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FULLY PLASTIC CRACK PROBLEMS IN BENDING AND TENSION

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INTRODUCTION

Continuing work done previously in Ref. [1-5], we will consider a class of small strain, fully plastic crack problems for incompressible power law materials. In simple tension the stress and strain are related by

$$\varepsilon/\varepsilon_0 = \alpha(\sigma/\sigma_0)^n \qquad (1)$$

where ε_0 and σ_0 are a reference strain and stress, respectively, and α is a constant. Equation (1) is generalized to multiaxial states of stress using J_2 deformation theory according to

$$\varepsilon_{ij}/\varepsilon_0 = (3/2)\alpha(\sigma_e/\sigma_0)^{n-1}s_{ij}/\sigma_0 \qquad (2)$$

where s_{ij} is the stress deviator and the effective stress is given by

$$\sigma_e = \sqrt{3s_{ij}s_{ij}/2} \qquad (3)$$

As first noted by Ilyushin [6], a solution to a boundary value problem involving a single load or displacement parameter which is increased monotonically has two important properties. First, all quantities increase in direct proportion to the load or displacement parameter raised to a power. For example, if P is a load parameter the stress at every point is proportional to P while the strain is proportional to P^n. The second property follows from the first. Because the deformation and stress history is proportional at every point a fully plastic solution based on Eq. (2) is also an exact solution to the same problem posed using J_2 flow (incremental) theory. Furthermore, by identifying ε_{ij} as the strain rate the fully plastic crack problems apply to a power law creeping material as discussed in [1].

As a consequence of the known simple functional dependence of any quantity of interest in a crack problem, such as the J-integral or the load point displacement, or the load or displacement parameter, it only

remains to determine how this dependence varies with geometry. It is feasible to tabulate this geometric dependence once and for all for a number of basic configurations, just as has been done for linear elastic crack problems. Analytic solutions to several problems in antiplane shear have been given by Amazigo [2]. But for plane problems numerical methods appear to be necessary for generating solutions. Numerical results for the center-cracked strip in tension and the edge-cracked strip in bending for plane stress have been given in Ref. [3] and these will be referred to further within the present chapter. The primary purpose of the present chapter is to present new results for the edged crack in plane strain bending and also to present more accurate results for the center-cracked strip in plane strain tension than those given previously in Ref. [1]. We will also make comparisons with recent results of Parks [4] and Ranaweera and Leckie [5] for center-cracked strip. Results for the double edge-cracked strip in tension have been given in Ref. [5] for plane stress and plane strain.

Fully plastic crack solutions have been employed together with linear elastic solutions to produce relatively simple approximate formulas for quantities such as J and the crack-opening displacement which interpolate over the range from small-scale yielding to large-scale yielding [3, 7]. Recently such interpolation formulas have been used in the investigation of the stability of small amounts of crack growth under J-controlled growth conditions which may involve large-scale plastic yielding [8-10]. These interpolation methods are in need of further development, but they hold out promise for reasonably accurate, simple estimates of quantities of interest in crack problems involving moderate to large amounts of plastic yielding.

NUMERICAL METHOD

In plane strain the incompressibility of the power law material (2) imposes a constraint on the in-plane displacement gradients which must be enforced in any numerical method. Goldman and Hutchinson [1] introduced a compressible material which reduced to Eq. (2) in the incompressible limit. They used a finite element method and attempted to extrapolate numerically to the limit of incompressibility, but only with limited success. Parks [4] and Ranaweera and Leckie [5] used a Lagrangian multiplier technique to enforce incompressibility in their finite element solutions. The disadvantage of introducing Lagrangian multipliers is the considerable increase in the number of unknowns and the associated increase in expense of the calculations. This disadvantage is largely overcome if the multipliers are eliminated algebraically prior to numerical computation. The results in plane strain reported below have been obtained using a computer program developed for incompressible materials by Needleman and Shih [11]. In addition to eliminated multipliers, the program makes use of a modified Newton method for values of n greater than unity.

Parameter tracking is employed using the solution for $n = 1$ as the starting point in the search for the solution at a somewhat larger value of n, which in turn is taken as a starting point for an even larger value of n. At the larger values of n (say, $n > 7$) some care must be exercised to ensure convergence of the Newton's method. In some cases it proved necessary to employ several linear iterations before using the Newton procedure.

Quadrilateral elements composed of four constant strain triangular elements are used with the internal degree of freedom condensed out consistent with the incompressibility requirement [11]. In the finest mesh used there were 12 quadrilateral elements in the circumferential direction about the crack tip (from $\theta = 0$ to π) and 24 elements in the radial direction from the tip to the external boundaries. This was the mesh used to calculate the numerical results presented below. A somewhat cruder mesh, with 8×18 elements, was used to obtain some insight to the effect of mesh refinement. In general, the J-values changed by only a few percent in going from the crude to the finer mesh, while the crack opening displacement δ, introduced below, was somewhat more sensitive to the mesh size. At the low n-values, for example in changing n from 3 to 5, typically three Newton iterations were required to achieve satisfactory convergence. From $n = 7$ to $n = 10$, eight iterations was more typical. The calculations were carried out on a CDC-7600 computer. For the finer mesh one iteration involved slightly less than 4 seconds, while for the cruder mesh one iteration took slightly over one second.

Numerical results will be presented for the J-integral, the load point displacement and for certain crack opening displacements. The J-integral was evaluated using the path-independent line integral of Rice [12]. As a check on the numerical accuracy a number of contours encircling the crack tip were taken. In all cases the numerically computed values of J were indeed found to be the same to within about a percent on all contours. Accuracy will be discussed along with the numerical results given below. We will also give some discussion of the range of dominance of the crack tip singularity fields in the last section.

EDGE-CRACKED STRIP IN BENDING

The geometry of the strip is shown in Fig. 1. An uncracked strip of width b and height $2h$ in plane strain pure bending undergoes a relative rotation of its ends given by

$$\theta(h)_{\text{no crack}} = \frac{2\sqrt{3}\alpha\varepsilon_0 h}{b}\left[\left(\frac{2n+1}{n}\right)\frac{\sqrt{3}\,M}{\sigma_0 b^2}\right]^n \tag{4}$$

where M is the resultant moment of the stresses per unit thickness, i.e.,

$$M = \int_0^b \sigma_{yy}(x,h)\,x\,dx \tag{5}$$

For the strip with crack of length a the stress distribution associated with the pure bending of the uncracked strip is imposed on the ends $y = \pm h$. The load point rotation, through which M works, is defined as

$$M\theta(h) = 2\int_0^b \sigma_{yy}(x,h)\,u_y(x,h)\,dx \tag{6}$$

where u_y is the displacement in the y direction. The contribution to the load point rotation due to the presence of the crack, $\theta_c(h)$, is defined by

$$\theta_c(h) = \theta(h) - \theta(h)_{\text{no crack}} \tag{7}$$

The numerical results presented below were computed with $h/b = 2$. Previous work indicates that $\theta_c(h)$ is then essentially the limiting value $\theta_c(\infty)$ for an infinitely long strip. Henceforth, θ_c, with J and δ, introduced below, will be assumed to be values for an infinite strip and the implied dependence on h will be dropped.

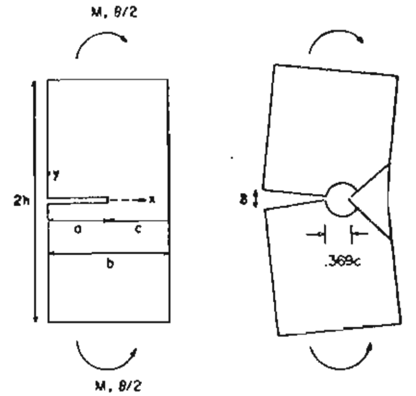

Fig. 1 Edge-cracked strip in bending

With $c = b-a$ denoting the width of the remaining ligament, the limit moment of a perfectly plastic $(n = \infty)$ strip in plane strain is

$$M_0 = 0.364\,\sigma_0 c^2 \tag{8}$$

as long as a/b is not too small. Following Ref. [3], we use M_0 to normalize the moment such that the general expressions for the J-integral, the crack opening displacement at the edge of the strip, $\delta = u_2(0,0^+) - u_2(0,0^-)$ (see Fig. 1), and θ_c are of the form

$$J = \alpha\sigma_0\varepsilon_0 c\, h_1(a/b,n)(M/M_0)^{n+1} \tag{9}$$

$$\delta = \alpha\varepsilon_0 a\, h_2(a/b,n)(M/M_0)^n \tag{10}$$

$$\theta_c = \alpha\varepsilon_0\, h_3(a/b,n)(M/M_0)^n \tag{11}$$

Here h_1, h_2, and h_3 are dimensionless functions of a/b and n.

Calculated values of h_1, h_2, and h_3 for plane strain are given in Table 1 for $a/b = 1/4$, 1/2, and 3/4 and for $n = 1,2,3,4,7,$ and 10. For fixed n, the normalizations in Eqs. (9)-(11) are such that the h's approach finite, nonzero limits in the deeply-cracked limit as $a/b \to 1$. It can be seen from Table 1 that h_1 at $a/b = 1/2$ differs by less than one percent from its values at $a/b = 3/4$ for all $n \geq 3$. This suggests that the deeply crackeo limit is already attained at $a/b = 1/2$ for $n \geq 3$. An independent check on this assertion can be had by using another formula for J derived in [13] under the assumption that the deeply cracked limit pertains. When the remaining ligament c is sufficiently small so that J depends on c, but not a, the deeply-cracked limit gives

$$J_{dc} = \frac{2}{c}\int_0^{\theta_c} M d\theta_c \qquad (12)$$

Using Eq. (11) in Eq. (12) we can evaluate J_{dc} in terms of h_3. The result is

$$J_{dc} = 0.728\alpha\left(\frac{n}{n+1}\right)h_3(a/b,n)\sigma_0\varepsilon_0 c\left(\frac{M}{M_0}\right)^{n+1} \qquad (13)$$

The values of h_3 in Table 1, at $a/b = 1/2$ are not quite as close to the values at $a/b = 3/4$ as is h_1; but, nevertheless, these values are within 10% of each other. At the same value of M, the ratio of J_{dc} to J given by (9) is

$$\frac{J_{dc}}{J} = 0.728\left(\frac{n}{n+1}\right)\frac{h_3(a/b,n)}{h_1(a/b,n)} \qquad (14)$$

Using the values from Table 1, we find, for example, that this ratio is 1.06 for $n = 5$ and 1.07 for $n = 10$ when $a/b = 1/2$ and 1.06 for $n = 5$ and 1.09 for $n = 10$ when $a/b = 3/4$. We believe that this difference from unity reflects numerical error in our results rather than a true departure from the deeply-cracked limit. In the plane stress problem the ratio J_{dc}/J was found to be within one or two percent of unity for all $n > 3$ even when $a/b = 1/2$. Thus we expect that there may be errors in the values in Table 1 of h_1 or possibly h_3 at the higher n values of as large as 10%.

Table 1 Plane Strain Strip in Bending

		n = 1	n = 2	n = 3	n = 5	n = 7	n = 10
$\frac{a}{b} = \frac{1}{4}$	h_1	1.27	1.25	1.22	1.15	1.08	.957
	h_2	5.96	5.62	5.25	4.63	4.13	3.52
	h_3	.866	1.24	1.49	1.74	1.81	1.72
$\frac{a}{b} = \frac{1}{2}$	h_1	1.49	1.33	1.21	1.04	.906	.742
	h_2	5.12	3.86	3.20	2.47	2.04	1.61
	h_3	2.68	2.51	2.26	1.82	1.51	1.20
$\frac{a}{b} = \frac{3}{4}$	h_1	1.40	1.34	1.23	1.03	.909	.747
	h_2	4.38	3.22	2.65	2.03	1.69	1.36
	h_3	3.59	2.82	2.34	1.81	1.52	1.23

For the purpose of comparing plane strain with plane stress, values of h_1, h_2, and h_3 for $a/b = 1/2$ for plane stress taken from Ref. [3] are repeated here in Table 2. Equations (9)-(11) still apply, except that in plane stress

$$M_0 = 0.2679\,\sigma_0 c^2 \qquad (15)$$

Plots of h_1, h_2, and h_3 against $1/n$ for $a/b = 1/2$ are shown in Fig. 2. For h_2 and h_3 the plane stress and plane strain values are within a few percent of one another for all n. The plane strain values of h_1 are about 30% higher than the corresponding plane stress values. (The difference between h_1 at $n = 1$ in the two cases is due not only to the Poisson ratio ($\nu = 1/2$) effect on J, but also to the different normalizations of M using Eq. (8) or Eq. (15).)

Table 2 Plane Stress Strip in Bending

		n = 1	n = 2	n = 3	n = 5	n = 7	n = 10
$\frac{a}{b} = \frac{1}{2}$	h_1	1.104	0.97	0.851	0.717	0.653	0.551
	h_2	5.13	3.64	2.95	2.26	1.95	1.61
	h_3	2.75	2.36	2.03	1.59	1.37	1.12

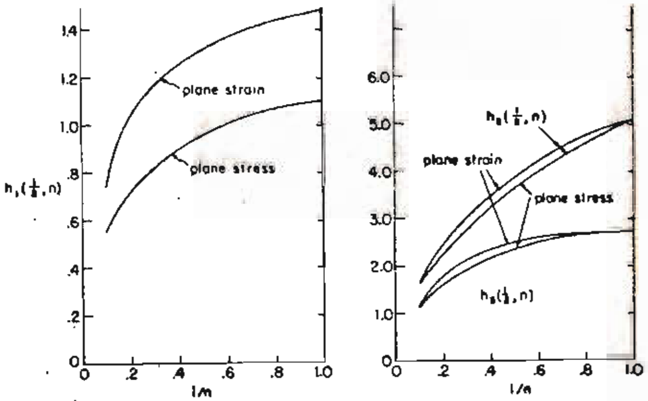

Fig. 2 Functions in Eqs. (9)-(11) for edge-cracked strip in bending for $a/b = 1/2$

A further check on the fully plastic plane strain results can be seen from the relation between δ and θ_c. For a rigid-perfectly plastic strip ($n = \infty$) the center of rotation of the slip line field is located at a distance 0.369c from the crack tip, as depicted in Fig. 1, so that

$$\frac{\delta}{a\theta_c} = 1 + 0.369\,\frac{c}{a} \tag{16}$$

Consider the ratio of δ to $a\theta_c$ from (10) and (11), i.e.,

$$\frac{\delta}{a\theta_c} = \frac{h_2}{h_3} \tag{17}$$

Using the values in Table 1 it is found that this ratio is within one or two percent of the limiting value (16) for all $n > 3$ for $a/b = 1/2$ and for $a/b = 3/4$.

CENTER-CRACKED STRIP IN TENSION

The calculations of Goldman and Hutchinson [1] are redone in this section. A strip of width 2b and length 2h with a centered crack of length 2a is considered,as shown in Fig. 3. A uniform stress distribution

$$\sigma_{yy}(x,\pm h) = P/(2b) \tag{18}$$

is imposed on the ends where P is the load per unit thickness carried by the strip. The load point displacement Δ is defined by

$$\Delta(h) = \int_{-b}^{b} [u_y(x,h) - u_y(x,-h)]dx \tag{19}$$

With

$$\Delta(h)_{\text{no crack}} = \sqrt{3}\alpha\varepsilon_0 h\left(\frac{\sqrt{3}\,P}{4b\sigma_0}\right)^n \tag{20}$$

as the load point displacement in the absence of a crack, we define Δ_c as

$$\Delta_c(h) = \Delta(h) - \Delta(h)_{\text{no crack}} \tag{21}$$

The calculations reported below were made with $h/b = 2$ and it is again expected that Δ_c can be regarded as being independent of h for $h/b \geq 2$.

The limit load per unit thickness P_0 for a cracked, perfectly plastic ($n = \infty$) strip is

$$P_0 = 4c\sigma_0/\sqrt{3} \tag{22}$$

Using the normalizations introduced in [1] and [3], we write

$$J = \alpha\sigma_0\varepsilon_0 a(c/b)g_1(a/b,n)(P/P_0)^{n+1} \tag{23}$$

$$\delta = \alpha\varepsilon_0 a\; g_2(a/b,n)(P/P_0)^n \tag{24}$$

$$\Delta_c = \alpha\varepsilon_0 a\; g_3(a/b,n)(P/P_0)^n \tag{25}$$

where now $\delta = u_y(0,0^+) - u_y(0,0^-)$ is the opening displacement at the center of the crack. Values of g_1 , g_2, and g_3 are given in Table 3.

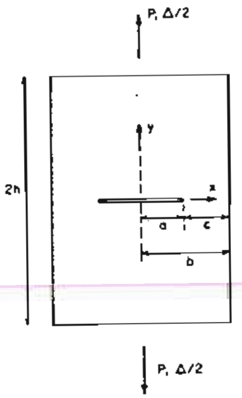

Fig. 3 Center-cracked strip in tension

Table 3 Plane Strain Strip in Tension

a/b		n = 1	n = 1.5	n = 2	n = 3	n = 5	n = 7	n = 10
$\frac{a}{b} = \frac{1}{4}$	g_1	2.57	2.90	3.11	3.35	3.49	3.43	3.23
	g_2	2.79	2.99	3.09	3.14	3.00	2.79	2.47
	g_3	.548	.752	.942	1.27	1.73	1.97	2.07
$\frac{a}{b} = \frac{1}{2}$	g_1	2.19	2.25	2.27	2.18	1.93	1.71	1.44
	g_2	2.09	1.99	1.87	1.61	1.23	.996	.776
	g_3	.798	.949	1.06	1.15	1.10	.959	.771
$\frac{a}{b} = \frac{3}{4}$	g_1	2.10	1.85	1.80	1.57	1.24	1.04	.799
	g_2	1.40	1.08	.899	.637	.401	.295	.204
	g_3	.814	.763	.733	.593	.399	.294	.204

These values differ by as much as 20% from those in Ref. [1] in some instances. Parks [4] gave results for the case $a/b = 1/2$ for n-values in the range $1 \leq n \leq 3$. He also noted that the results in Ref. [1] for J were substantially higher than those he determined. The results for h_1 in Table 3 for $a/b = 1/2$ and $n = 2$ and 3 are within 5% of the appropriately converted values given by Parks. The results of Ranaweera and Leckie [5] are also in reasonable agreement with the results of Table 3.[1]

In the deeply-cracked limit, $c/b \ll 1$, J can be expressed as [13]

$$J_{dc} = \frac{1}{2c}\left\{2\int_0^{\Delta_c} P d\Delta_c - P\Delta_c\right\} \quad (26)$$

Using Eq. (25) in Eq. (26) one can obtain

$$J_{dc} = \frac{2}{\sqrt{3}}\alpha\sigma_o\epsilon_o a\left(\frac{n-1}{n+1}\right)g_3(a/b,n)\left(\frac{P}{P_o}\right)^{n+1} \quad (27)$$

The ratio of this expression for J based on the deeply-cracked assumption and J from Eq. (23) is

$$\frac{J_{dc}}{J} = \frac{2}{\sqrt{3}}\left(\frac{b}{c}\right)\left(\frac{n-1}{n+1}\right)\frac{g_3}{g_1} \quad (28)$$

From the values in Table 3 this ratio is found to be within 3% of unity for $n \geq 5$ when $a/b = 3/4$ and for $n \geq 7$ when $a/b = 1/2$. This internal consistency in the results attests to the numerical accuracy and gives some indication as to the values of a/b and n needed for the deeply-cracked limit to prevail.

In plane stress the formulas for J , δ, and Δ_c Eqs. (23)-(25), still pertain with

$$P_o = 2\sigma_o c \quad (29)$$

instead of Eq. (22). Values of g_1, g_2, and g_3 for plane stress taken from Ref. [3] are repeated here in Table 4 for the one case $a/b = 1/2$ (additional cases can be found in Ref. [3]). Plots of g_1, g_2, and g_3 against $1/n$ are shown in Fig. 4 for $a/b = 1/2$. The difference between these functions in plane strain and plane stress is even less than for the corresponding functions for the edge-cracked strip in bending.

Table 4 Plane Stress Strip in Tension

		n = 1	n = 1.5	n = 2	n = 3	n = 5	n = 7	n = 10
$\frac{a}{b} = \frac{1}{2}$	g_1	2.21	2.24	2.20	2.06	1.81	1.64	1.47
	g_2	2.38	2.18	2.00	1.70	1.31	1.08	.892
	g_3	.924	1.09	1.18	1.25	1.18	1.05	.888

[1] A promising stress-based method for solving this class of incompressible plane strain problem has also been developed (F. A. Leckie, private communication).

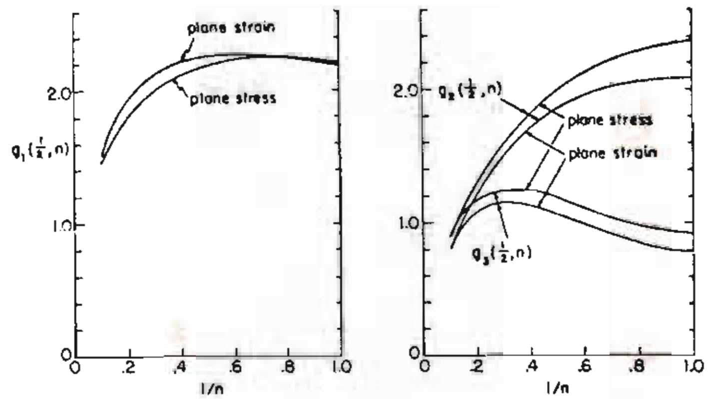

Fig. 4 Functions in Eqs. (23)-(25) for center-cracked strip in tension for $a/b = 1/2$

DOMINANCE OF THE CRACK TIP SINGULARITY FIELDS

The usual argument which is made to justify the use of J as a fracture parameter is based on the fact that J can be thought of as the amplitude of the crack tip singularity fields. For the power law material Eq. (2) the so-called HRR singularity fields [14, 15] are of the form

$$\sigma_{ij} = \sigma_o\left(\frac{J}{\alpha\sigma_o\epsilon_o I_n}\right)^{\frac{1}{n+1}}\left(r\right)^{-\frac{1}{n+1}}\tilde{\sigma}_{ij}(\theta,n) \quad (30)$$

$$\epsilon_{ij} = \alpha\epsilon_o\left(\frac{J}{\alpha\sigma_o\epsilon_o I_n}\right)^{\frac{n}{n+1}}\left(r\right)^{-\frac{n}{n+1}}\tilde{\epsilon}_{ij}(\theta,n) \quad (31)$$

where I_n is a numerical constant depending on n . Asymptotically as the crack tip is approached, the singularity fields become a better and better approximation to the actual stress and strain fields as long as $n \neq \infty$, i.e., some hardening is present, and as long as finite strain effects do not invalidate the small strain assumptions on which Eqs. (30) and (31) are based.

The size of the region over which the singularity fields Eqs. (30) and (31) dominate is of some importance in assessing the validity of a J-based approach. Recent work by McMeeking and Parks [16] has indicated that the size of the region dominated by the crack tip singularity fields is much larger in the plane strain bend specimen under fully yielded conditions than in the center-cracked strip under tension. Indeed, the zone of dominance in the center-cracked configuration is so small under fully yielded conditions that doubt would seem to be cast on the validity of using J for this configuration except for applications involving long cracks. Our study suggests the same conclusion. Figure 5 shows the tensile stress directly ahead of the crack, $\sigma_{yy}(r)$, as calculated for the strip in bending, for the center-cracked strip in tension, and from the singularity field (30), all for $n = 3$ and at the same value of J . Corresponding results for $n = 10$ are shown in Fig. 6. Because of the pure power stress-strain relation, the scale of the stress axis in these figures increases linearly with increasing applied load so that the relative position of the three curves remains fixed. It is evident that the region over which the tensile stress ahead of the crack is given with reasonable accuracy by the singularity field is significantly smaller in the center-cracked configuration and, in fact, does not appear to even exist when $n = 10$ to the extent that our numerical results can resolve the near-tip behavior. Amazigo [17] has recently closely analyzed his exact fully plastic solutions for antiplane shear [2] to discover the extent of dominance of the crack tip singularity fields. He finds a strong dependence on the hardening index n with the size of the dominant region decreasing rapidly for n-values above 10. As noted by Rice [18], a number of important questions connected with this issue, and with the significance of J , if any, independent of its role as the amplitude of the crack tip singularity fields, remain to be answered.

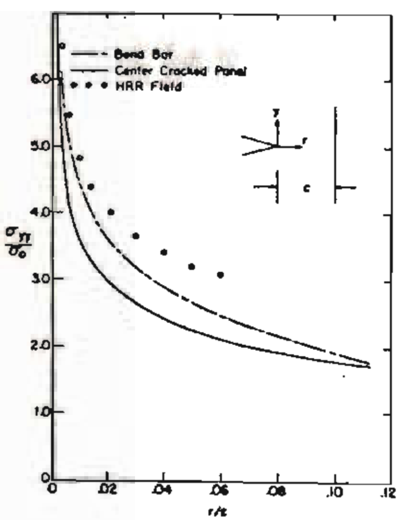

Fig. 5 Comparisons of stress ahead of crack tip with stress of HRR singularity field at same value of J for $n = 3$

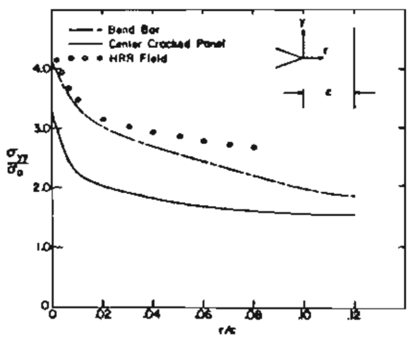

Fig. 6 Comparisons of stress ahead of crack tip with stress of HRR singularity field at same value of J for $n = 10$

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