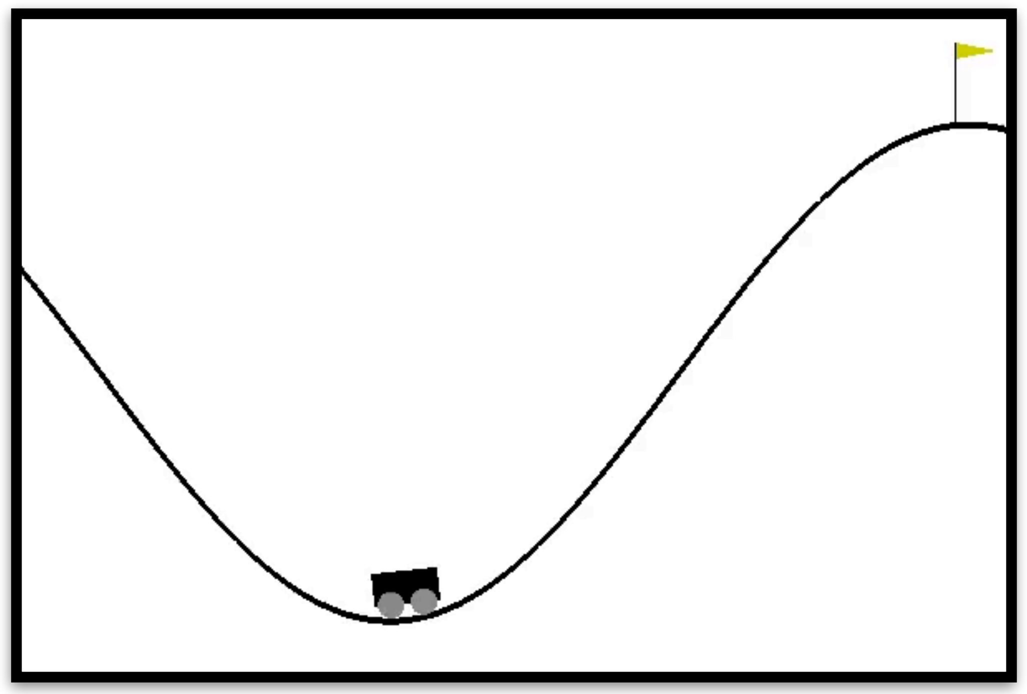
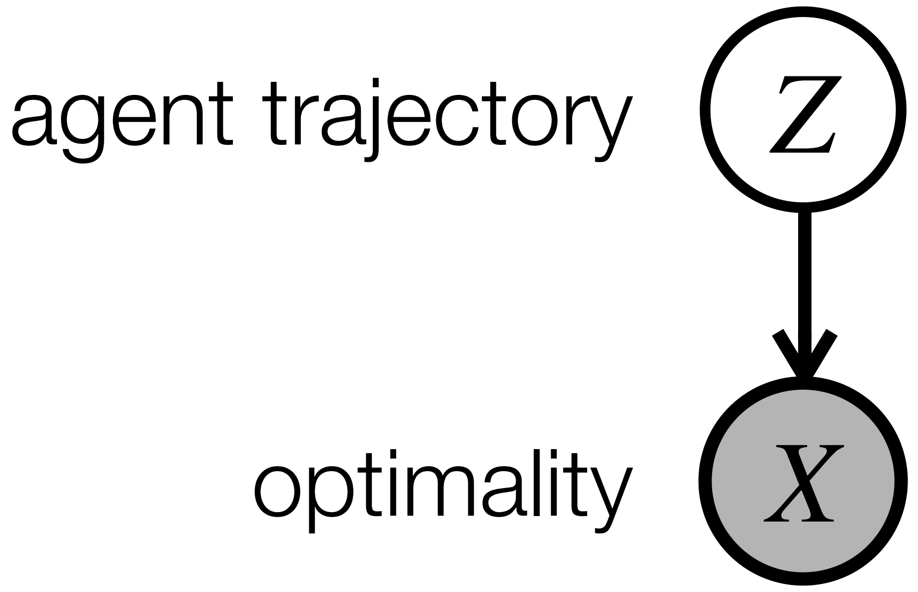
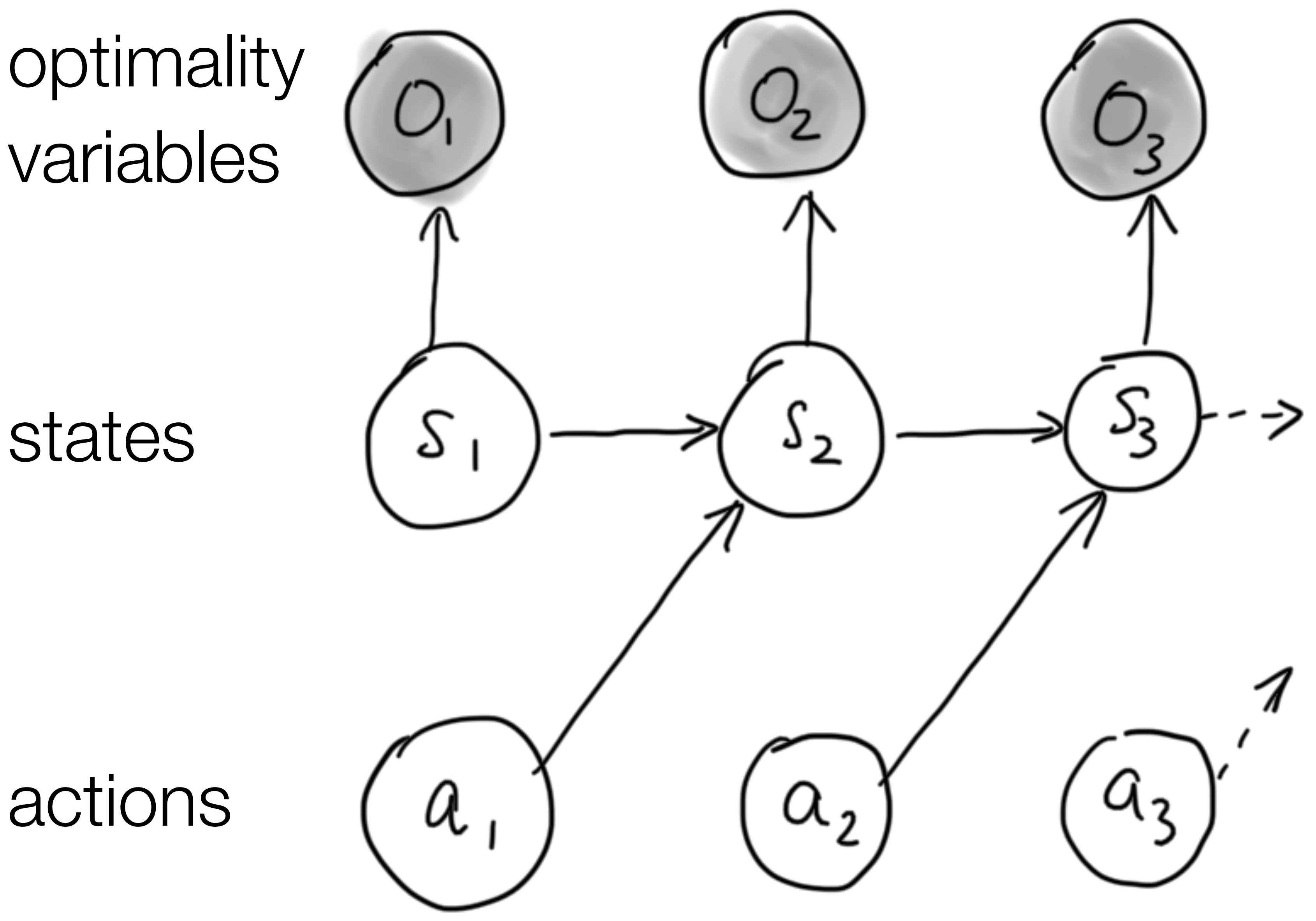


Example: Reinforcement Learning





$$p(O_t = 1 \mid s_t) \stackrel{def}{=} ?$$

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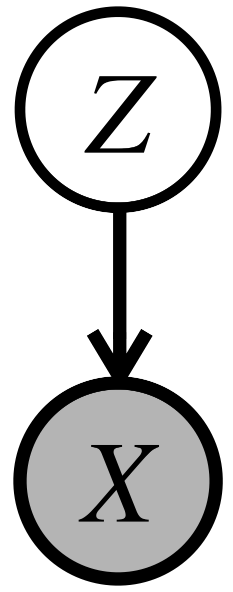
Example: Reinforcement Learning

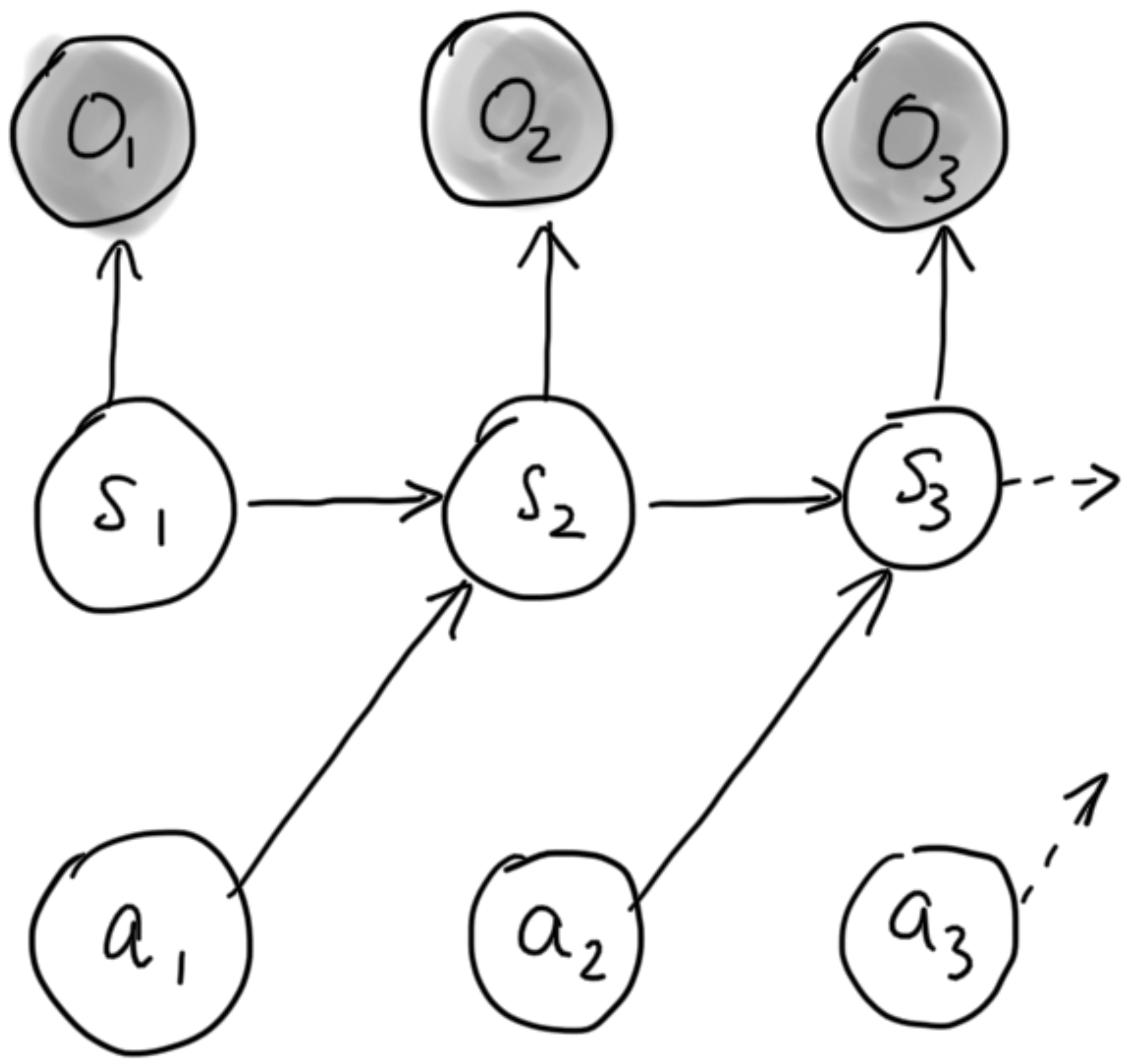
agent trajectory


optimality

optimality variables

states

actions



$$p(O_t = 1 \mid s_t) \stackrel{def}{=} \exp(r(s_t))$$

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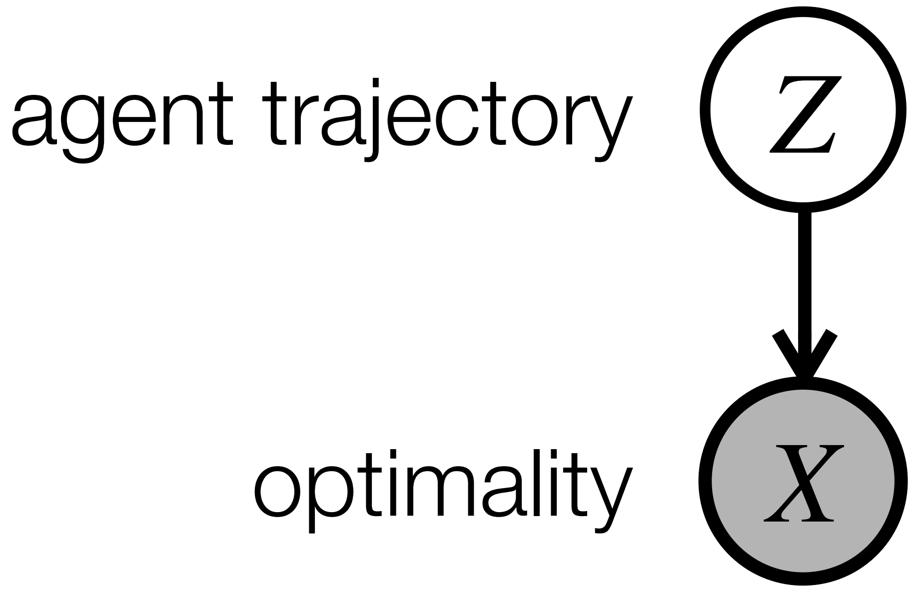
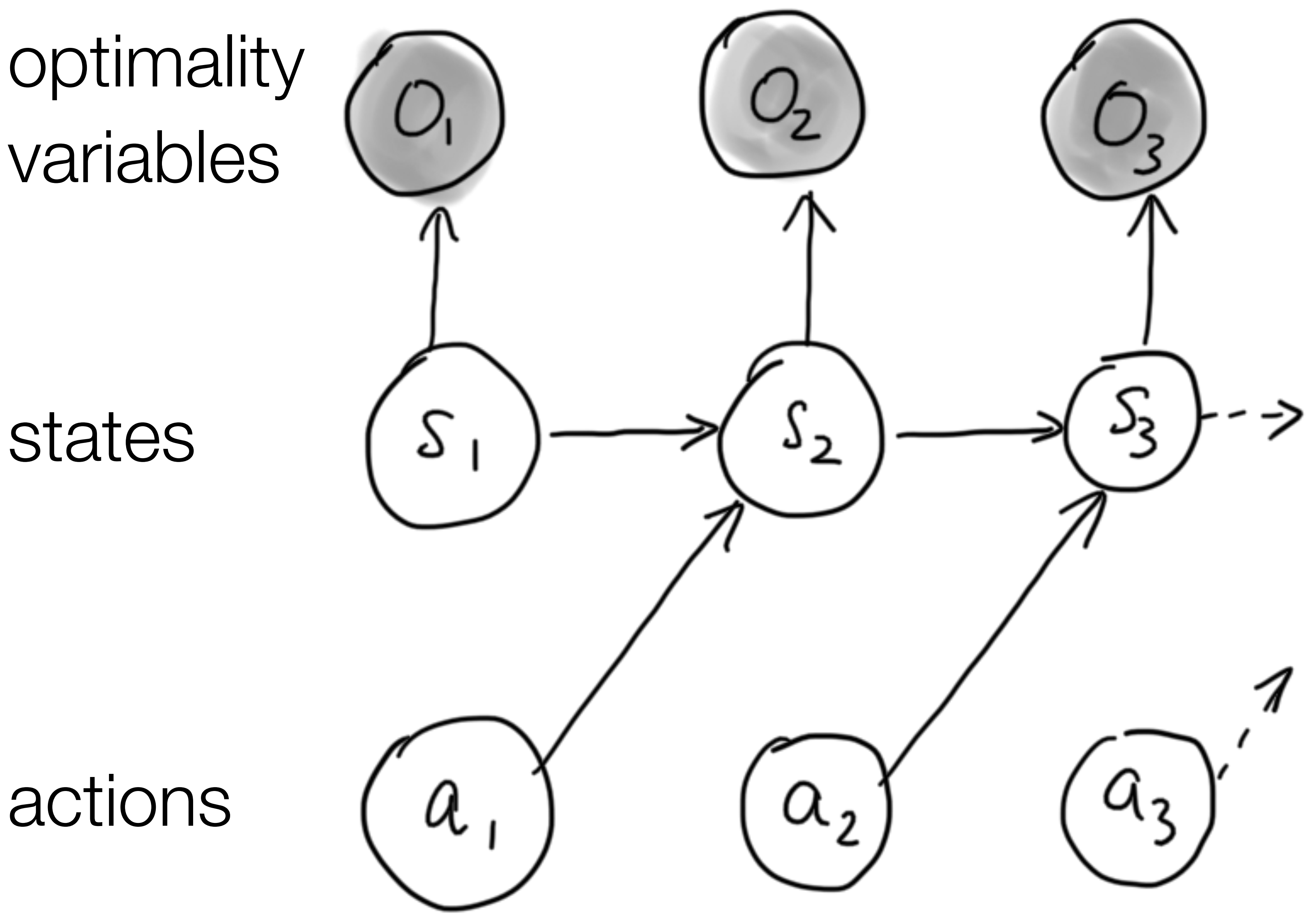
Example: Reinforcement Learning





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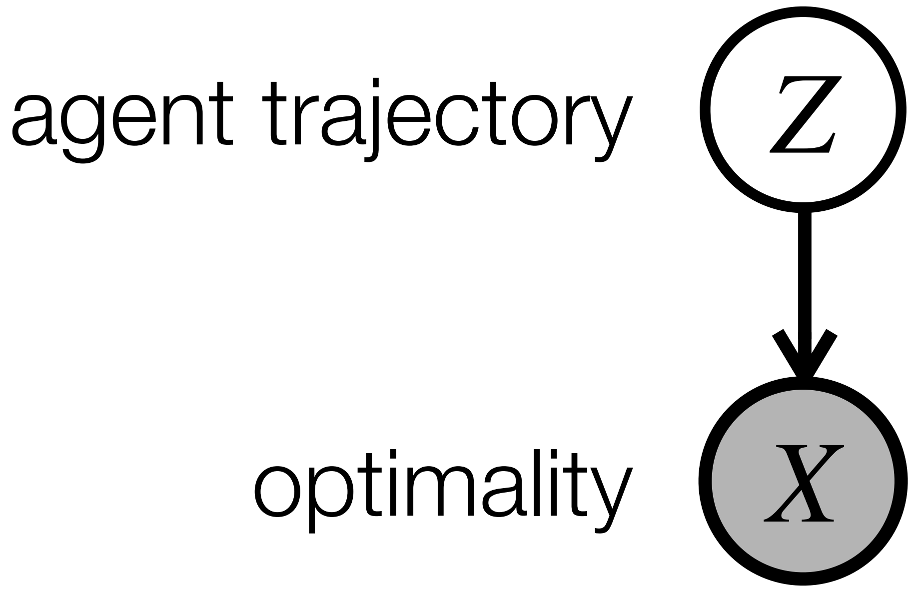

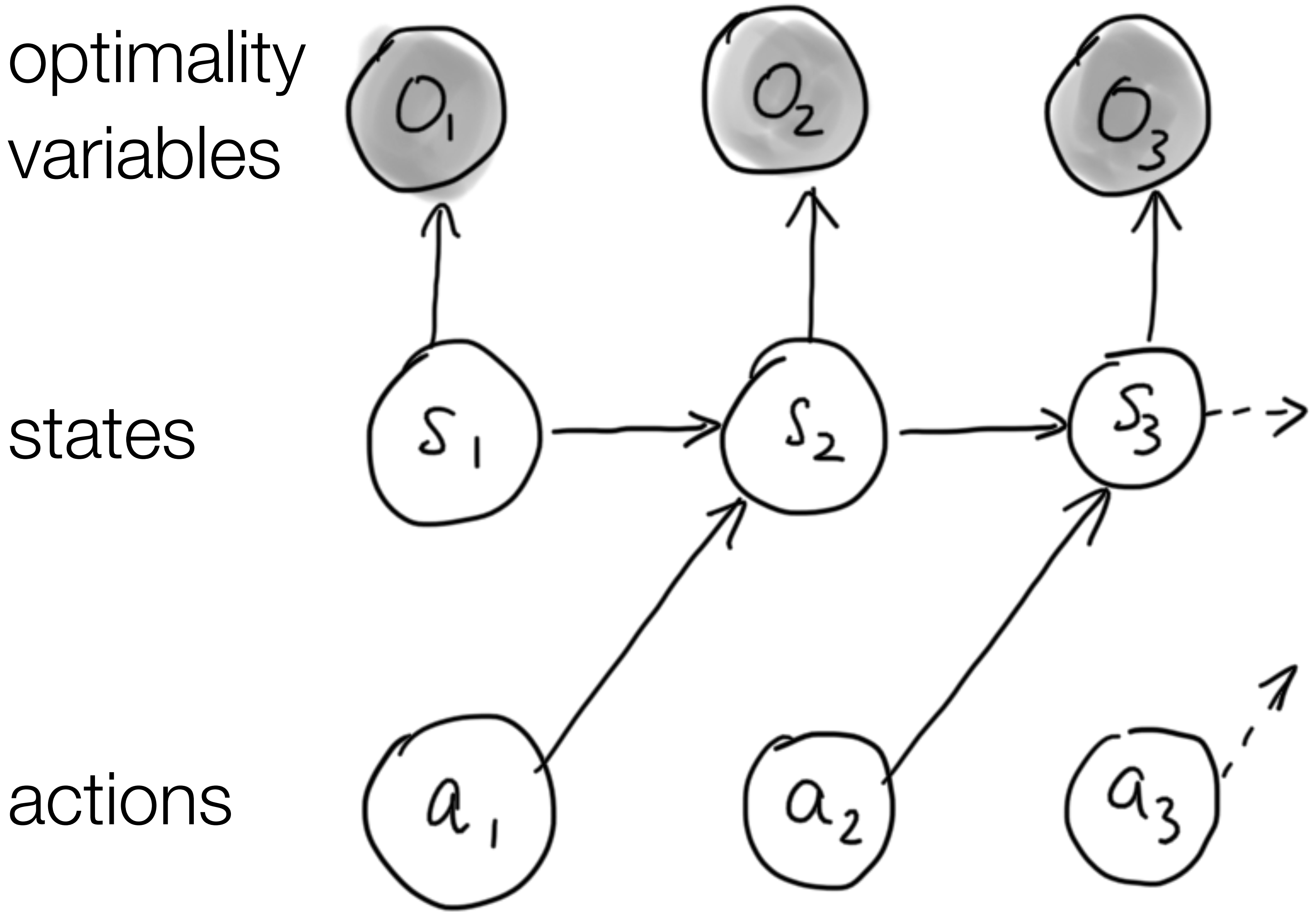

Goal of inference:
a policy function $\pi : \mathrm{State} \rightarrow \mathrm{Action}$ leading to optimal trajectory, rather than a trajectory per se

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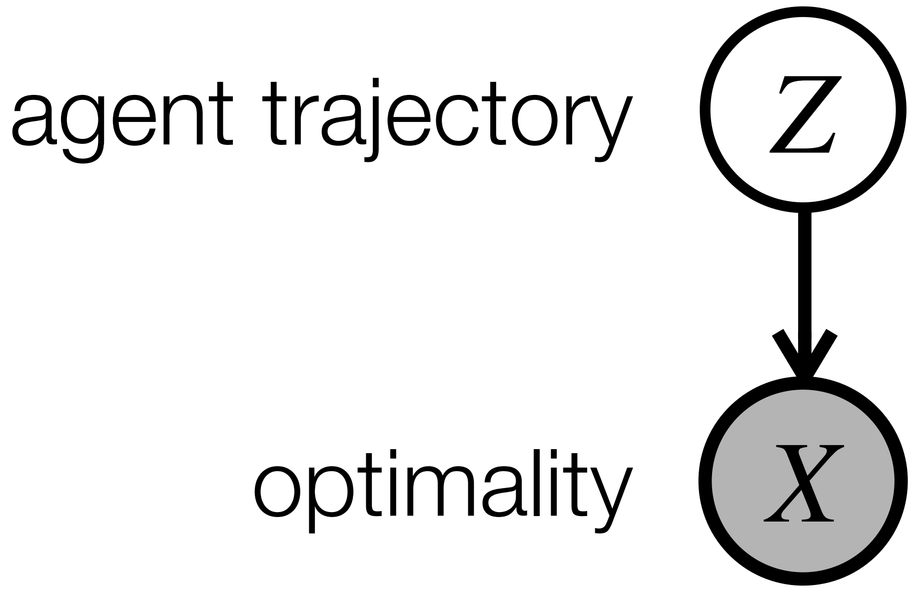

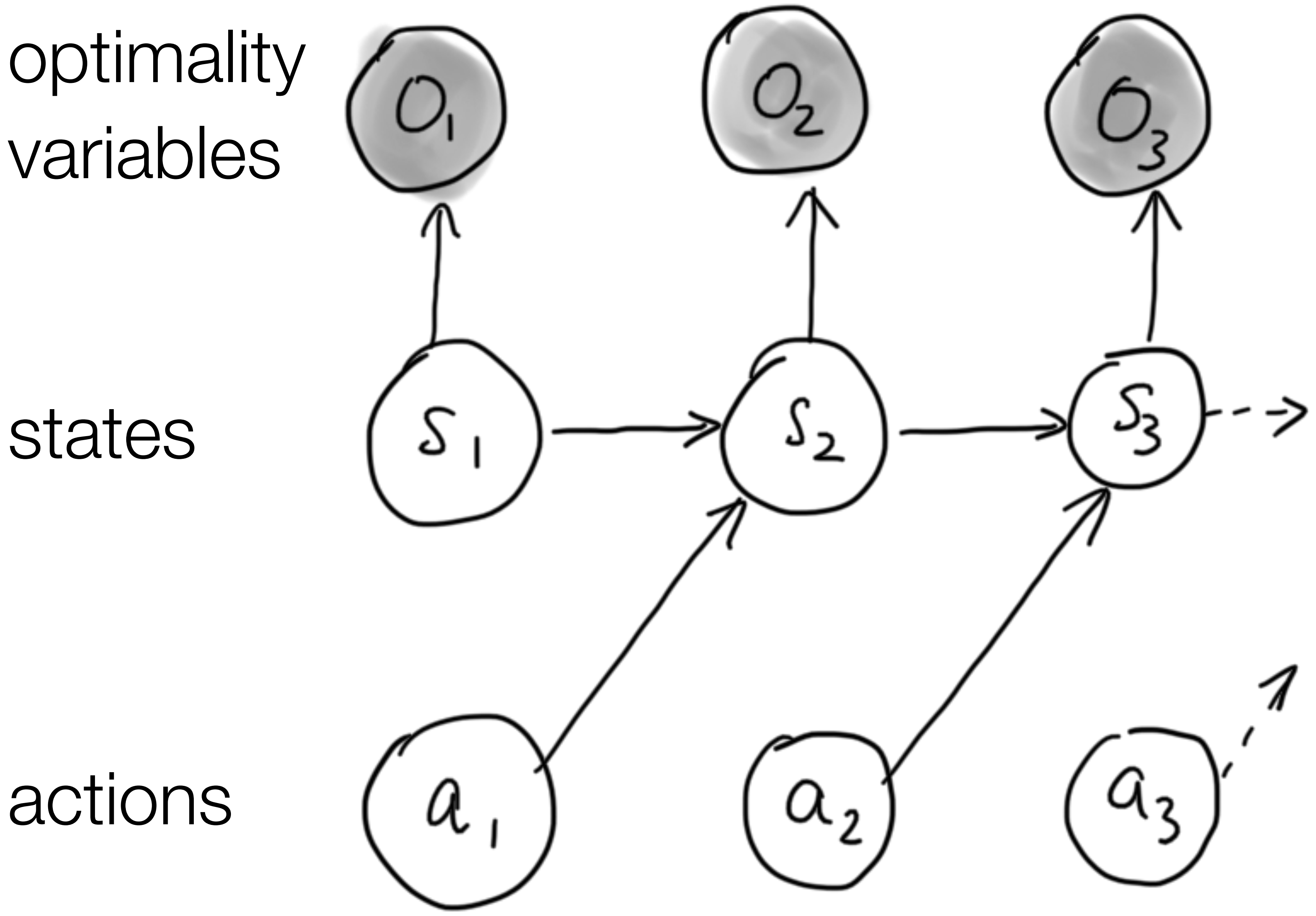

Goal of inference:
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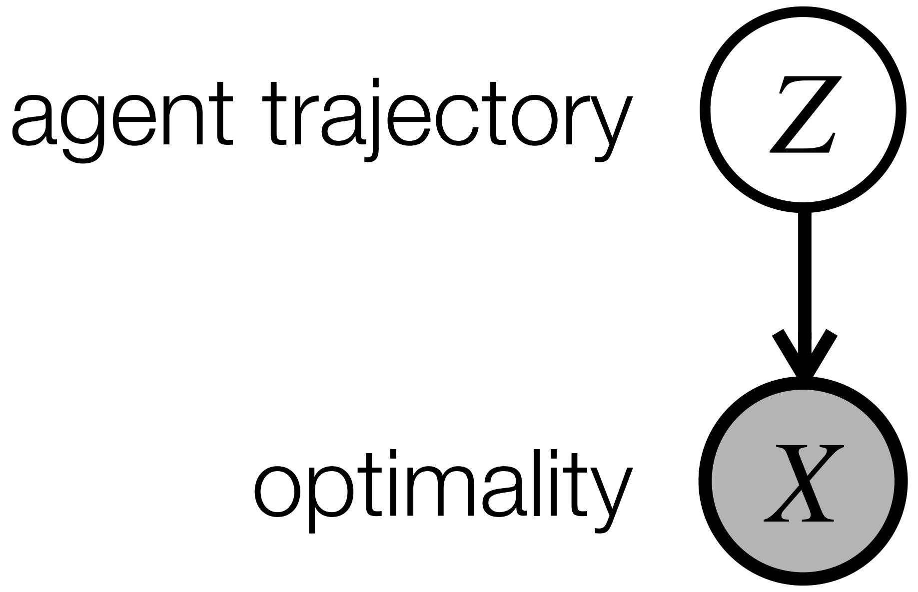

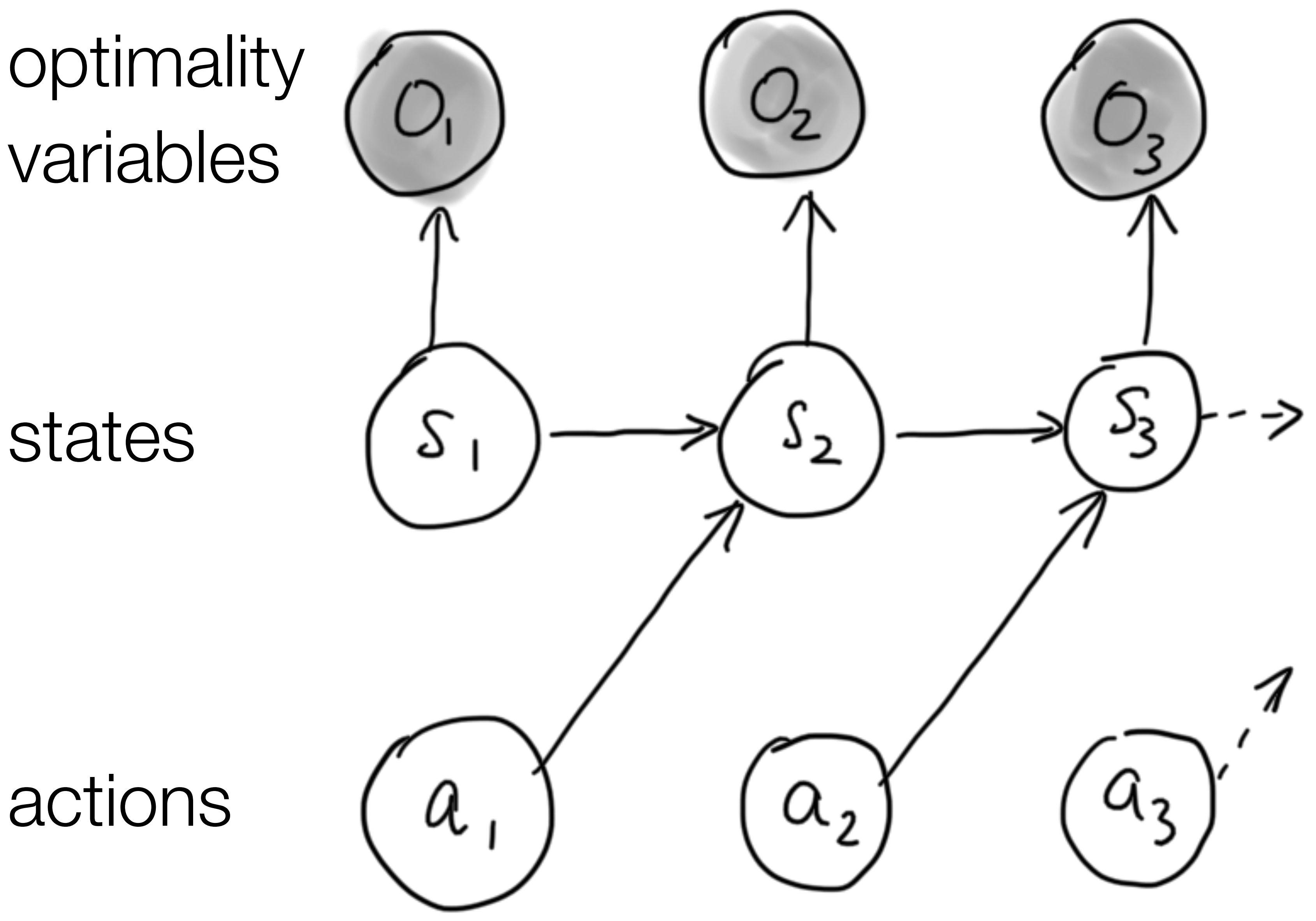

Goal of inference: $\pi(s) = q(a \mid s; \phi)$

a policy function $\pi : \mathrm{State} \to \mathrm{Dist[Action]}$ leading to optimal trajectory, rather than a trajectory per se

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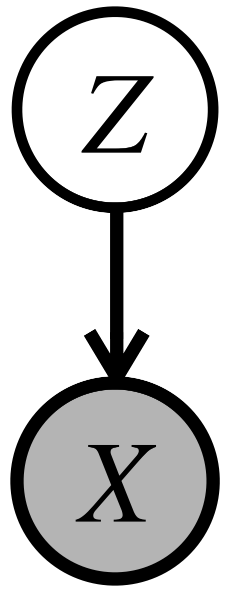
$p(O_t = 1 \mid s_t) \stackrel{def}{=} \exp(r(s_t))$

Example: Reinforcement Learning

agent trajectory

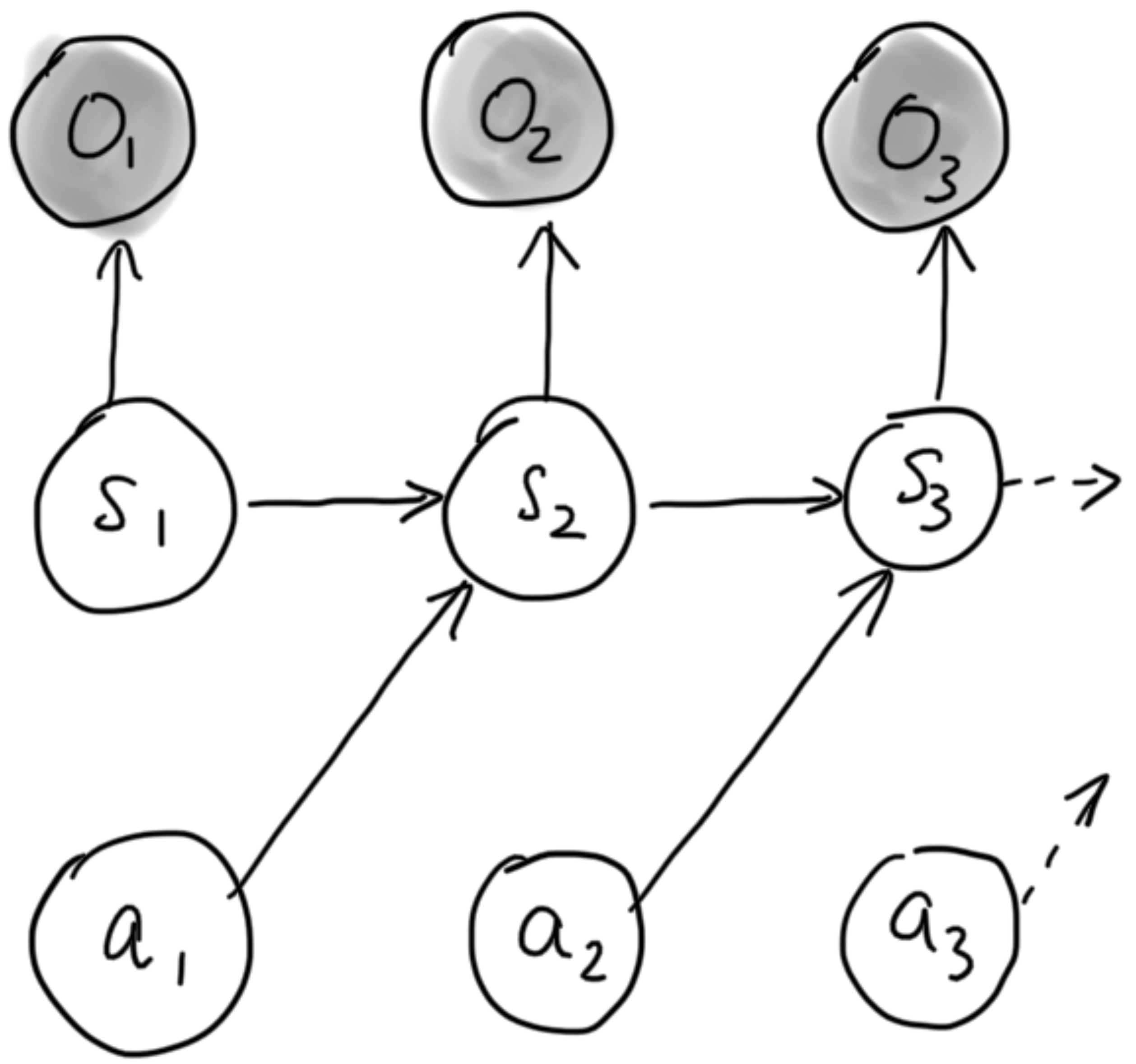


optimality

optimality variables

states

actions



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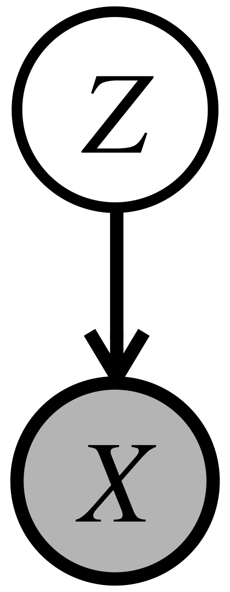
$p(s_{t+1} \mid s_t, a_t)$

$p(O_t = 1 \mid s_t) \stackrel{def}{=} \exp(r(s_t))$

Example: Reinforcement Learning

agent trajectory



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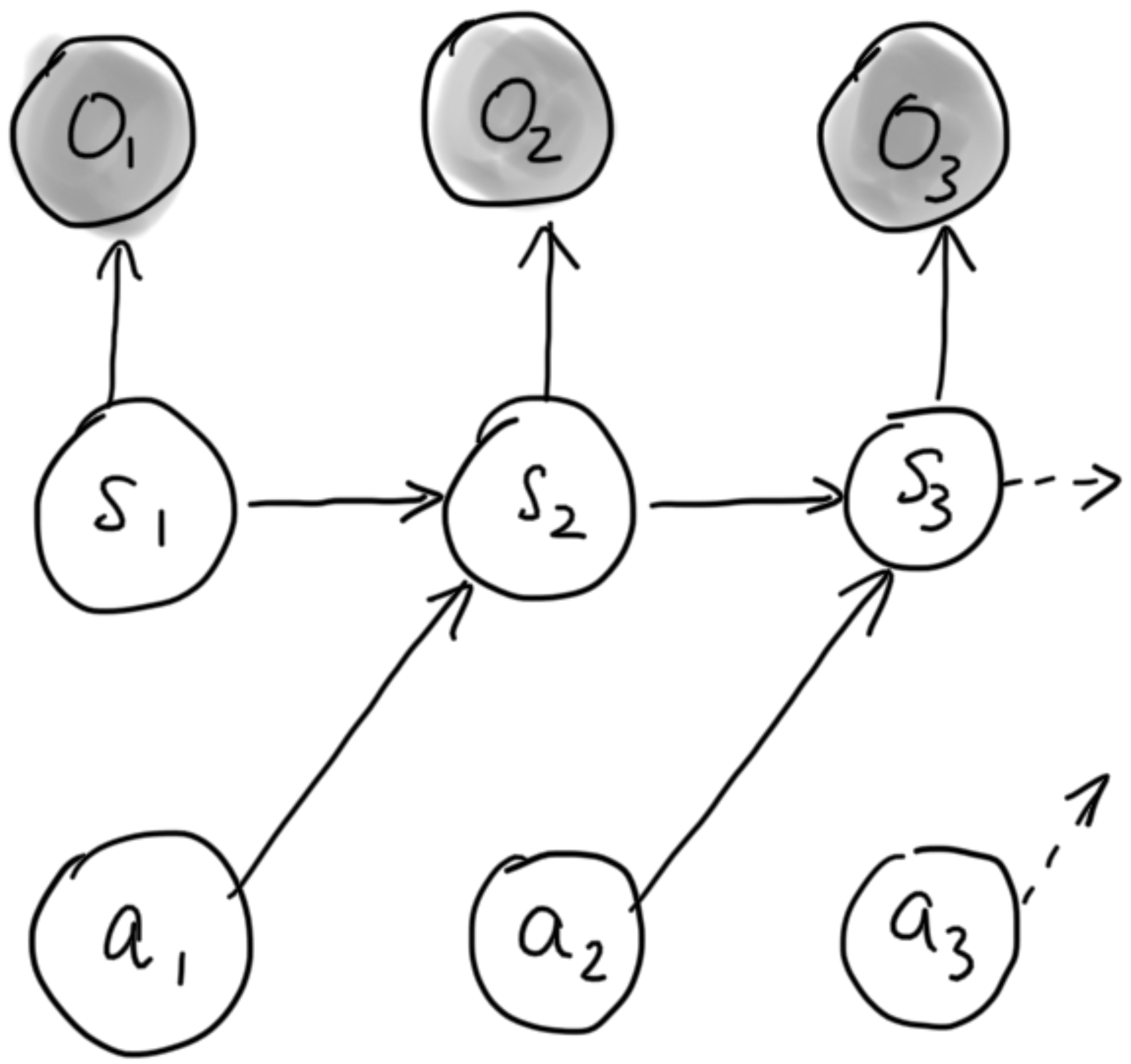

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a policy function $\pi : \mathrm{State} \rightarrow \mathrm{Dist[Action]}$ leading to optimal trajectory, rather than a trajectory per se

$$p\big(s_1, a_1, \ldots \mid \text{optimality}\big) \propto p\big(s_1\big) \prod_t p\big(a_t\big)\, p\big(s_{t+1} \mid s_t, a_t\big)\, p\big(O_t = 1 \mid s_t\big)$$

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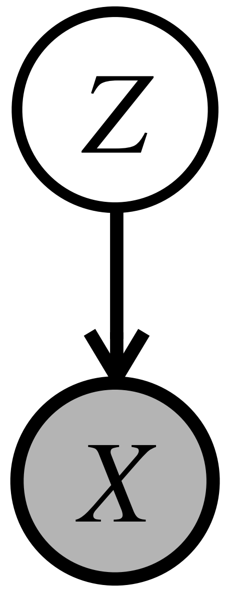
Example: Reinforcement Learning

agent trajectory



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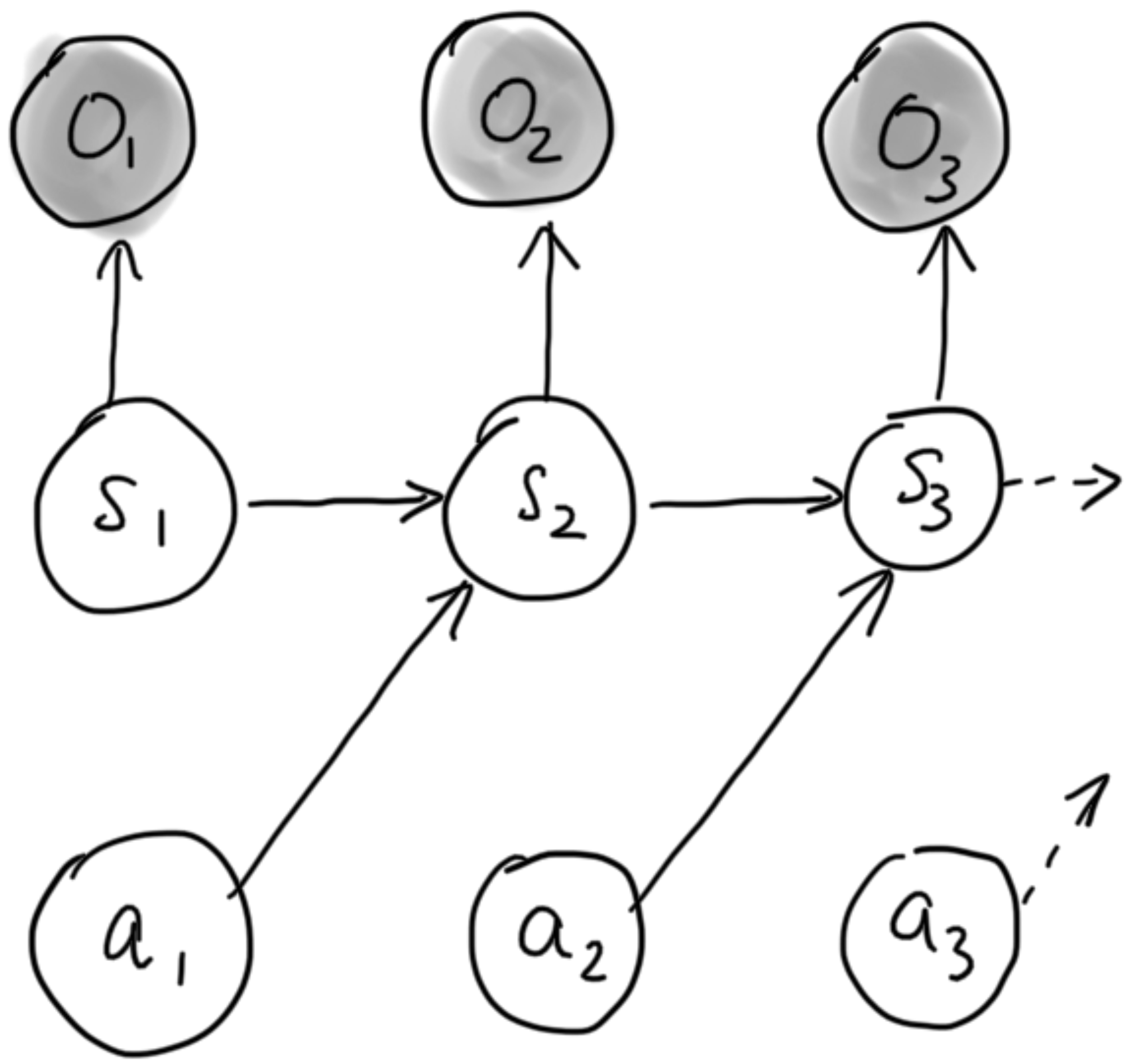

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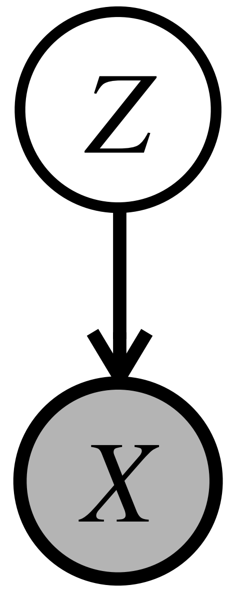

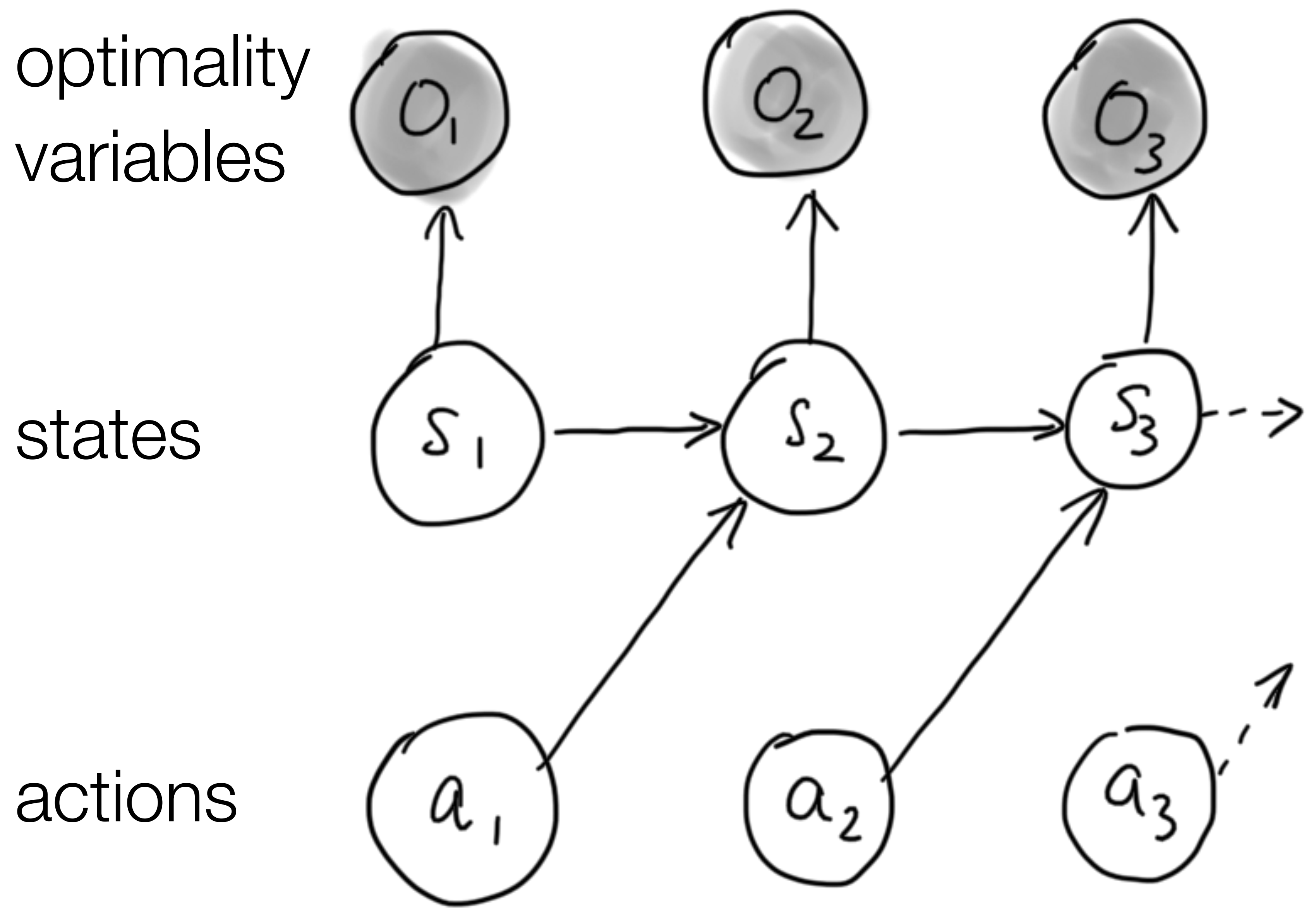

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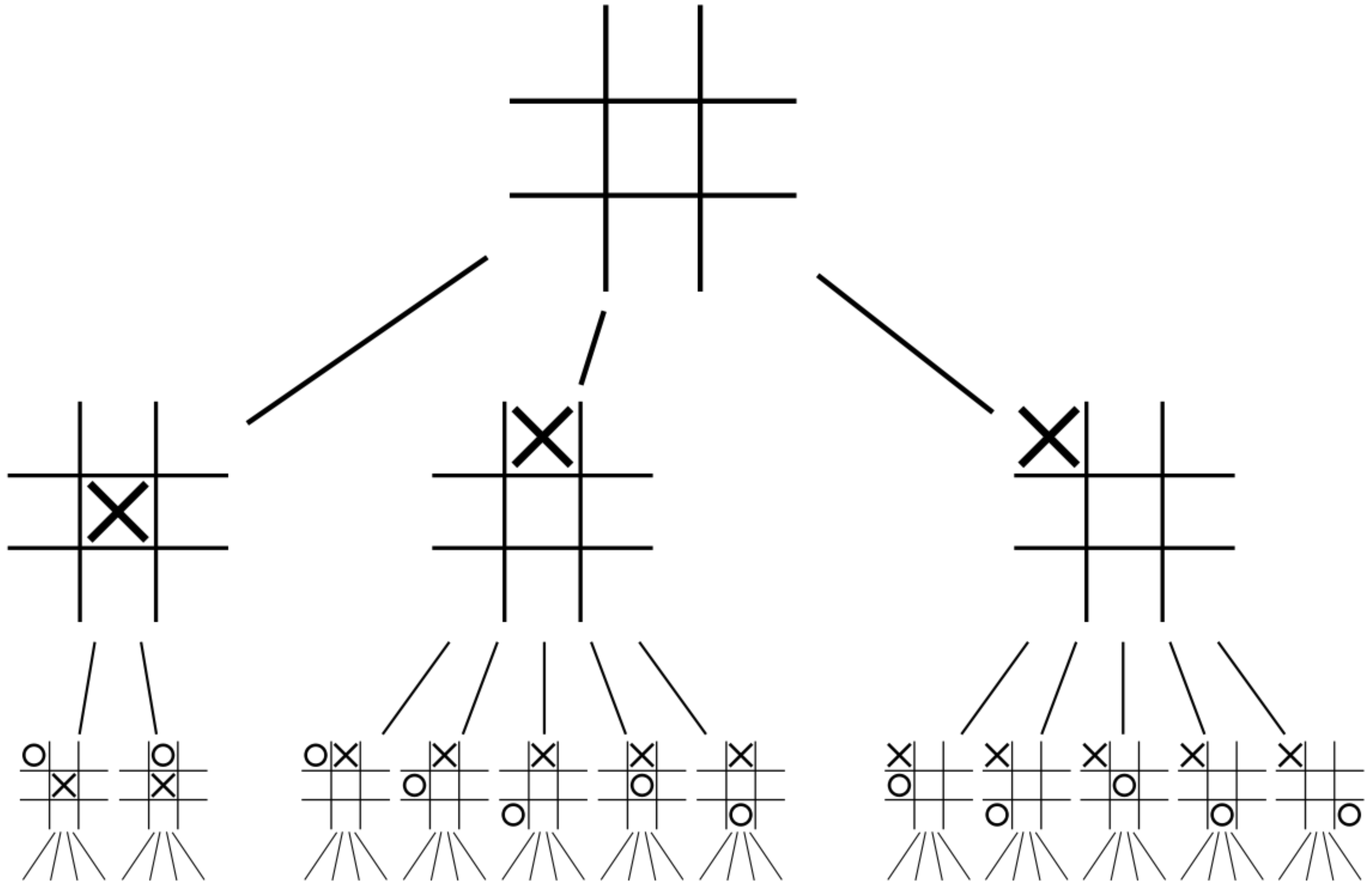
$$q(s_1, a_1, \ldots, s_t, a_t; \phi) = p(s_1) \prod_t q(a_t \mid s_t; \phi)\, p(s_{t+1} \mid s_t, a_t)$$

$$\min_{\phi} D_{\mathrm{KL}}\Big(q(s_1, a_1, \ldots, s_t, a_t; \phi) \,||\, p(s_1, a_1, \ldots \mid \text{optimality})\Big)$$

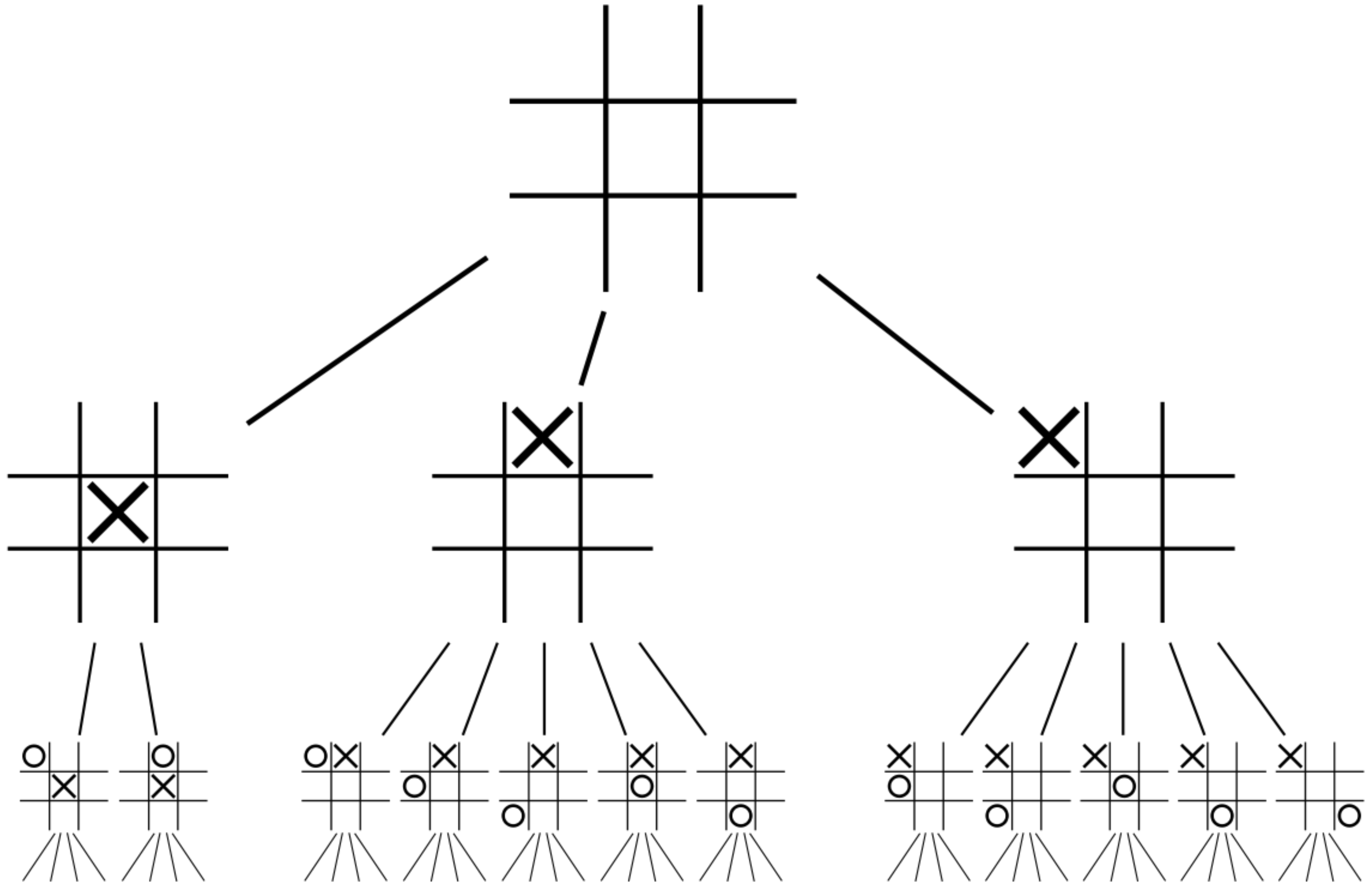
Example: Two-Player Game



Stuhlmüller and Goodman. Reasoning about reasoning by nested conditioning: Modeling theory of mind with probabilistic programs. CogSci. 2014.
https://agentmodels.org/chapters/7-multi-agent.html

Example: Two-Player Game

```
var act = function(state, player) {
    var move = sample(movePrior(state));
    var u = utility(state, move, player);
    factor(alpha * u);
    return move;
};
```



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    else if (isDraw(state)) { return 0; }
    else { return -10; }
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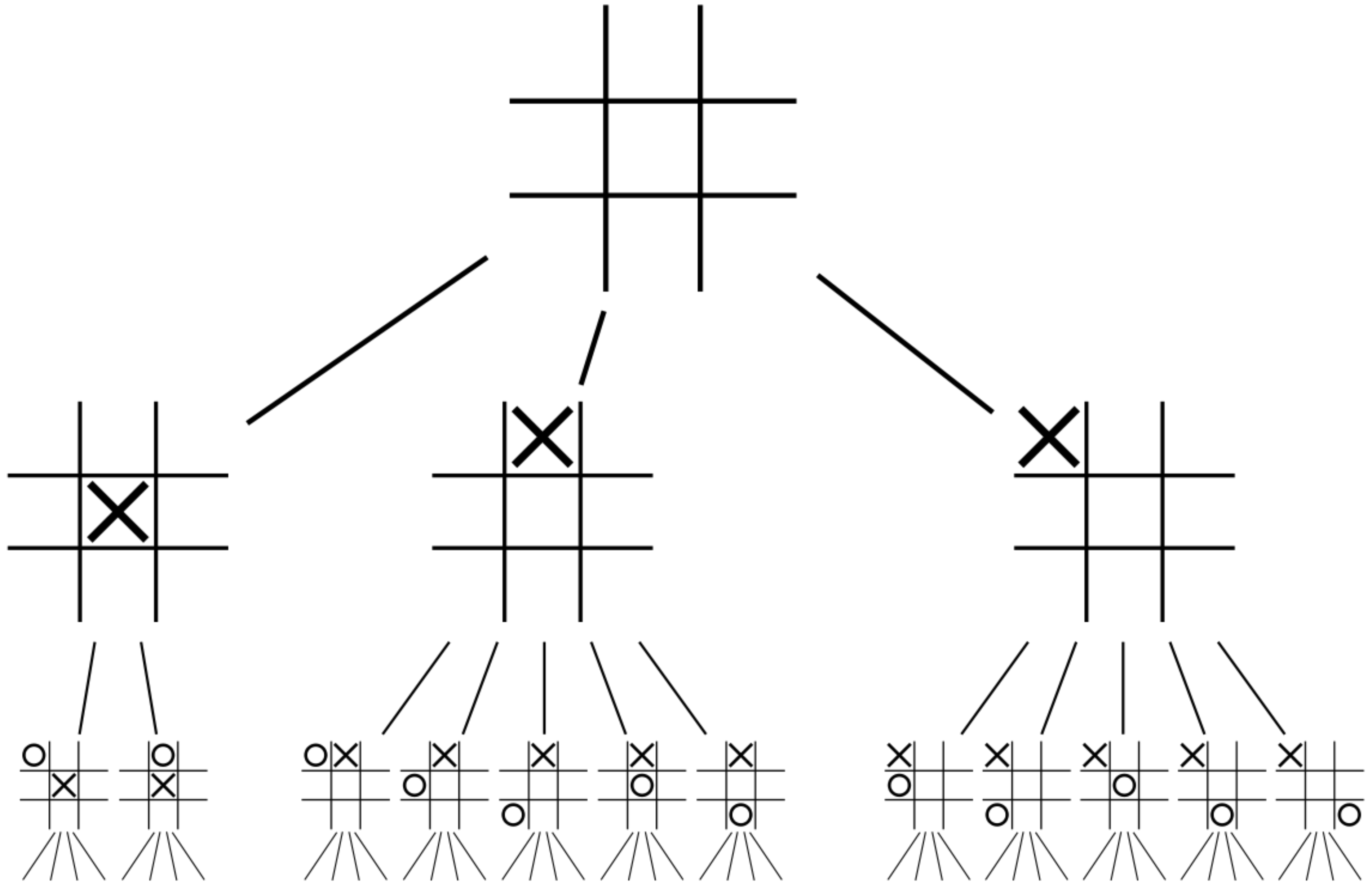

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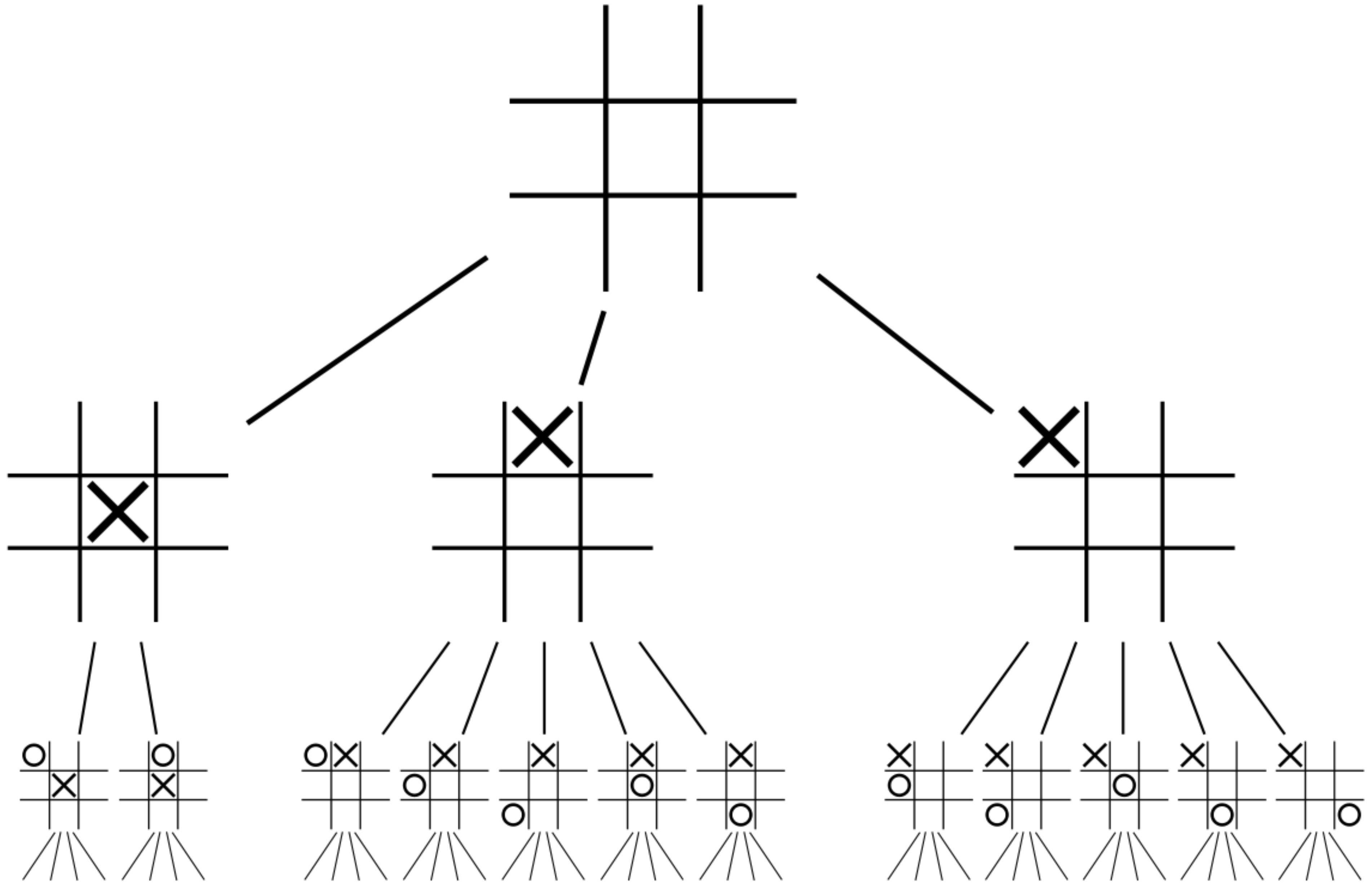
}

var simulate = function(state, action, player) {
    var nextState = transition(state, action, player);
    if (isTerminal(nextState)) {
        return nextState;
    } else {
        var nextPlayer = otherPlayer(player);
        return sample(Infer({ model() {
            var nextMove = act(nextState, nextPlayer);
            return simulate(nextState, nextMove nextPlayer);
        }}));
    }
};
```



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    var move = sample(movePrior(state));
    var u = utility(state, move, player);
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var utility = function(state, move, player) {
    var outcome = simulate(state, move, player);
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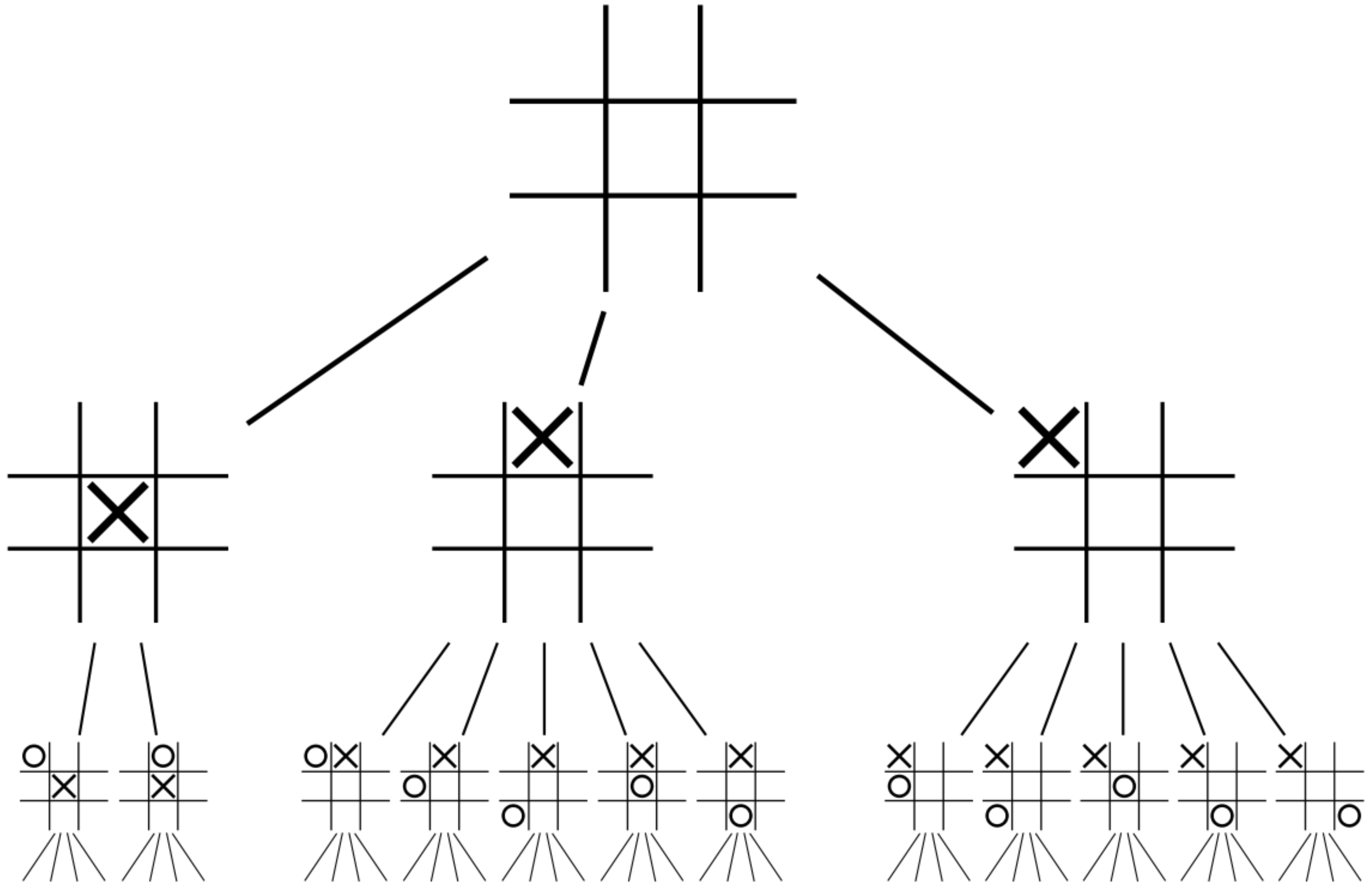

Stuhlmüller and Goodman. Reasoning about reasoning by nested conditioning: Modeling theory of mind with probabilistic programs. CogSci. 2014.
https://agentmodels.org/chapters/7-multi-agent.html

Example: Two-Player Game

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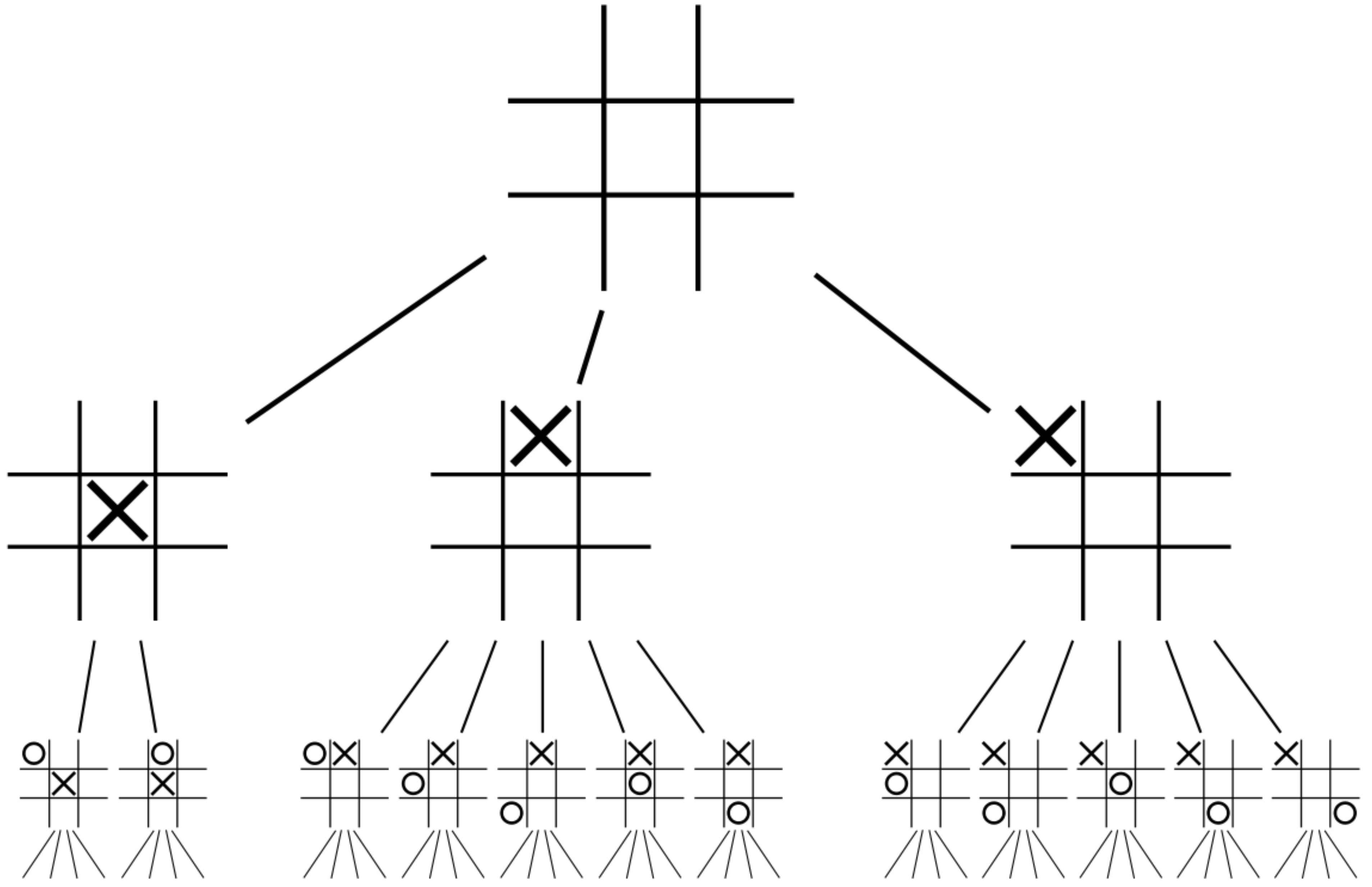

Inference inside inference => Thinking about thinking

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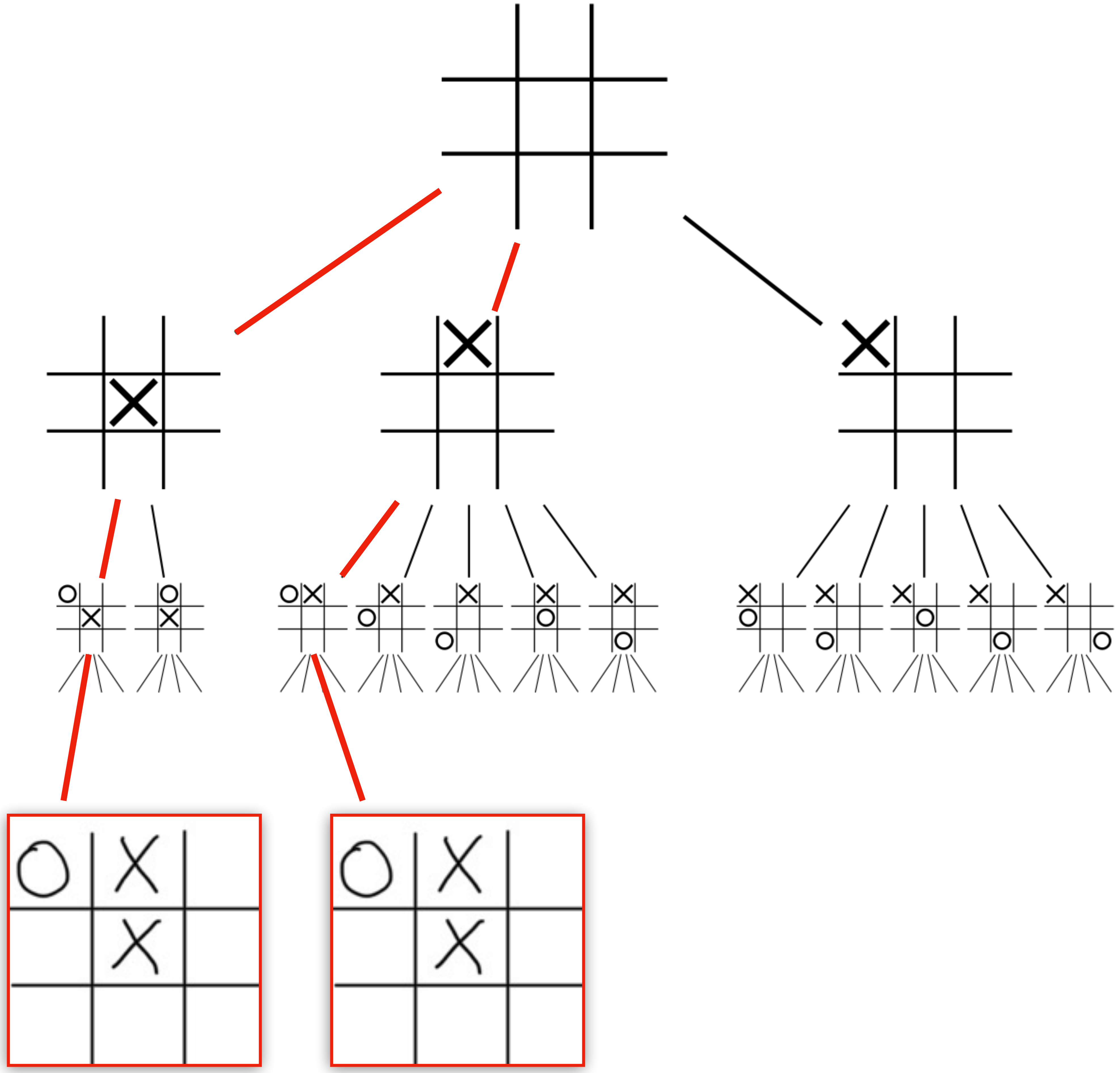
Amortizing nested inference



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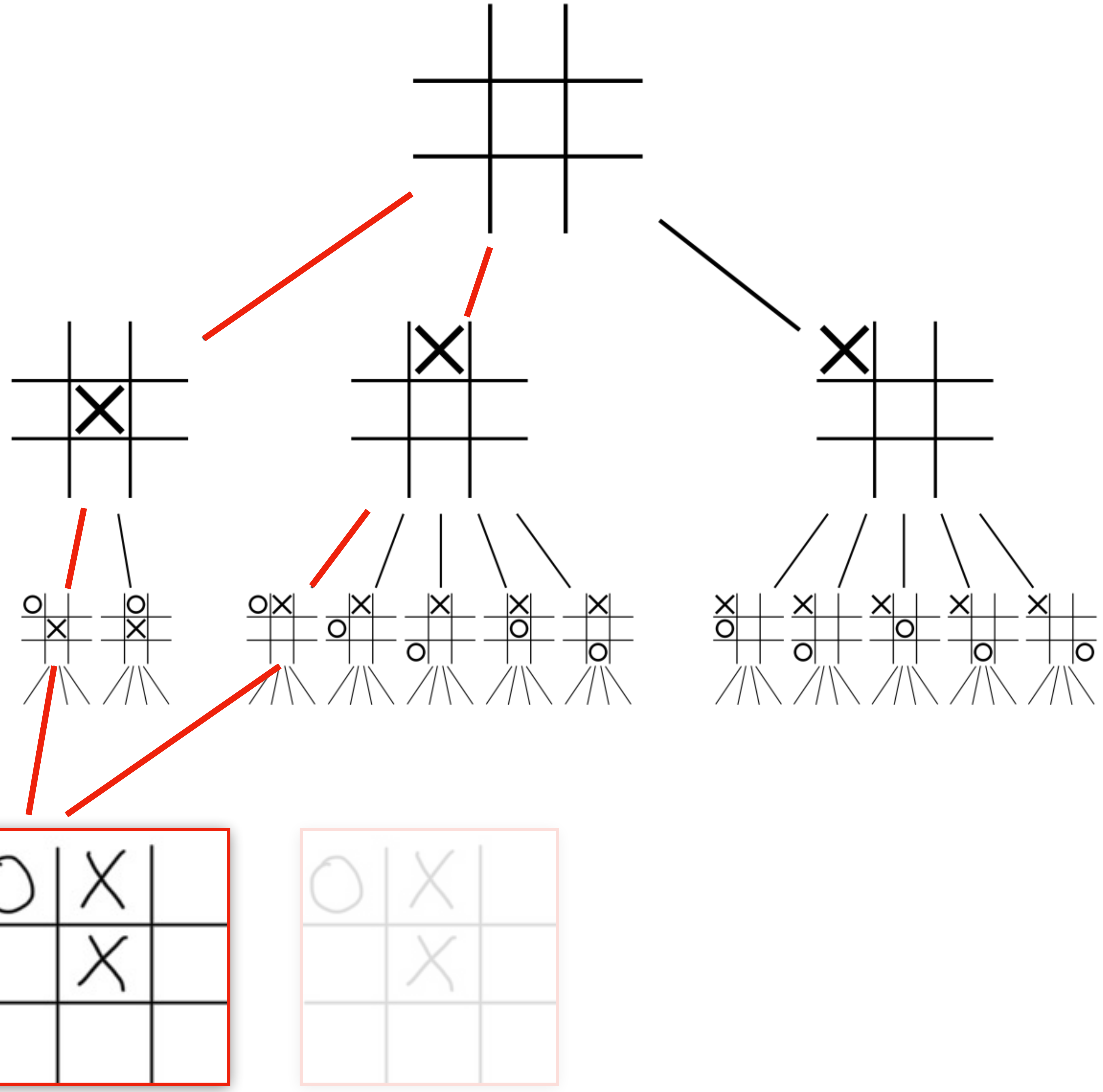

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A Glimpse into Formal Semantics

terms	t	$::=$	x	variable
		$\mid$	$\lambda x.\, t$	abstraction
		$\mid$	$t_1\ t_2$	application
		$\mid$	r	real number
		$\mid$	$\mathsf{op}_n(t_1, \ldots, t_n)$	n-ary operation invocation
		$\mid$	**sample**	sampling
		$\mid$	**factor** t	conditioning

$$\rho_n(\ell, t, V) \stackrel{def}{=} \begin{cases} r & \langle \ell \mid t \rangle \longrightarrow_n \langle \epsilon \mid v \rangle \bullet r \text{ and } v \in V \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_n(t, V) \stackrel{def}{=} \int \rho_n(\ell, t, V)\, \mathrm{d}\ell$$

$$\mu(t, V) \stackrel{def}{=} \lim_{n \to \infty} \mu_n(t, V)$$

$$\boxed{t_1 \longrightarrow t_2}$$

[KTX] $$\frac{t_1 \longrightarrow t_2}{K[t_1] \longrightarrow K[t_2]}$$

[BETA] $$(\lambda x.\, t)\ v \longrightarrow t\,\{v/x\}$$

[OP] $$\frac{[\![\mathsf{op}_n]\!](r_1, \ldots, r_n) = r}{\mathsf{op}_n(r_1, \ldots, r_n) \longrightarrow r}$$

$$\boxed{\langle \ell_1 \mid t_1 \rangle \longrightarrow \langle \ell_2 \mid t_2 \rangle \bullet r}$$

[PURE] $$\frac{t_1 \longrightarrow t_2}{\langle \ell \mid t_1 \rangle \longrightarrow \langle \ell \mid t_2 \rangle \bullet 1}$$

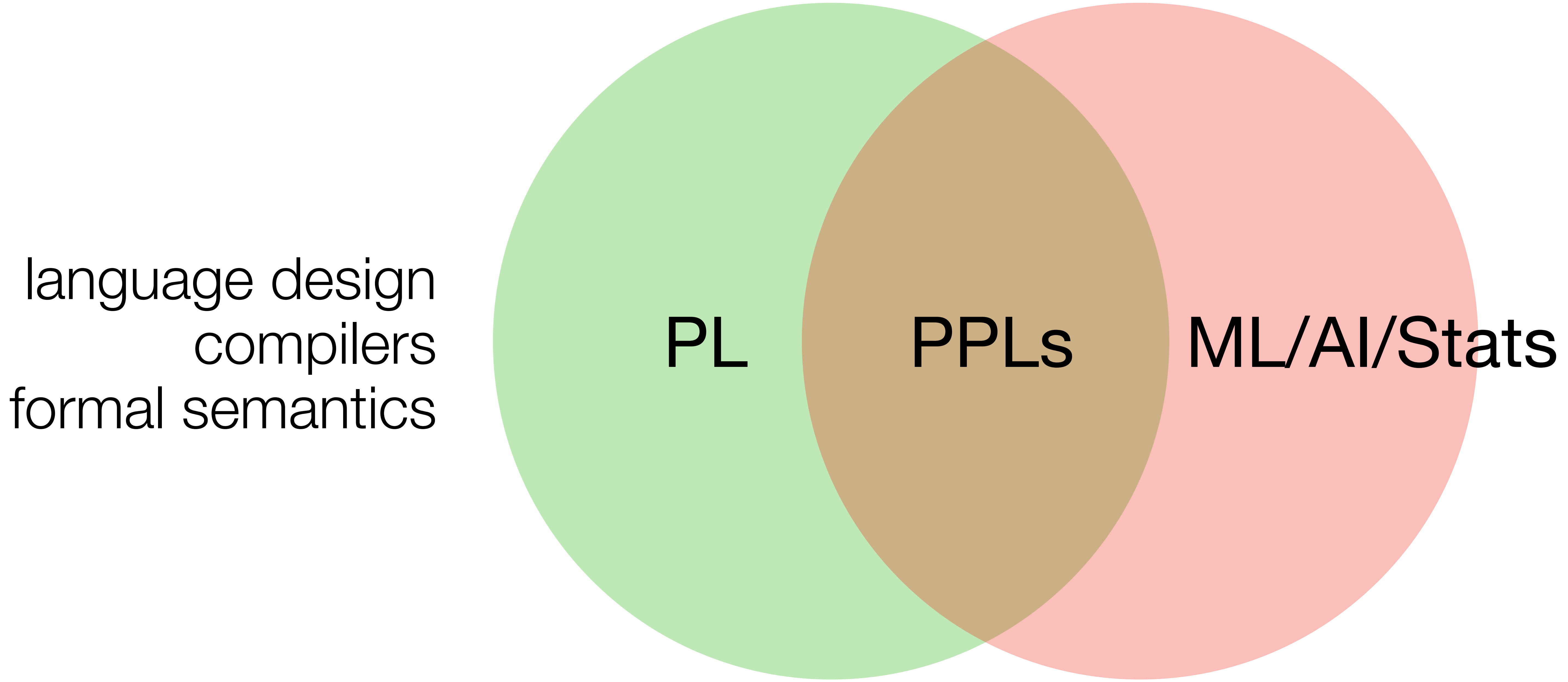
[SAMPLE] $$\frac{0 \leq r \leq 1}{\langle r, \ell \mid \textbf{sample} \rangle \longrightarrow \langle \ell \mid r \rangle \bullet 1}$$

[FACTOR] $$\frac{0 < r}{\langle \ell \mid \textbf{factor}\ r \rangle \longrightarrow \langle \ell \mid 0 \rangle \bullet r}$$

Takeaway messages

PPLs are powerful tools for probabilistic modeling and inference

Exciting area of ongoing research



CS 152 Programming Languages

Probabilistic Programming & Probabilistic Programming Languages

Yizhou Zhang
University of Waterloo