ES 151 Assignment #6

Professor: Donhee Ham
Date: March 26th, 2015
Due: **12:55pm + 10 min grace period**, April 2nd, 2015; slide your work under through the door at Maxwell-Dworkin 131.

Problem 1 (100pt)

(a) Consider (classically) a proton with a magnetic dipole moment $\vec{m}$ and an angular momentum $\vec{L}$, both of which arise from the proton spin. Let the gyromagnetic ratio $|\vec{m}|/|\vec{L}|$ be γ. Assume that $\vec{m}$ initially makes an angle θ_0 with a homogenous static magnetic field $\vec{B_0} = B_0\hat{z}$. Show that the magnetic dipole moment exhibits a precession motion about the direction of $\vec{B_0}$ with an angular precession frequency of $\omega_0 = \gamma B_0$ (*Larmor frequency*). To show this, you can solve the following equation of motion:

$$\frac{1}{\gamma}\frac{d\vec{m}}{dt} = \vec{m} \times B_0\hat{z}. \tag{1}$$

(b) Now, let a small (perturbing) time-varying magnetic field of angular frequency ω be applied to the proton perpendicularly to the homogenous static magnetic field $\vec{B_0} = B_0\hat{z}$. Without loss of generality, x-axis can be assigned to the direction of the time-varying field, which then can be written as $\vec{B_1}(t) = \hat{x}B_1\cos(\omega t)$, where $B_1 \ll B_0$. The equation of motion for $\vec{m}$ is then given by

$$\frac{1}{\gamma}\frac{d\vec{m}}{dt} = \vec{m} \times \left[B_0\hat{z} + B_1\cos(\omega t)\hat{x}\right]. \tag{2}$$

By solving this equation for $\omega = \omega_0$ (resonance condition), describe the motion of $\vec{m}$, with the initial condition of $\vec{m}$ lined up with $B_0\hat{z}$. In particular, you should be able to observe the followings:

- The projection of $\vec{m}$ onto the xy-plane will exhibit a precession motion with the angular frequency of ω_0;
- The z-component of $\vec{m}$ will exhibit an oscillation with the angular frequency of $\omega_1 \equiv \gamma B_1/2$ (this frequency is called *Rabi frequency*).

Can you picture the overall motion of $\vec{m}$ and how its potential energy in the $B_0\hat{z}$ field changes with time?

Problem 2 (60pt)

As shown in Fig. 1(a), a closed circular coil of N-turns, radius a, and total resistance R is rotated with a uniform angular frequency of ω_0 about its vertical diameter that lies on the z-axis. There is a horizontal, constant magnetic field $\vec{B_0} = B_0\hat{\vec{x}}$ along the x-axis.

(a) Express the electromotive force induced in the coil in terms of the given parameters (and of course, feel free to use all the universal constants such as ϵ_0, μ_0, etc). Also compute the time-averaged power $\overline{P}$ required for maintaining the rotational motion of the coil with the angular frequency of ω_0. Neglect the coil's self-inductance.

Now, as shown in Fig. 1(b), a small magnetic compass is placed at the center of the coil. It is free to turn slowly around the z-axis in the x-y plane, but it cannot follow the fast, ω_0-rotation of the coil.

(b) The compass will eventually settle at a stationary direction, forming an angle θ with $\vec{B_0}$, as depicted in Fig. 1(b). Compute this final angle.

Problem 3 (30pt)

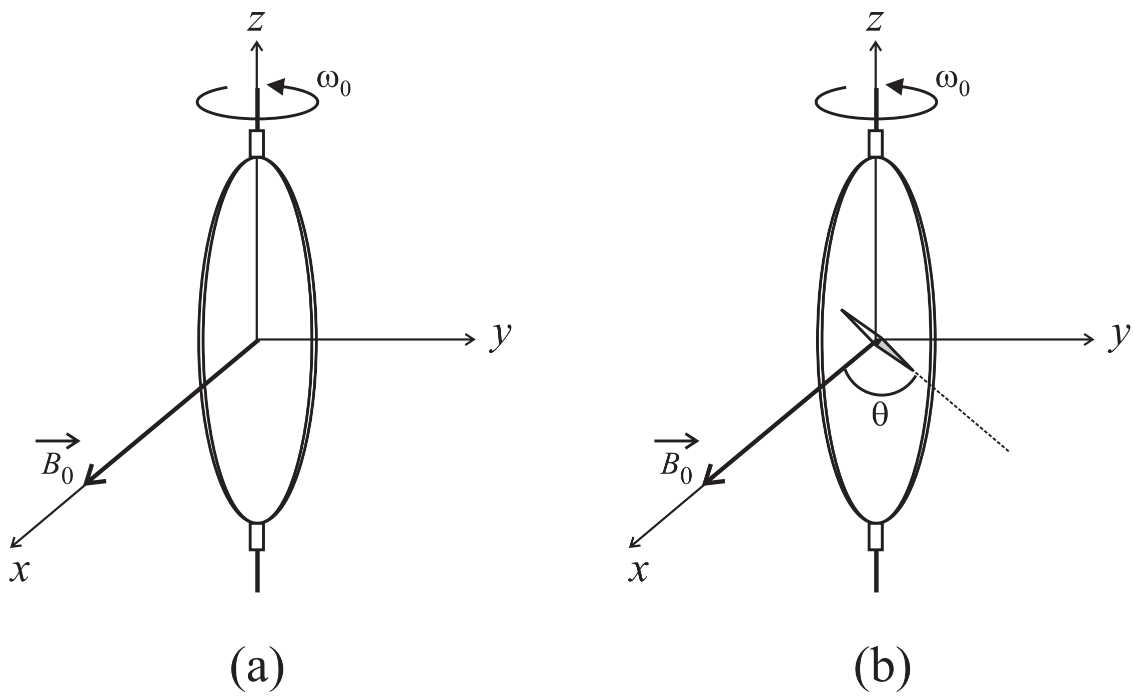

Figure 1:

A thin ring of radius a and of mass M carries a static charge q. This ring is supported to its axis so that it is free to rotate about that axis (*no friction, no loss*). Now a uniform magnetic field of strength B_0 is switched on, in the direction parallel to the ring's axis. How much angular momentum will be added/subtracted to/from the ring? How much angular velocity will the ring acquire, if it was initially at rest? Show that the result depends only on the final value of the field strength, but not on the detailed manner the change is brought about.

Problem 4 (40pt)

Following the derivation on Page 3 of Lecture Note #15, prove that the bound current of magnetic origin is given by

$$\vec{J}_{bound} = \vec{\nabla} \times \vec{M} \tag{3}$$

where $\vec{M}$ is the magnetization. Prove this by comparing only the x-component of both terms.

Problem 5 (30pt)

Using a single loop of a conducting wire and the setting shown on Page 1 of Lecture Note # 16, prove that the magnetic energy density per unit volume, u_B, is given by

$$u_B = \frac{1}{2}\vec{H} \cdot \vec{B} \tag{4}$$