ES 272 Assignment #2

Due: March 3rd, 2009

Instructor: Donhee Ham
Teaching fellow: Nan Sun

(Problem 1) Artificial transmission line (40pt)

The artificial transmission line is a lattice of "lumped" LC sections [Fig. 1(a)]. Differently from the transmission line, the artificial transmission line, even in the lossless case, possesses a cutoff frequency and a frequency-dependent characteristic impedance due to its lumped periodicity (non-smoothness). In this problem, we derive these properties of the artificial line.

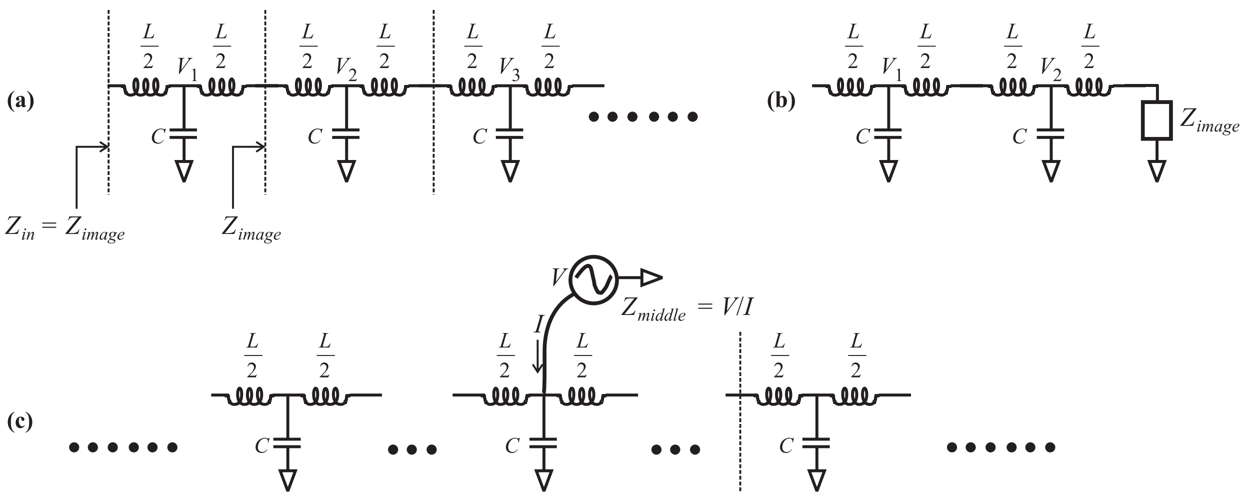

Figure 1: Artificial transmission line

(a) The input impedance of an infinitely long artificial line [Fig. 1(a)] is its characteristic impedance. This impedance is also called image impedance, Z_{image}. Show that Z_{image} is given by

$$Z_{image} = Z_0 \cdot \sqrt{1 - \frac{\omega^2}{\omega_c^2}} \tag{1}$$

where $Z_0 \equiv \sqrt{L/C}$ and $\omega_c \equiv 2/\sqrt{LC}$. You can solve this problem by noticing that adding another LC section at the beginning of the artificial line would not change the image impedance. What happens to Z_{image} at angular frequencies in excess of ω_c? Can you explain why?

(b) From the definition of Z_{image}, we know that Z_{image} provides a perfect termination (no reflection) to an artificial transmission line, and the infinitely long artificial line [Fig. 1(a)] is equivalent to a finite artificial line terminated with Z_{image} [Fig. 1(b)]. Using this notion, show that an input impedance, Z_{middle}, seen at an arbitrary node in an artificial line infinitely long in both directions [Fig. 1(c)] is given by

$$Z_{middle} = \frac{1}{2} \cdot \frac{Z_0}{\sqrt{1 - \omega^2/\omega_c^2}} \tag{2}$$

What happens to Z_{middle} at angular frequencies in excess of ω_c? Can you explain why?

(c) Any sinusoidal signal whose angular frequency is larger than ω_c cannot be propagated on the artificial line (hence the name, cutoff frequency, for ω_c). To prove this, first show that the wave equation in the artificial line is given by the following equation

$$LC\frac{d^2}{dt^2}V_n(t) = V_{n-1}(t) - 2V_n(t) + V_{n+1}(t) \tag{3}$$

where $V_n(t)$ is the voltage across the n-th capacitor at time t. For a sinusoidal wave, $V_n(t)$ can be written as

$$V_n(t) = A_0 \cdot e^{j(\omega t - \beta n)} \tag{4}$$

where A_0 is the amplitude, ω is the angular frequency, and β is the wave number[1]. By substituting (4) into the wave equation of (3), show that ω and β are related through

$$\omega = \omega_c \left| \sin\left(\frac{\beta}{2}\right) \right| \tag{5}$$

This ω-β relation indicates that the velocity[2] $v = \omega/\beta$ is a function of frequency ω, that is, different frequency components travel at different speeds. This "dispersion" phenomenon is another characteristic of the artificial line. Using the ω-β dispersion relation, show that any sinusoidal wave whose angular frequency is in excess of ω_c cannot be propagated down the artificial line.

(Problem 2) Distributed amplifier (80pt)

This problem revisits the distributed amplifier discussed in class, for your review and also to fill in some steps omitted in class. We will also consider a case more general than what we discussed in class. As we learned in class, the artificial transmission line plays an important role in the operation of the distributed amplifier. Therefore, the results of Problem 1 will be helpful here.

(a) Consider the distributed amplifier with N distributed MOS transistors in Fig. 2. The input and output artificial lines are identical (the same C and L). This condition guarantees the synchronized propagation velocities in the input and output lines. Parasitic capacitors of MOS transistors are absorbed into the artificial lines, contributing to the capacitors (the gate-drain parasitic caps that cannot be absorbed into the lines are ignored). Each transistor is magically biased and has a transconductance of g_m. Note the terminations of the input and output lines with the image impedances.

Let the small-signal gate and drain voltages of the k-th (k=1, 2, 3, ..., N) MOS transistor be a_k and b_k, respectively. Assume sinusoidal waves at an angular frequency ω. Derive the following recursive relations for $k = 1, 2, 3, ..., N-1$

$$a_{k+1} = a_k e^{-j\beta} \tag{6}$$

$$b_1 = -a_1 g_m Z_{middle} \tag{7}$$

$$b_{k+1} = b_k e^{-j\beta} - a_{k+1} g_m Z_{middle} \tag{8}$$

where Z_{middle} is in (2) and $\beta = \cos^{-1}[1 - 2(\omega/\omega_c)^2]$ is equivalent to (5). Using these recursive relations, show that the voltage gain $A_v = b_N/a_1$ is given by

$$A_v = -N g_m Z_{middle} \cdot e^{-j(N-1)\phi} \tag{9}$$

and its magnitude is hence given by

$$|A_v(\omega)| = \frac{N g_m Z_0}{2} \cdot \frac{1}{\sqrt{1 - \omega^2/\omega_c^2}} \tag{10}$$

In class, we already studied the implication of (10): as one distributes more of smaller transistors, reducing the degree of lumpedness, the cutoff frequency of the artificial lines is increased while maintaining the same

[1] While in the continuous transmission line, $\beta/(2\pi)$ in the unit of [1/length] signifies how many wavelengths a unit length contains, in the artificial line, $\beta/(2\pi)$ is unitless, and signifies how many wavelengths a unit lattice ($\Delta n = 1$) contains.

[2] This velocity in the artificial line, in the unit of [1/time], is the number of lattices the wave's phase passes per unit time.

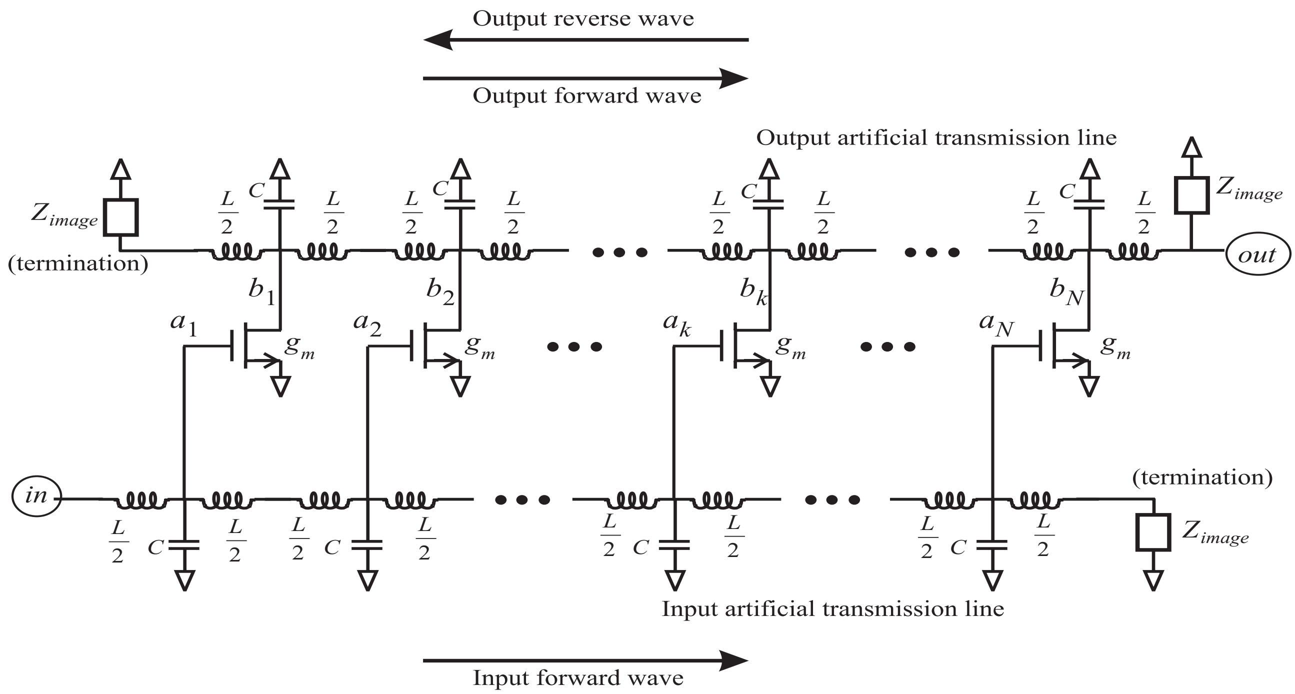

Figure 2: Distributed amplifier.

gain, hence enhancing the gain-bandwidth product in a very fascinating way.

(b) The amplifier gain $A_v(\omega)$ in (10) diverges to infinity at the cutoff frequency. This problem, however, can be mitigated, that is, $A_v(\omega)$ can be made attenuate near cutoff frequencies, by deliberately mismatching the phase delay of the input line and that of the output line. To see this, let the lumped inductors and capacitors in the input line be L_g and C_g, and those in the output line be L_d and C_d. The phase delay, cutoff frequency, image impedance, etc., of the input line are now all different from those of the output line. Derive $A_v(\omega)$ as in Part (a). Plot $A_v(\omega)$ versus ω for various phase delay mismatches. Can you make $A_v(\omega)$ maximally flat by properly adjusting the phase delay mismatch (you don't have to be analytically precise here)?

(Problem 3) Q enhancement technique (40pt)

To tackle the problem of low Q of spiral inductors in standard silicon (CMOS) technology, a Q enhancement technique utilizing magnetic energy coupling in a transformer is often used. The transformer structure is illustrated in Fig. 3. The primary coil is what is going to be used as an inductor, while the magnetic flux of the primary coil is coupled with that of the secondary coil. The primary and secondary coils are identical in that each coil has the same inductance of L and the same parasitic resistance of R. We will assume that the coupling between the two coils is perfect and hence the mutual inductance, M, is equal to L, *i.e.*, $M = L$. In the system, most of the RF input energy goes into the primary coil while its small replica (voltage coupling constant, $\alpha \ll 1$: also the power coupling constant is very small.), obtained through a directional coupler, drives the secondary coil after amplification by voltage gain of G and phase shift by ϕ. The voltage across, and the current flowing into, the primary coil are denoted as v_1 and i_1, respectively. Likewise, the voltage across, and the current flowing into, the secondary coil are denoted as v_2 and i_2, respectively. The relation between v_1 and v_2 are $v_2 = \alpha G v_1 e^{j\phi}$.

(a) The input impedance, $Z(\omega)$, shown in the figure can be expressed as $Z(\omega) \approx v_1(\omega)/i_1(\omega)$ with a very good approximation since most RF input energy flows into the primary coil. Show that $Re\{Z(\omega)\} = 0$ if the overall voltage gain αG and the phase shift ϕ in the path to the secondary coil satisfy:

$$\alpha G = \frac{2Q_0^2 + 1}{2Q_0^2 \cos\phi - Q_0 \sin\phi} \tag{11}$$

where $Q_0 \equiv \omega L/R$ is the quality factor of each coil. Since Q of the primary coil, or equivalently, Q of the transformer system *measured from Port 1* is given by $Q(\omega) = Im\{Z(\omega)\}/Re\{Z(\omega)\}$, when αG and ϕ satisfy (11), the primary coil Q becomes very large (infinite in theory) with very small (zero in theory) net power dissipation in the primary coil. This effect has been experimentally demonstrated and reported many times, with Q values up to several thousands.

(b) Although this technique seems to be an attractive solution for enhancing inductor Q, one can show that there is no actual benefit obtained with this technique as far as the overall energy efficiency is concerned. To see the overall energetics of the system clearly, prove that, with the arrangement of $Re\{Z(\omega)\} = 0$, the power dissipation in the primary coil R is exactly compensated by the magnetic energy/power coupled from the secondary coil. This is why you see zero net power dissipation in the primary coil. In connection with this proof, also show that, with the arrangement of $Re\{Z(\omega)\} = 0$, the power delivered into the secondary coil is the sum of the power dissipation in the secondary coil R and the aforementioned magnetic energy/power coupled into the primary coil to compensate the dissipation in the primary coil R. In summary, with the arrangement of $Re\{Z(\omega)\} = 0$, while there still exists power dissipation in the primary coil R, this power comes through the secondary coil, making the primary coil appear lossless. The power that flows into the secondary coil is obtained from the (power) amplifier (since the coupled power at Port 2 is very small), and portion of the *dc* power dissipation in the amplifier explains the zero dissipation in the primary coil.[3]

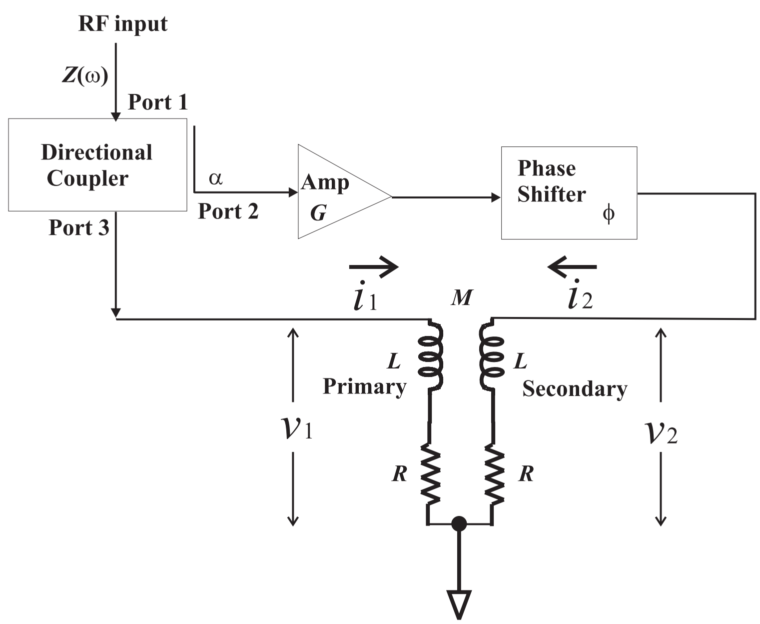

Figure 3: Q enhancement technique.

(*Note*) As discussed in class, the quality factor, Q, is originally defined for a "resonator" as

$$Q \equiv \omega_0 \frac{\text{Stored energy}}{\text{Power dissipation}}\Big|_{\omega=\omega_0} \tag{12}$$

[3]Since no power amplifier can have 100 % efficiency in converting *dc* energy into *RF* energy, actually the overall energy efficiency is even worse with the transformer scheme, as compared to the stand-alone inductor. However, this technique may be useful in obtaining a narrow-band transfer function when power dissipation is not critical.

where ω_0 is the resonance frequency. For a resonator whose resonance frequency is fixed, Q is *not* a function of frequency. Since inductors are not resonators, the original Q definition above cannot be used for inductors. However, the following frequency-dependent Q definition may be used instead as the quality factor for inductors:

$$Q(\omega) \equiv \omega \frac{\text{Stored energy } (\omega)}{\text{Power dissipation } (\omega)} \tag{13}$$

It can be easily shown that $Q(\omega)$ in (13) is on the same order as, but not exactly the same as,

$$Q(\omega) \equiv \frac{Im\{Z(\omega)\}}{Re\{Z(\omega)\}} \tag{14}$$

where $Z(\omega)$ is the frequency-dependent input impedance of a given inductor. RF engineers traditionally choose to use (14) over (13) due to the simplicity of (14).

(Problem 4) Inductive peaking (40pt)

Figure 4 shows a shunt-peaked MOS amplifier. The transistor is biased in the pinch-off regime with a transconductance of g_m. The transistor size and the values of R and C are fixed, and the inductance, L, is the sole design parameter (Neglect all the parasitics of the transistor). As discussed in class, one can control the frequency response of the shunt-peaked amplifier by changing the value of L. This problem concerns obtaining the maximum-bandwidth and maximally-flat responses of the amplifier.

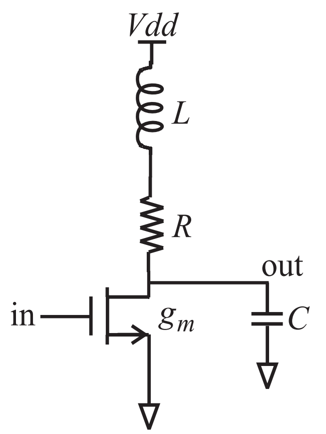

Figure 4: Shunt-peaked amplifier. Bias is not shown.

(a) Show that the amplifier's small-signal voltage gain (frequency response) normalized to its *dc* gain, $g_m R$, is given by

$$H(x) \equiv \frac{|A_v|}{g_m R} = \sqrt{\frac{1 + (\alpha x)^2}{(1 - \alpha x^2)^2 + x^2}} \tag{15}$$

Here $x \equiv \omega/\omega_{3dB,0}$ is the normalized frequency and $\alpha \equiv L \cdot (\omega_{3dB,0}/R)$ is the design parameter directly proportional to L where $\omega_{3dB,0} \equiv (RC)^{-1}$ is the 3dB bandwidth of the uncompensated ($L = 0$) amplifier. Plot $H(x)$ versus x for α =0, 0.2, 0.4, 0.6, and 0.8.

(b) Express the 3dB bandwidth, ω_{3dB}, of the shunt-peaked amplifier in terms of $\omega_{3dB,0}$ and α.

(c) Find the value of α at which ω_{3dB} becomes maximum. Calculate the ratio of this maximum bandwidth to the uncompensated bandwidth, $\omega_{3dB,0}$. Estimate the peaking in the frequency response (maximum of $H(x)$) in this maximum-bandwidth design.

(d) If $H(x) \leq 1$ for all $x \geq 0$, we call the frequency response "flat". Find the maximum α with which $H(x)$ is flat. This α corresponds to the so-called "maximally-flat" response. What is ω_{3dB} in this maximally-flat design?