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by

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It has been suggested that multilayered sheets, in which alternating layers have different elastic moduli, might lend themselves to *tailoring* to reduce, or even eliminate, harmful stress concentrations at holes or other stress raisers. Such tailoring could be implemented by making the sheet thickness spatially nonuniform, varying the *number* of layers, but keeping the layering pattern unchanged; or, keeping the total thickness unchanged, by varying the pattern of layer locations and thicknesses; or by a combination of these two approaches. We will call the first method "thickness tailoring", and the second "modulus tailoring". Tailored fabrication of such nonuniform layered sheets seems particularly well suited to masked deposition techniques.

This note provides a preliminary analytical assessment of the theoretical feasibility of designing a tailored, layered sheet that would alleviate the stress concentration induced by a circular hole in a field of balanced biaxial tension (see Fig. 1). If the stress concentration is actually eliminated, the result is a so-called "neutral" hole. It should be emphasized at the outset that reducing the *average* circumferential stress at the boundary of the hole is definitely not necessarily the desired goal. As we shall see, if modulus tailoring with constant overall thickness is exploited, and only the relative volumes of the layer constituents are changed, the stresses within the individual layers can be reduced while the average stress goes up! (This seemingly paradoxical result takes a little getting used to; the reason it's right is that while the stress in the stiffer material drops, there is more of it, so the average rises.) Conversely, a misguided reduction of the average hole-boundary stress by means of modulus tailoring can lead to higher stress concentrations within the layers.

We consider a two-constituent layered sheet, with Young's moduli E_α (α=1,2) in the alternating layers, and for simplicity, we assume the same Poisson's ratio ν in each layer. The effective sheet modulus is $E=f_1E_1 + f_2E_2$, where the f's are volume fractions. At each r, denote the average radial and circumferential stresses by σ_r and σ_θ, and let $\sigma_r^{(\alpha)}$, $\sigma_\theta^{(\alpha)}$ (α=1,2) be the stresses in the layers. The stress-strain relations are

$$\varepsilon_r = \frac{\sigma_r^{(\alpha)} - \nu\sigma_\theta^{(\alpha)}}{E_\alpha} = \frac{\sigma_r - \nu\sigma_\theta}{E(r)}$$
$$\varepsilon_\theta = \frac{\sigma_\theta^{(\alpha)} - \nu\sigma_r^{(\alpha)}}{E_\alpha} = \frac{\sigma_\theta - \nu\sigma_r}{E(r)} \qquad (1)$$

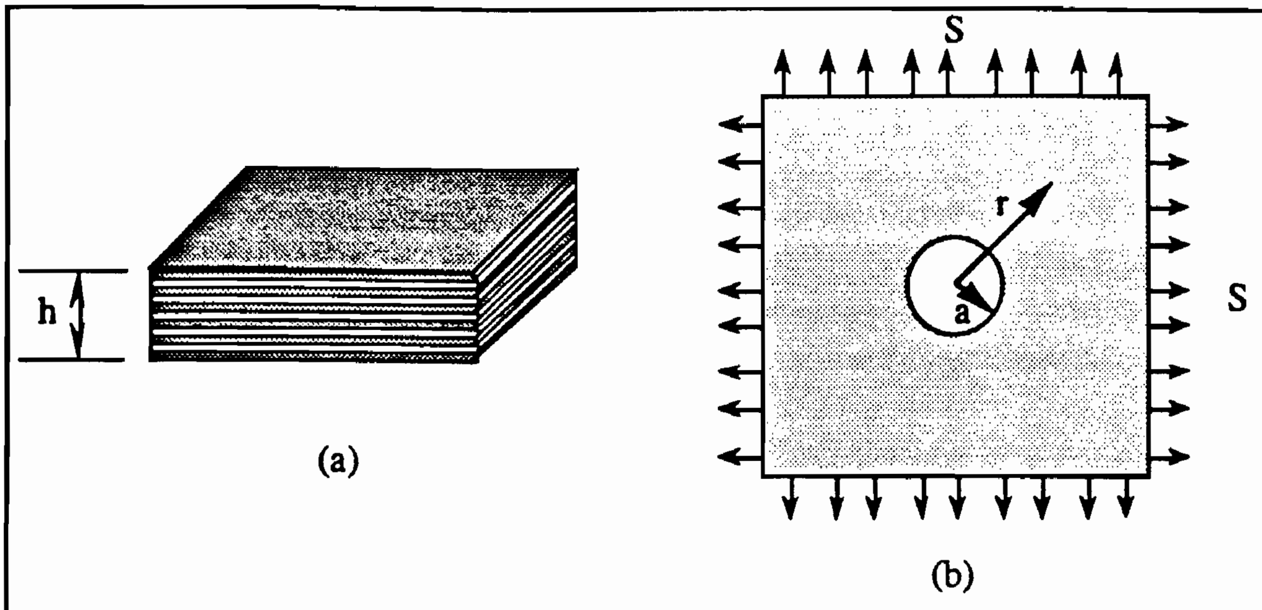

Fig. 1. (a) Layered sheet. (b) Hole in sheet under balanced biaxial tension.

Let

$$\sigma_r = \frac{E(r)}{E(\infty)} s_r, \quad \sigma_\theta = \frac{E(r)}{E(\infty)} s_\theta \tag{2}$$

where $E(\infty)$ is the untailored sheet modulus far from the hole. Then

$$\sigma_r^{(\alpha)} = \frac{E_\alpha}{E(\infty)} s_r, \quad \sigma_\theta^{(\alpha)} = \frac{E_\alpha}{E(\infty)} s_\theta \tag{3}$$

and so the layer stresses are proportional to s_r and s_θ. Hence, it is the value of s_θ at r=a that we must seek to lower by tailoring E(r), or the sheet thickness h(r), or both. Note that while the stress concentration factor (SCF) for the *average* sheet stress σ_θ is $\sigma_\theta(a)/S$, the *layer* concentration factors are

$$\frac{\sigma_\theta^{(\alpha)}(a)}{\sigma_\theta^{(\alpha)}(\infty)} = \frac{s_\theta(a)}{s_\theta(\infty)} = \frac{s_\theta(a)}{S} \tag{4}$$

For a *uniform* layered sheet, these layer concentration factors are equal to the classical stress concentration factor $\sigma_\theta(a)/S = 2$.

The equations of equilibrium and compatibility are

$$\frac{d(rh\sigma_r)}{dr} = h\sigma_\theta \tag{5}$$

and

$$\frac{d(r\varepsilon_\theta)}{dr} = \varepsilon_r \tag{6}$$

respectively. These may be rewritten as

$$[\lambda \rho s_r]' = \lambda s_\theta \tag{7}$$

$$[\rho(s_\theta - \nu s_r)]' = s_r - \nu s_\theta \tag{8}$$

in terms of $\rho \equiv r/a$, and the *tailoring function* defined by

$$\lambda(r) \equiv \frac{E(r)}{E(\infty)}\frac{h(r)}{h(\infty)} \tag{9}$$

Primes denote derivatives with respect to ρ.

We proceed in a semi-inverse fashion by asserting the spatial distribution

$$s_r = S\left(1-\rho^{-n}\right) \tag{10}$$

and solving the compatibility equation (8) for s_θ to get

$$s_\theta = S\left[1+\frac{1-\nu(n-1)}{(n-1-\nu)\rho^n}-\frac{C}{\rho^{(1+\nu)}}\right] \tag{11}$$

where C is a constant. The only value of C that leads to a bounded tailoring function is

$$C=\frac{2n-n^2}{n-1-\nu} \tag{12}$$

and this gives the *layer* stress concentration factor $s_\theta/S=n$ at $\rho=1$. The tailoring formula

$$\lambda(r)=\exp\left[\frac{n(2-n)}{n-1-\nu}\int_0^{a/r}\frac{x^\nu-x^{n-1}}{1-x^n}dx\right] \tag{13}$$

follows from the equilibrium equation (7). In all cases the peak value of $\lambda(r)$, as expected, occurs at $r=a$, and is given by

$$\begin{aligned}\lambda(a)&=\exp\left[\frac{n(2-n)}{n-1-\nu}\int_0^1\frac{x^\nu-x^{n-1}}{1-x^n}dx\right] \quad (n\neq 1+\nu)\\ &=\exp\left[-(1-\nu^2)\int_0^1\frac{x^\nu\log x}{1-x^n}dx\right] \quad (n=1+\nu)\end{aligned} \tag{14}$$

For $\nu=0$ this last result equals $\exp(\pi^2/6)$.

Fig. 2 shows how the peak tailoring magnitude varies with the layer stress concentration factor n, for several values of ν. We remark that if only thickness tailoring is used, the SCF for average stress is the same as that for the layers, and so is also reduced below 2. But for pure modulus tailoring, the SCF for the average stress is given by $n\lambda(a)$, and this always exceeds 2 for $n<2$.

To get a neutral hole, we set n=1 in the formula for $\lambda(r)$, and find

$$\lambda_{neutral}(r)=\exp\left[\frac{1}{\nu}\int_0^{a/r}\frac{1-x^\nu}{1-x}dx\right] \tag{15}$$

For $\nu=0$, this result becomes

$$\lambda_{neutral}(r)=\exp\left[-\int_0^{a/r}\frac{\log x}{1-x}dx\right] \quad (\nu=0) \tag{16}$$

Fig. 3 shows how $\lambda_{neutral}$ varies with r/a for $\nu=0, 1/4$, and $1/2$.

We should check the values of $\sigma_\theta^{(\alpha)}(r)/\sigma_\theta^{(\alpha)}(\infty)=s_\theta(r)/S$ away from the hole. In the case of the neutral hole, we find

$$s_\theta / S = 1 - \left(\rho^{-1} - \rho^{-(1+\nu)}\right)/\nu \quad (\nu \neq 0)$$
$$= 1 - \rho^{-1} \log\rho \qquad (\nu = 0) \tag{17}$$

and so the peak layer stress does indeed occur at the hole.

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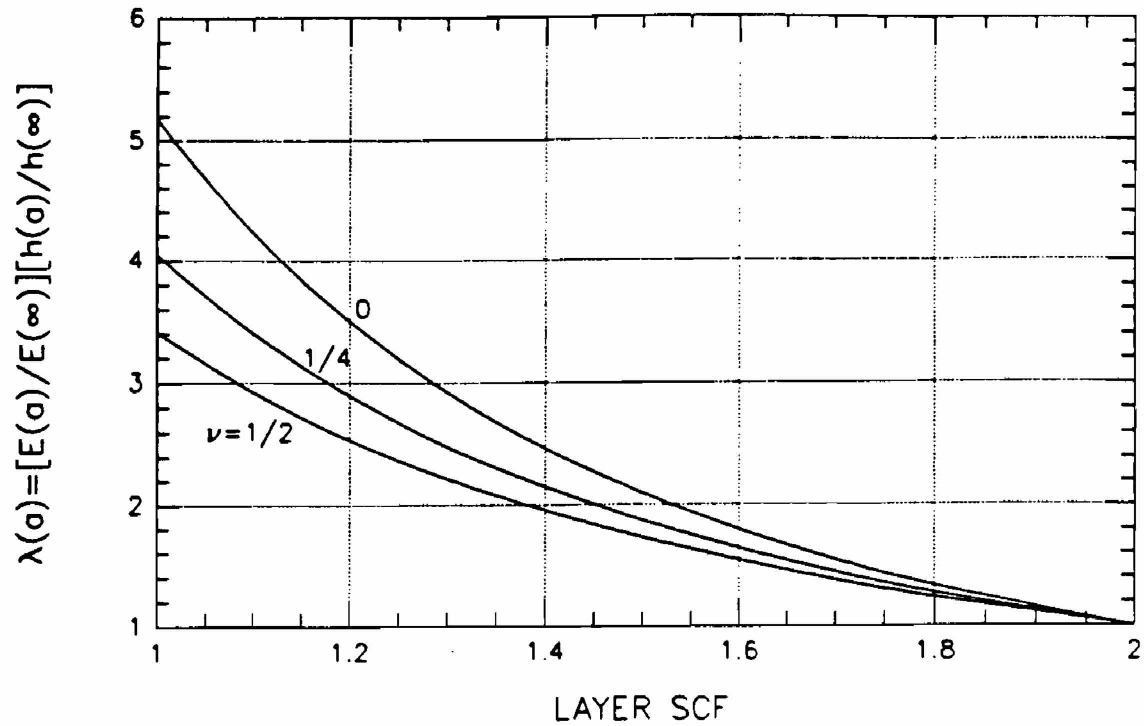

Fig. 2. Tailoring function at hole boundary vs. layer stress concentration factor.

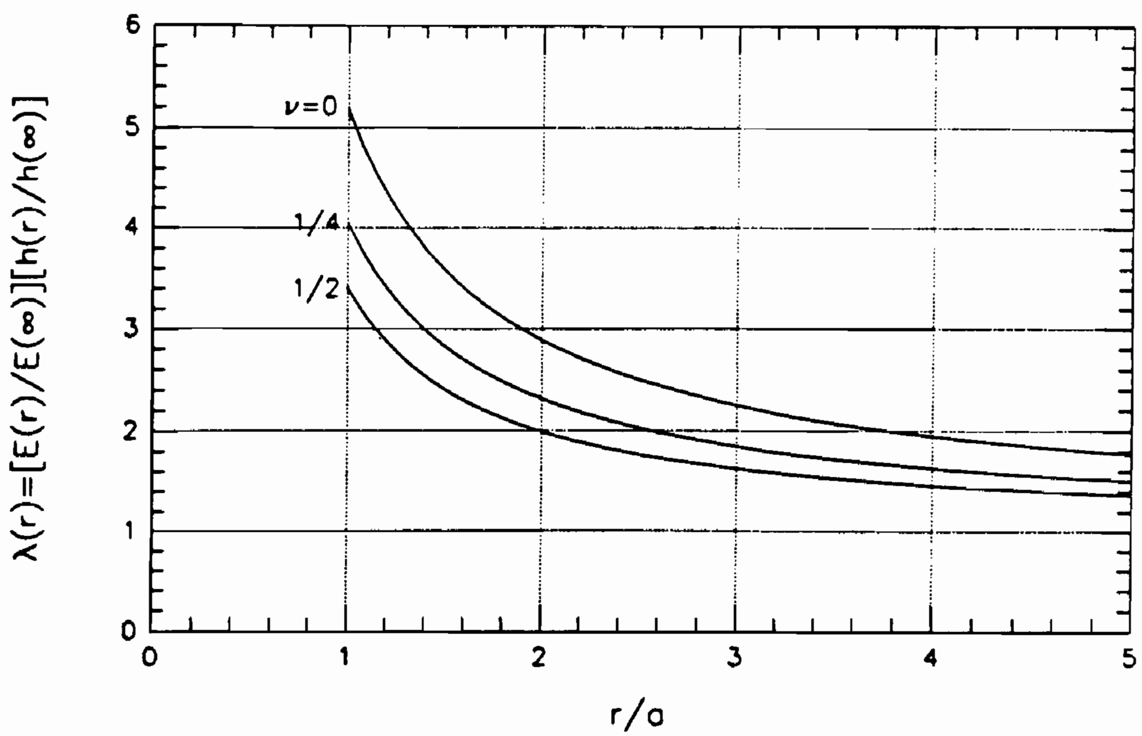

Fig. 3. Spatial variation of tailoring function for a neutral hole.