ES 272 Assignment #5

Due: April 2nd, 2009

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(Problem 1) Energetics in Van der Pol oscillators (30pt)

As we saw in class, the dynamics of a self-sustained oscillator consisting of a parallel LRC resonator and active devices may be described by the Van der Pol differential equation:

$$\ddot{v} + \omega_0^2 v + \omega_\tau f(v)\dot{v} = 0 \tag{1}$$

where v is the voltage across the LC tank, $f(v) < 0$ for small enough $|v|$, $f(v) > 0$ for large enough $|v|$, $\omega_0 = 1/\sqrt{LC}$, and $\omega_\tau = 1/(RC)$. We will assume that the quality factor $Q = RC\omega_0$ of the resonator is much larger than 1: $Q \gg 1$.

(a) Show that the time derivative of the instantaneous tank energy, $E_{tank}(t) = (1/2) \cdot [Cv^2(t) + Li^2(t)]$, is approximately given by[1]

$$\frac{dE_{tank}}{dt} \approx -\frac{\dot{v}^2 f(v)}{R\omega_0^2} \tag{2}$$

As can be seen, when $|v|$ is small enough, $f(v) < 0$ and the tank energy increases, and when $|v|$ is large enough, $f(v) > 0$, the tank energy decreases.

(b) For $Q \gg 1$, we can approximate the steady-state voltage output of the oscillator as $v(t) \approx v_0 \cos\omega_0 t$ (argue why - one sentence should be sufficient), where v_0 is the amplitude. One form of $f(v)$ that can be used for the Van der Pol oscillator is $f(v) = av^2 - b$ where $a > 0$ and $b > 0$. Show that v_0 is given by $v_0 = 2\sqrt{b/a}$ noting that in steady state the net tank energy change per period is zero.

(Problem 2*) Cross-coupled LC voltage-controlled oscillator (80pt)

Using *nmos2v* and/or *pmos2v* devices in the *tsmc18rf* library, design an LC voltage-controlled oscillator of Fig. 1: you may design a variational form as far as it includes at least one-pair of cross-coupled MOS transistors. The oscillator should satisfy the following specifications:

- Supply voltage, V_{dd}: 1.8 V
- Oscillation frequencies: $f_{min} < 5$ GHz $< f_{max}$
- frequency tuning range: $(f_{max} - f_{min})/5$ GHz $\geq 10\%$ (Control voltage, V_{cont}, between 0 and 1.8 V.)
- In Fig. 1, R models parasitic loss of the inductors. Minimum R and maximum L that you can use are 10 Ω and 3 nH, respectively.
- Variable capacitors, C, in Fig. 1: use *nmos2v* or *pmos2v* as varactors.
- Maximum bias current in the oscillator core: 30 mA - do *not* use an ideal current source.
- Minimum differential voltage amplitude across the whole frequency tuning range: 500 mV

[1] i is the current in the inductor. For a high-Q case, i can be approximated as $i \approx C\dot{v}$.

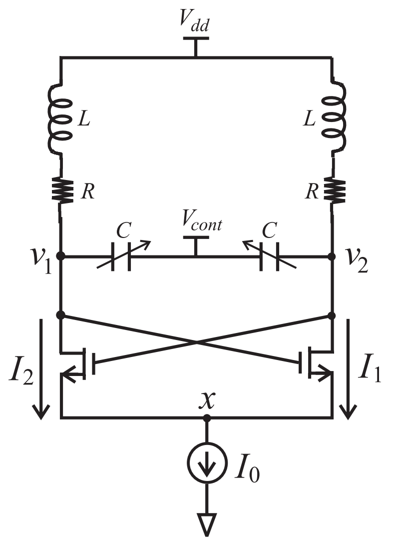

Figure 1: Cross-coupled LC voltage-controlled oscillator

(Problem 3*) Quadrature synthesis via superharmonic coupling (80pt)

Accurate quadrature synthesis is prerequisite for vector modem. Most widely-used methods of quadrature generation may be categorized into: (1) Feeding the differential signals of an oscillator into an RC polyphase filter (See Assignment #4); (2) a ring oscillator consisting of 4 differential inverters; (3) Quadrature synthesis via coupling two symmetric LC oscillators. One example in this 3rd category by Asad Abidi's group from UCLA[2] will be discussed in class. In this problem, we will analyze and design a quadature LC oscillator shown in Fig. 2(a), which is yet another circuit that belongs in the 3rd category above.

(a) Show that voltages x and y have to be either in the same phase (even-mode) or in the opposite phase (odd-mode) due to the inductive coupling through the two identical inductors, L_c, using KCL and a symmetry argument. Show that in the even-mode, $x = y = z$ while in the odd-mode, $x = -y$ and $z = 0$. This $z = 0$ in the odd-mode corresponds to a virtual ac ground.

(*Operating principle*) Let's assume that $v_1(t) \propto \cos(\omega_0 t)$ and $v_2(t) \propto -\cos(\omega_0 t)$ where we dropped any dc component for the sake of brevity. Then $x(t) \propto \cos(2\omega_0 t + \theta)$ where $2\omega_0$ is due to the symmetry of the single LC oscillator and θ is the phase difference introduced by transistors' effective resistance, C_c, and L_c. In the odd-mode, $y(t) \propto \cos(2\omega_0 t + \theta + \pi)$, and hence, $v_3(t) \propto \cos(\frac{2\omega_0 t+\theta+\pi-\theta}{2}) = -\sin(\omega_0 t)$ and similarly, $v_4(t) = \sin(\omega_0 t)$. Therefore, in the odd-mode, $v_1(t) - v_2(t)$ and $v_3(t) - v_4(t)$ oscillate in quadrature phase.[3] You can similary show that in the even-mode, $v_1(t) - v_2(t)$ and $v_3(t) - v_4(t)$ oscillate in the same phase (or opposite phase depending upon the order of subtraction to obtain the differential voltages.).

Now we need to make the odd-mode quadrature oscillation prevail over the even-mode in-phase oscillation. This can be done by appropriately selecting the values of L_c, C_c, and I_0. To figure out how to properly choose the three design parameters, let's consdier equivalent odd-mode and even-mode half-circuits depicted in Figs. 2(b) and (c), respectively. In the odd-mode half circuit [Fig. 2(b)], the phase θ mentioned in the previous paragraph can be made equal to zero by selecting C_c and L_c appropriately to make the C_c-L_c tank resonate at around $2\omega_0$. With this resonance, $v_1 - v_2$ and x will be related through the effective transistor resistance with no reactive elements, and hence $\theta = 0$ and the minima of the x will be aligned

[2] A. Rofougaran, J. Rael, M. Rofougaran, and A. Abidi, "A 900 MHz CMOS LC-Oscillator with Quadrature Outputs," *IEEE International Solid-State Circuits Conference (ISSCC), 1996.*

[3] Note that the odd-mode quadrature oscillation is made possible through the coupling of the two second-order harmonic voltages, x and y, with phase shift by π (odd-mode), and hence this quadrature oscillator is called "superharmonically-coupled" quadrature generator.

with minima of v_1 and v_2, as shown in the inset of Fig. 2(b). The resonance at $2\omega_0$ in the L_c-C_c tank is possible because z is a virtual *ac* ground. Figure 2(c) illustrates the even-mode half-circuit. Since $x = z$ in the even-mode, the coupling inductance L_c is not playing any role and hence x is directly connected to the current source. The C_c value was already chosen using the odd-mode half circuit as described shortly before. In the even-mode, $v_1 - v_2$ and x are related through the effective transistor resistance and C_c, and hence $\theta \neq 0$ where θ is again the phase difference between v_1 (or v_2) and x as mentioned earlier. This leads to misalignment of the minima of x and minima of v_1 and v_2 in the even-mode as shown in the inset of Fig. 2(c).

The alignment between the minima of x and v_1, v_2 in the odd-mode and the misalignment between the minima of x and v_1, v_2 in the even-mode make the odd-mode oscillation have a larger voltage swing if the current I_0 is large enough. This is because in the case of the misalignment, v_1 and v_2 swings get clamped by x if the current is large enough to make the oscillation amplitude in x significant. Therefore, with an increasing current, the even-mode oscillation reaches the voltage-limited regime earlier than the odd-mode oscillation as illustrated in Fig. 2(d). As far as $I_0 > I_{0,critical}$, the odd-mode voltage swing will be larger than the even-mode voltage swing, and the loop nonlinearity inherent in the oscillator selects the odd-mode oscillation while suppressing the even-mode oscillation. This is how the quadrature oscillation is chosen over the in-phase oscillation. When $I_0 < I_{0,critical}$, both the even-mode and the odd-mode oscillations are possible, but the loop nonlinearity often seems to prefer the even-mode oscillation (the reason for this is not clear to me.).

(b) Design the superharmonically-coupled quadrature oscillator of Fig. 2(a) using *nmos2v* devices in the *tsmc18rf* library, while meeting the following specs:

- Supply voltage = 1.8 volt.
- Minimum $R = 2\Omega$, Maximum $L = 5$ nH.
- Maximum $I_0 = 10$ mA
- Oscillation frequency, $f_0 = 2$ GHz.
- $C_c = 2$ pF.
- You may use an ideal current source.

You should be able to see in your design that as the tail current I_0 increases, the oscillator goes into the quadrature generation regime (odd-mode). For a smaller I_0, the circuit may oscillate in the in-phase regime (even-mode).

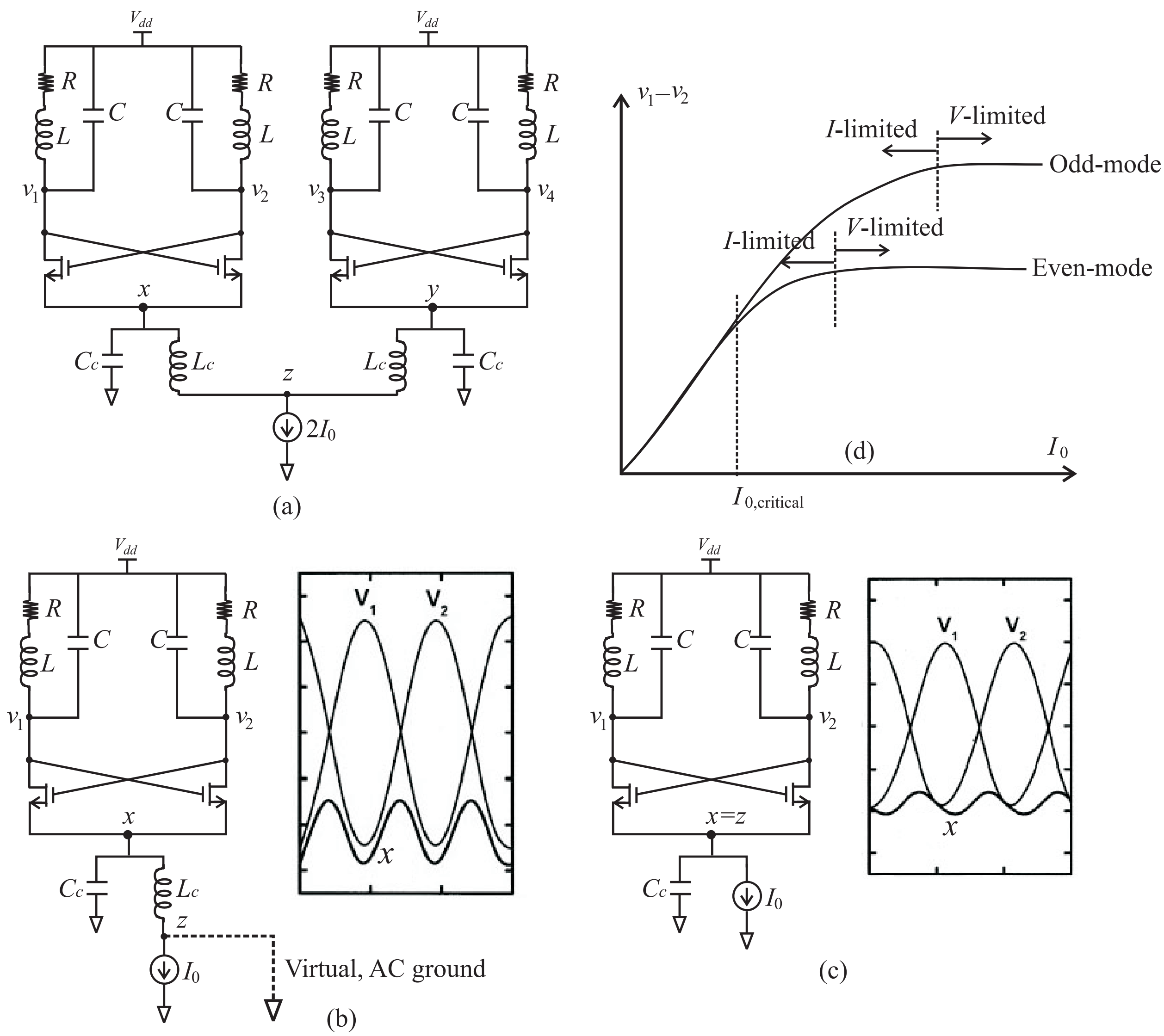

Figure 2: (a) *LC* superharmonically-coupled quadrature oscillator. (b) Odd-mode half circuit. (c) Even-mode half circuit. (d) Voltage swing versus I_0 for odd- and even-modes.

(Problem 4) Phase noise of an oscillator (60pt)

As seen in class, the voltage output of a noisy oscillator can be expressed as

$$v(t) = v_0 \cos(\omega_0 t + \phi(t)) \tag{3}$$

around its fundamental oscillation frequency $\omega_0 = 2\pi f_0$, where the oscillator phase, $\phi(t)$, is a random process. In the presence of only white noise which we will assume in this problem, $\phi(t)$ is a diffusion process (random walk process). The statistical properties of $\phi(t)$ are given by[4]

$$\langle \phi(t) \rangle = 0 \tag{4}$$

$$\langle \phi(t_1)\phi(t_2) \rangle = 2D \min\{t_1, t_2\} \tag{5}$$

which are key signatures of any diffusion process subject to white noise, where D is the phase diffusion constant. $\phi(t)$ is also a Gaussian process for any given time t.

(a) Express $\langle v(t) \rangle$ in terms of v_0, ω_0, D, and t. (See Hint below.) What is $\lim_{t\to\infty}\langle v(t)\rangle$? Why?

(b) Express the autocorrelation of $v(t)$, *i.e.*, $\langle v(t_1)v(t_2)\rangle$, in terms of v_0, ω_0, D, and t. What is $\lim_{t\to\infty}\langle v^2(t)\rangle$? Why?

(c) Show that the single-side band (SSB) power spectral density of the oscillator, $S_v(\omega)$, is given by

$$S_v(\omega) = v_0^2 \cdot \frac{D}{(\Delta\omega)^2 + D^2} \tag{6}$$

where $\Delta\omega \equiv \omega - \omega_0$.

(d) The phase diffusion constant, D, of an LC oscillator (with parallel LC tank) is roughly approximated as

$$v_0^2 \cdot D \approx \frac{kT}{C} \cdot \frac{\omega_0}{Q} \tag{7}$$

where Q is the quality factor of the LC tank, k is Boltzmann's constant, and T is the ambient absolute temperature.[5] Using (6), (7), and the definition of phase noise, show that in the $1/f^2$ region where $\Delta\omega \gg D$ the phase noise of the LC oscillator is given by

$$\mathcal{L}\{\Delta\omega\} \sim \frac{kT}{P_{diss}} \cdot (\frac{\omega_0}{Q\Delta\omega})^2 \tag{8}$$

where P_{diss} is the averaged RF power dissipated in the tank. This is Leeson's formula[6] very useful in estimating the oscillator phase noise (Leeson did not derive it the way you did in this problem). Note that the oscillator phase noise is inversely proportional to Q^2 for a given RF power dissipation in the tank. This explains the oscillator designers' obsession with the resonator Q.

(3) Calculate the power spectral density (PSD), $S_\phi(\omega)$, of $\phi(t)$, and show that $S_\phi(\omega) \propto D/\omega^2$. This $S_\phi(\omega)$ is what can be used for the noise analysis in a PLL loop later. Make a quantitative comparison of $S_\phi(\omega)$ to $S_v(\omega)$, and give a meaningful interpretation to their relation.

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[4] The notation, $\langle\cdot\rangle$, signifies an ensemble average.

[5] This formula can be interpreted in analogy with the well-known particle diffusion (Brownian motion) theorized by Einstein. In Brownian motion, a particle of mass m is suspended in a liquid whose viscosity is γ, and the particle's displacement, x, satisfies $\langle x^2(t)\rangle = 2Dt$ where D is the diffusion constant. Einstein showed that D is given by $D = (kT/m)\cdot(1/\gamma)$, and this should be familiar in connection with the physics of pn junction devices. kT/C and ω_0/Q in (7) correspond to kT/m and $1/\gamma$ in the Brownian motion. In (7), $v_0^2 D$ signifies the diffusion rate of the oscillation trajectory diffusion on the limit cycle while D is the diffusion rate for the phase diffusion.

[6] D. B. Leeson, "A simple model of feedback oscillator noise spectrum," *Proc. IEEE*, vol. 54, pp. 329 - 330, Feb. 1966.

(*Hint*) In answering (a) and (b), you will have to calculate $\langle \cos\phi(t) \rangle$, $\langle \sin\phi(t) \rangle$, $\langle \cos\phi(t_1)\cos\phi(t_2) \rangle$, etc. These calculations can be simplified by noting that the $n-th$ order cumulants for any Gaussian distribution are zero for $n > 3$.

(*Note*) The sequence of calculations in this problem is a well-known formulation in calculating not only oscillator phase noise but also laser linewidth in optics.