# BASIC STRAIN GRADIENT PLASTICITY THEORIES WITH APPLICATION TO CONSTRAINED FILM DEFORMATION[1]

Christian F. Niordson* and John W. Hutchinson**

*Technical University of Denmark, Lyngby, Denmark

**Harvard University, Cambridge, MA, USA

## Abstract

A family of basic rate-independent strain gradient plasticity theories is considered that generalize conventional $J_2$ deformation and flow theories of plasticity to include a dependence on strain gradients in a simple way. The theory builds on three recent developments: (i) the work of Gudmundson (2004) and Gurtin & Anand (2009) proposing constitutive relations for flow theories consistent with requirements of positive plastic dissipation; (ii) the work of Fleck & Willis (2009a,b) who clarified the structure of the new flow theories and presented the underlying variational formulation; and (iii) observations of Evans & Hutchinson (2009) related to preferences for specific functional compositions of strains and strain gradients. The starting point in this paper is the deformation theory formulation of Fleck & Hutchinson (2001) which provides the clearest insights into the role of strain gradients and serves as a template for the flow (incremental) theory. The flow theory is constructed such that it coincides with the deformation theory under proportional straining, analogous to the corresponding coincidence in the conventional $J_2$ theories. The generality of proportional straining is demonstrated for pure power-law materials, and the utility of power-law solutions is illustrated for the constrained deformation of thin films: the compression or extension of a finite layer joining rigid platens. Full elastic-plastic solutions are obtained for the same problem based on a finite element method devised for the new class of flow theories. Potential difficulties and open issues associated with the new class of flow theories are identified and discussed.

*Keywords:* Plasticity, strain gradient plasticity, size effects, metallic bonding layers

---

[1] This paper is dedicated to Charles and Marie-Louise Steele for their exceptional contributions to publication in the field of solids and structures, and, particularly, for founding this journal.

## 1. Introduction

Gudmundson (2004) and Gurtin & Anand (2009) have called attention to the fact that the phenomenological flow (incremental) strain gradient plasticity theory of Fleck & Hutchinson (2001) does not guarantee that plastic dissipation will be positive for all straining histories. The advantage of the Fleck-Hutchinson theory of 2001 is that it generalizes the most widely used conventional plasticity theories, $J_2$ deformation and flow theories, in a straightforward way that can be readily implemented numerically. In its simplest form, the theory introduces only a single new material length parameter with no other inputs than those of conventional $J_2$ theory. By invoking viscous plasticity, Gudmundson (2004) and Gurtin & Anand (2009) suggest ways that the flow can be modified such that the thermodynamic requirement that plastic dissipation be non-negative is met. Following up on the work of these authors, Fleck & Willis (2009a,b) have proposed specific forms of the constitutive law for both viscous and rate-independent materials. The simplest version proposed by Fleck and Willis: (i) coincides with the original theory of Fleck & Hutchinson (2001) under proportional straining, and (ii) again introduces only a single length parameter. In addition, Fleck & Willis (2009a,b) call attention to the fact that the new form of the constitutive relation for the rate-independent flow theory has close parallels to aspects of conventional rigid plasticity in the sense that some of the higher order stress quantities are not determined by prior history but depend on the current strain rates. Most importantly, Fleck and Willis establish variational principles for the new flow theory formulations.

In this paper, a unified family of small strain, rate-independent, gradient plasticity theories will adopted from Fleck & Willis (2009a,b) which meet the thermodynamic requirements detailed by Gurtin & Anand (2009). Consideration of the extended family is motivated by a recent assessment of basic aspects of strain gradient theories by Evans & Hutchinson (2009) who examined the manner in which plastic strain gradients are incorporated into the theories in light of experimental trends. Specifically, with $\varepsilon_P$ as a measure of the amplitude of the plastic strain and $\varepsilon_P^*$ as a positive measure of the plastic strain gradients, Evans and Hutchinson assessed the family of generalized effective plastic strains defined by

$$E_P = \left( \left( \varepsilon_P \right)^\mu + \left( \ell \varepsilon_P^* \right)^\mu \right)^{1/\mu} \tag{1.1}$$

as introduced originally by Fleck & Hutchinson (1997). The material length parameter, $\ell$, arises due to dimensional consistency.

The composition with $\mu = 2$,

$$E_P = \sqrt{\left( \varepsilon_P \right)^2 + \left( \ell \varepsilon_P^* \right)^2}\ , \tag{1.2}$$

has generally been preferred, primarily for mathematical reasons, as by Fleck & Hutchinson (2001) and also in the new formulation of Fleck & Willis (2009a). A quadratic dependence on $\ell$ is also implicit in the various formulations of Gudmundson (2004), Fredriksson & Gudmundson (2005) and Gurtin & Anand (2009). The argument for the linear composition ($\mu = 1$),

$$E_P = \varepsilon_P + \ell \varepsilon_P^*, \tag{1.3}$$

rests on the fact that it leads to gradient effects proportional to $\ell / h$ for a sequence of objects of increasing size $h$ as the conventional limit is approached. This trend in experimental indentation hardness data has been noted for many metals, and it has been invoked by Nix & Gao (1998) in their constitutive equation. By contrast, compositions depending quadratically on $\ell \varepsilon_P^*$, as in (1.2), predict gradient effects proportional to $(\ell / h)^2$ as the conventional limit is approached for many problems such as indentation and would appear unable to capture experimental trends as well as the linear composition as the conventional limit is approached.

This paper takes as its starting point the postulation of the deformation theory of plasticity (i.e., effectively, a strain gradient theory of small strain nonlinear elasticity) as proposed by Fleck & Hutchinson (2001). Then, following Fleck and Willis, the flow theory is constructed to coincide with the deformation theory for proportional straining. This approach unifies the theories, as in the conventional $J_2$ theories, and it illuminates the role gradients play in the clearest possible manner. In addition, it provides a simpler framework for solving problems when use of deformation theory is justified, and it underpins theoretical developments such as the J-integral of crack mechanics which rely on the existence of a deformation theory. The objective here continues to be a theory generalizing conventional rate-independent $J_2$ theory in the simplest meaningful way building, in particular, on the earlier work of Fleck & Hutchinson (2001) and the more recent developments in Fleck & Willis (2009a,b). As in the work of Idiart et al. (2009), the homogeneous composition of the generalized effective plastic strain in (1.1) will be considered with $\mu$ regarded as having a fixed value based on mechanistic considerations such as those discussed by Evans & Hutchinson (2009). A parallel approach developed for an alternative effective strain composition favored by Nix & Gao (1998) is given in the Appendix.

Following introduction of the underlying theory, application to pure power-law materials is discussed. Deformation theory and flow theory solutions for these materials coincide because proportional straining is precisely satisfied. Compression or separation of a finite length thin film bonding rigid platens is analyzed for the insights it provides into strain gradient effects on thin metallic bonding layers and as a possible test configuration for obtaining material data. Deformation and flow theory solutions will be generated, illustrating their correspondence and revealing the role of the material length parameter and higher order boundary conditions. In solving specific problems, some seemingly anomalous predictions have been found for the flow theory. These issues are addressed at the end of the paper.

## 2 Preliminaries

The paper lays out a small strain, isotropic, rate-independent phenomenological theory. The notation and theoretical framework is similar to that in Fleck & Hutchinson (2001). The conventional Cauchy stress is denoted by $\sigma_{ij}$, its deviator stress by $s_{ij}$, the total strain by $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ with $u_i$ as the displacement, and the elastic strain by $\varepsilon_{ij}^e = \varepsilon_{ij} - \varepsilon_{ij}^P$ with $\varepsilon_{ij}^P$ as the plastic strain. The conventional effective stress is defined by $\sigma_e = \sqrt{3s_{ij}s_{ij}/2}$, and a dimensionless deviator tensor co-directional to the deviator stress is defined by $m_{ij} = (3/2)s_{ij}/\sigma_e$. The stress is given by $\sigma_{ij} = \mathcal{L}_{ijkl}\varepsilon_{kl}^e$, with isotropic elastic moduli determined by the Young's modulus, $E$, and Poisson's ratio, $\nu$. The other input to conventional $J_2$ theory is the uniaxial tensile stress-strain curve. Denote the relation between the stress and the plastic strain in uniaxial tension by $\sigma(\varepsilon_P)$. Furthermore, denote the plastic work density to deform a material element in uniaxial tension to plastic strain $\varepsilon_P$ by

$$U_P(\varepsilon_P) = \int_0^{\varepsilon_P} \sigma(\varepsilon_P) d\varepsilon_P \tag{2.1}$$

**(a) Measures of plastic strain and plastic strain gradient in the deformation theories**

With $\varepsilon_{ij}^P = \varepsilon_{ij} - \varepsilon_{ij}^e$ and $\varepsilon_P = \sqrt{2\varepsilon_{ij}^P \varepsilon_{ij}^P / 3}$ as the plastic strain amplitude, the plastic strain is required to be co-directional with the stress deviator such that $\varepsilon_{ij}^P = \varepsilon_P m_{ij}$. The *one-parameter* measure of plastic strain gradient,

$$\varepsilon_P^* = \sqrt{\varepsilon_{P,i}\varepsilon_{P,i}}\ , \tag{2.2}$$

is the same as that employed by Aifantis (1984) and Muhlhaus & Aifantis (1991). More general measures based on the three quadratic invariants of the plastic strain gradient, $\varepsilon_{ij,k}^P = \varepsilon_{P,k} m_{ij} + \varepsilon_P m_{ij,k}$, and involving three length parameters, $\ell_i \ \ (i = 1,3)$ have been considered by Fleck and Hutchinson (2001) and Fleck and Willis (2009a), but here attention is confined to the one-parameter measure (2.2) which will be referred to as the *scalar measure* following the terminology of Fleck and Willis. A one-parameter *tensor measure*, in the Fleck-Willis terminology, is

$$\varepsilon_P^* = \sqrt{2\varepsilon_{ij,k}^P \varepsilon_{ij,k}^P / 3} \tag{2.3}$$

The definition in (2.3) is such that it coincides with (2.2) for a shearing with only one non-zero gradient, e.g., $\varepsilon_{12,2}^P = \varepsilon_{21,2}^P$. For other strain gradients the two measures will generally not coincide. For both (2.2) and (2.3), the generalized effective plastic strain, $E_P$ is defined by (1.1).

The length parameter must be determined by experiment, as for other material properties. The value of the parameter depends on the specific theory, as discussed in some detail by Evans and Hutchinson (2009), and possibly on the specific experiment. The three-parameter versions of strain gradient plasticity have greater flexibility in accurately encompassing a wide array of problems, but that is not the main concern here.

**(b) Measures of plastic strain and plastic strain gradient in the flow theories**

The fundamental measures for flow theory are defined for increments (rates), otherwise, they are similar to those above. Specifically, with $\dot{\varepsilon}_{ij}^{P} = \dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^{e}$, $\dot{\varepsilon}_P = \sqrt{2\dot{\varepsilon}_{ij}^{P}\dot{\varepsilon}_{ij}^{P}/3}$ and with a plastic strain rate co-directional with the stress deviator, $\dot{\varepsilon}_{ij}^{P} = \dot{\varepsilon}_P m_{ij}$. For the one-parameter scalar version, the gradient rate measure is

$$\dot{\varepsilon}_P^* = \sqrt{\dot{\varepsilon}_{P,i}\dot{\varepsilon}_{P,i}} \tag{2.4}$$

and for the one-parameter tensor version it is

$$\dot{\varepsilon}_P^* = \sqrt{2\dot{\varepsilon}_{ij,k}^{P}\dot{\varepsilon}_{ij,k}^{P}/3} \tag{2.5}$$

The generalized effective plastic strain rate is taken as

$$\dot{E}_P = \left( \left( \dot{\varepsilon}_P \right)^{\mu} + \left( \ell \dot{\varepsilon}_P^* \right)^{\mu} \right)^{1/\mu} \tag{2.6}$$

Current values of the plastic strain measures are integrals over the history:

$$\varepsilon_P = \int \dot{\varepsilon}_P \quad \text{and} \quad E_P = \int \dot{E}_P \tag{2.7}$$

Throughout this paper, the notion of proportional straining requires that the plastic strain, $\varepsilon_{ij}^{P}$, and its gradient, $\varepsilon_{ij,k}^{P}$, change monotonically and proportionally. For proportional straining the definitions for the flow theories coincide with those for the respective deformation theories.

It is important to be cognizant of restrictions inherent to the effective strain measures $\varepsilon_P$ and $E_P$ in the flow theory. Both measures are non-decreasing since their rates are intrinsically positive. The theories proposed below are similar to conventional $J_2$ theory in that they take hardening and the yield surface to depend on the total plastic strain. The theory invokes an isotropic expansion of the yield surface which should not be expected to correctly reproduce histories involving significant stress and strain reversal. As with conventional $J_2$ theory, the strain gradient version proposed here is not intended for application to problems involving significant stress reversal without due consideration to yield surface shape changes.

### (c) The Principle of Virtual Work and the equilibrium equations

The principle of virtual work for the scalar theory is (Fleck & Hutchinson, 2001)

$$\int_V \left(\sigma_{ij}\left(\delta\varepsilon_{ij} - \delta\varepsilon_P m_{ij}\right) + Q\delta\varepsilon_P + \tau_i \delta\varepsilon_{P,i}\right) dV = \int_S \left(T_i \delta u_i + t\delta\varepsilon_P\right) dS \tag{2.8}$$

for all admissible $\delta u_i$ and $\delta\varepsilon_P$. The associated equilibrium equations are

$$\sigma_{ij,j} = 0 \;, \qquad \tau_{i,i} - Q + \sigma_e = 0 \tag{2.9}$$

and the boundary data pairs are (with $n_i$ as the surface normal)

$$\left(T_i = \sigma_{ij} n_j \,,\, u_i\right) \text{ and } \left(t = \tau_i n_i , \varepsilon_P\right) \tag{2.10}$$

In conventional theory, $Q = \sigma_e$, but in the higher order theory $Q$ and $\tau_i$ are new stress variables that are work conjugate to $\varepsilon_P$ and $\varepsilon_{P,i}$, respectively. With stress quantities replaced by their increments, (2.8)-(2.10) apply for incremental equilibrium.

The corresponding principle of virtual work for the tensor formulation is

$$\int_V \left(\sigma_{ij}\left(\delta\varepsilon_{ij} - \delta\varepsilon_{ij}^P\right) + Q_{ij}\delta\varepsilon_{ij}^P + \tau_{ijk}\delta\varepsilon_{ij,k}^P\right) dV = \int_S \left(T_i \delta u_i + t_{ij}\delta\varepsilon_{ij}^P\right) dS \tag{2.11}$$

with equilibrium equations

$$\sigma_{ij,j} = 0 \,, \;\; \tau_{iij,k} - Q_{ij} + s_{ij} = 0 \tag{2.12}$$

and boundary pairs

$$\left(T_i = \sigma_{ij} n_j \,,\, u_i\right), \left(t_{ij} = \tau_{ijk} n_k , \varepsilon_{ij}^P\right) \tag{2.13}$$

The reader is referred to Fleck & Willis (2009b) for complete details of the tensor version which parallel those for the scalar version. In what follows details will only be presented for the scalar version, but selected results for the tensor version will be given.

### (d) The role of the gradients of plastic strain in the theory

The physical interpretation underlying $E_P$ in (1.1) is that it represents the collective sum of the movement of statistically stored dislocations tied to $\varepsilon_P$ and of the geometrically necessary dislocations associated with $\varepsilon_P^*$. The specific composition of $E_P$ is phenomenological, but it is intended to measure the history of all the dislocation motion. It reduces to the conventional measure when the gradients are small. In conventional $J_2$ deformation theory the energy density of a material element following proportional straining is the sum of elastic and "plastic" parts according to $\frac{1}{2}\mathcal{L}_{ijkl}\varepsilon_{ij}^e\varepsilon_{kl}^e + U_P(\varepsilon_P)$ with $U_P$ given by (2.1). Following Fleck & Hutchinson (2001), the energy density for strain gradient deformation plasticity is taken as $\frac{1}{2}\mathcal{L}_{ijkl}\varepsilon_{ij}^{e}\varepsilon_{kl}^{e}+U_P(E_P)$. The replacement of $\varepsilon_P$ by $E_P$ in $U_P(\varepsilon_P)$ reveals the essence of the plastic strain gradient in this family of theories. In words, the plastic work needed to deform the material element in the presence of strain gradients under proportional straining as measured by $E_P$ in (1.1) is taken equal to that at the same strain, $\varepsilon_P = E_P$, in the absence of gradients, consistent with the notion that $E_P$ and $\varepsilon_P$ measure the dislocation motion under the two circumstances.

## 3. The scalar theory

### (a) Deformation theory

The potential energy functional for the deformation theory is

$$\Phi(\mathbf{u},\varepsilon_P)=\int_V\left(\frac{1}{2}\mathcal{L}_{ijkl}\varepsilon_{ij}^{e}\varepsilon_{kl}^{e}+U_P(E_P)\right)dV-\int_{S_T}\left(T_iu_i+t\varepsilon_P\right)dS \tag{3.1}$$

with $\varepsilon_{ij}^{e}=\varepsilon_{ij}-\varepsilon_P m_{ij}$ and where $T_i$ and $t$ are prescribed on $S_T$. A solution minimizes the functional with respect to admissible $\mathbf{u}$ and $\varepsilon_P$ assuming the tensile stress-strain curve, $\sigma(\varepsilon_P)$, defining $U_P$ in (2.1) is monotone. The higher order stresses are

$$Q=\frac{\partial U_P}{\partial \varepsilon_P}=\sigma(E_P)\frac{(\varepsilon_P)^{\mu-1}}{(E_P)^{\mu-1}} \quad \text{and} \quad \tau_i=\frac{\partial U_P}{\partial \varepsilon_{P,i}}=\sigma(E_P)\frac{\ell^2(\ell\varepsilon_P^{*})^{\mu-2}\varepsilon_{P,i}}{(E_P)^{\mu-1}} \tag{3.2}$$

One can easily show that the following generalized effective stress satisfies

$$\Sigma\equiv\left(Q^{\frac{\mu}{\mu-1}}+\left(\tau/\ell\right)^{\frac{\mu}{\mu-1}}\right)^{\frac{\mu-1}{\mu}}=\sigma(E_P) \tag{3.3}$$

with $\tau=\sqrt{\tau_i\tau_i}$ .

### (b) Flow theory

With definitions of rates and integrated quantities defined in *Section 2b,* the stress quantities $Q$ and $\tau_i$ are chosen such that they coincide with those of deformation theory in (3.2) when the straining is proportional, i.e.,

$$Q = \sigma(E_P)\frac{(\dot{\varepsilon}_P)^{\mu-1}}{(\dot{E}_P)^{\mu-1}} \quad \text{and} \quad \tau_i = \sigma(E_P)\frac{\ell^2(\ell\dot{\varepsilon}_P^*)^{\mu-2}\dot{\varepsilon}_{P,i}}{(\dot{E}_P)^{\mu-1}} \tag{3.4}$$

This choice guarantees positive plastic dissipation rate because, as shown by direct calculation,

$$Q\dot{\varepsilon}_P + \tau_{,i}\dot{\varepsilon}_{P,i} = \sigma(E_P)\dot{E}_P \tag{3.5}$$

The choice (3.4) is a special case of the wide range of constitutive possibilities outlined by Gudmundson (2004) and Gurtin & Anand (2009) and it is identical to that of Fleck & Willis (2009a) for $\mu = 2$ and to that of Idiart et al. (2009) for arbitrary $\mu$. As Fleck & Willis (2009a) have emphasized, (3.4) specifies the higher order stresses, $Q$ and $\tau_i$, in terms of the rate of the plastic strain and its gradient. In this respect it is akin to conventional rigid plasticity. One important consequence of (3.4) is that $Q$ and $\tau_i$ are not determined by the prior stress history; rather, they depend the strain-rates and thus on the boundary conditions of the incremental problem.

The fact that $Q$ and $\tau_i$ are not predetermined in the current state is seen more clearly when the higher order equilibrium equation in (2.9) is expressed in terms of the plastic strain rate using (3.4):

$$\left(\sigma(E_P)\frac{\ell(\ell\dot{\varepsilon}_P^*)^{\mu-2}\ell\dot{\varepsilon}_{P,i}}{(\dot{E}_P)^{\mu-1}}\right)_{,i} - \sigma(E_P)\frac{(\dot{\varepsilon}_P)^{\mu-1}}{(\dot{E}_P)^{\mu-1}} = -\sigma_e \tag{3.6}$$

The distributions of conventional effective stress, $\sigma_e$, and $E_P$ *are* predetermined by the prior history. Eq. (3.6) is a nonlinear, second order partial differential equation that is homogeneous of degree zero in $\dot{\varepsilon}_P$. The solution $\dot{\varepsilon}_P$ depends on the boundary conditions with prescribed $t = \tau_{,i}n_i = t^0$ on $S_T$ and $\dot{\varepsilon}_P$ on $S_u$. Neither these boundary conditions, nor even $S_T$ and $S_u$, need depend on prior history. Consequently, $Q$ and $\tau_i$ from (3.4) can undergo discontinuous changes when incremental boundary conditions are abruptly altered. However, when the boundary conditions for a sequence of incremental problems change continuously, as in the case for many problems of interest, $Q$ and $\tau_i$ may also vary continuously.

It is readily verified that (3.6) is the Euler equation associated with stationary

$$H(\dot{\varepsilon}^P) = \int_V \left( \sigma(E_P)\dot{E}_P - \sigma_e \dot{\varepsilon}_P \right) dV - \int_{S_T} t^0 \dot{\varepsilon}_P dS \tag{3.7}$$

with respect to all positive $\dot{\varepsilon}_P$ satisfying prescribed conditions on $S_u$. This is Minimum Principle I of Fleck & Willis (2009a). In each zone in which $\dot{\varepsilon}_P$ is non-zero, $\dot{\varepsilon}_P$ is fully determined by (3.6) if non-zero $\dot{\varepsilon}_P$ is prescribed on any portion of $S$. Otherwise, $\dot{\varepsilon}_P$ is determined to within a multiplicative constant within each non-overlapping yielded zone.

From (3.5) it follows that

$$\dot{Q}\dot{\varepsilon}_P + \dot{\tau}_{,i}\dot{\varepsilon}_{P,i} + Q\ddot{\varepsilon}_P + \tau_{,i}\ddot{\varepsilon}_{P,i} = h(E_P)\dot{E}_P{}^2 + \sigma(E_P)\ddot{E}_P \tag{3.8}$$

where $h(E_P) = d\sigma(E_P)/dE_P$. This equation holds even when $Q$ and $\tau_i$ change discontinuously, as described above. Based on (3.4) and on the derivatives of the plastic strain-rate quantities, one can show: $Q\ddot{\varepsilon}_P + \tau_{,i}\ddot{\varepsilon}_{P,i} = \sigma(E_P)\ddot{E}_P$. Consequently, whether $Q$ and $\tau_i$ change continuously or discontinuously,

$$\dot{Q}\dot{\varepsilon}_P + \dot{\tau}_{,i}\dot{\varepsilon}_{P,i} = h(E_P)\dot{E}_P{}^2 \tag{3.9}$$

The quadratic work terms for the incremental problem are therefore

$$\frac{1}{2}\left\{\dot{\sigma}_{ij}\dot{\varepsilon}^e_{ij} + \dot{Q}\dot{\varepsilon}_P + \dot{\tau}_{,i}\dot{\varepsilon}_{P,i}\right\} = \frac{1}{2}\left\{\mathcal{L}_{ijkl}\left(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_P m_{ij}\right)\left(\dot{\varepsilon}_{kl} - \dot{\varepsilon}_P m_{kl}\right) + h(E_P)\dot{E}_P{}^2\right\}$$

The quadratic work terms are combined with the prescribed incremental surface tractions to form the functional whose minimum with respect to admissible distributions $(\dot{u}_i, \dot{\varepsilon}_P)$ provides the solution:

$$J\left(\dot{u}_i, \dot{\varepsilon}_P\right) = \frac{1}{2}\int_V \left\{\mathcal{L}_{ijkl}\left(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_P m_{ij}\right)\left(\dot{\varepsilon}_{kl} - \dot{\varepsilon}_P m_{kl}\right) + h(E_P)\dot{E}_P{}^2\right\} dV - \int_{S_T} \left(\dot{T}_i\dot{u}_i + \dot{t}\dot{\varepsilon}_P\right) dS \tag{3.10}$$

with $\dot{T}_i$ and $\dot{t}$ prescribed on $S_T$ and with $\dot{u}_i$ and $\dot{\varepsilon}_P$ prescribed on $S_u$. If the complete distributions of $(\dot{u}_i, \dot{\varepsilon}_P)$ were unknown, as in the original version of Fleck & Hutchinson (2001), then the minimum of $J$ with respect to all admissible $(\dot{u}_i, \dot{\varepsilon}_P)$ produces the entire incremental solution. However, due to the special nature of the constitutive law (3.4) discussed above, $\dot{\varepsilon}_P$ is either known, or known to within a multiplicative constant, within each yielded region of the body. With $\dot{\varepsilon}_P$ constrained by the results Minimum Principle I, Fleck & Willis (2009a) have proved that the minimum of (3.10) delivers the unknown multiplicative constants of $\dot{\varepsilon}_P$ as well as $\dot{u}_i$ and $\dot{\sigma}_{ij}$. This is Minimum Principle II. Fleck and Willis also prove uniqueness and discuss bounding principles for the incremental problem. The incremental problem is fully characterized, apart from a condition for distinguishing loading and elastic unloading, as will be discussed shortly. The conventional stress is updated using $\sigma_{ij} \to \sigma_{ij} + \dot{\sigma}_{ij}$ in the standard manner, as is $E_P$ based on (2.7); these and the distribution of $m_{ij}$ are known entering the repetition of the solution process for the next incremental step.

Equation (3.4) implies the existence of a surface in terms of $Q$ and $\tau_i$ given explicitly by

$$\Sigma \equiv \left( Q^{\frac{\mu}{\mu-1}} + \left(\tau / \ell\right)^{\frac{\mu}{\mu-1}} \right)^{\frac{\mu-1}{\mu}} = \sigma(E_P) \quad \text{with} \quad \tau = \sqrt{\tau_i \tau_i} \tag{3.11}$$

For $\mu = 2$, this same surface, or ones analogous to it, are found in the theories of Fleck and Hutchinson (1997), Gudmundson (2004) and Fleck and Willis (2009a). The surface is plotted in Fig. 1 for five values of $\mu$, including the limit for $\mu$ =1 which has a corner. The plastic strain rate and its gradient are normal to the surface:

$$\dot{\varepsilon}_P = \dot{E}_P \frac{\partial \Sigma}{\partial Q} = \dot{E}_P \left(\frac{Q}{\sigma}\right)^{\frac{1}{\mu-1}}, \quad \dot{\varepsilon}_{P,i} = \dot{E}_P \frac{\partial \Sigma}{\partial \tau_i} = \ell^{-1} \dot{E}_P \left(\frac{\tau}{\ell \sigma}\right)^{\frac{1}{\mu-1}} \frac{\tau_i}{\tau} \tag{3.12}$$

These equations are equivalent to (3.4). Plastic loading requires $\dot{\varepsilon}_P \geq 0$. From (3.11) it follows that for any plastic loading increment $\dot{\Sigma} = h(E_P)\dot{E}_P \geq 0$ because $\dot{E}_P$ is intrinsically non-negative and material softening is not considered, i.e. $h(E_P) \geq 0$. In the present theory, the surface (3.11) is not a yield surface in the conventional sense because any field $\dot{\varepsilon}_P > 0$ satisfying (3.4) always produces $Q$ and $\tau_i$ that lie on the surface.

[Place Fig. 1 around here]

While this formulation guarantees positive plastic work, no attempt has been made here to identify whether the contributions of the gradients of plastic strains to the plastic work are energetic or dissipative in the terminology of Gurtin and Anand (2009). In this respect, the situation is analogous to conventional $J_2$ flow theory where the fractional proportions of the plastic work dissipated as heat and stored as the elastic energy of the dislocations are largely irrelevant to the formulation and, indeed, can differ from material to material without being reflected in the theory. It was noted earlier that the flow theories laid out here with isotropic hardening are not usually expected to be good models for histories involving reversed plastic straining due to disregard of Bauschinger effects. Efforts to associate energetic contributions with a component of kinematic hardening and dissipative contributions with the isotropic component of hardening, as some have attempted, fall outside the aims of this paper.

## 4. The tensor theory

The deformation and flow theory versions employing the measures of strain and strain-rate based on the tensor theory are analogous to those laid out above for the scalar version and they can be found in Fleck & Willis (2009b). One major difference between the two versions concerns the manner in which plastic flow is constrained in the scalar version contrasted with the tensor version. In the scalar version the plastic strain increment is constrained to be proportional to the deviator of the Cauchy stress, $s_{ij}$, which is known (and fixed) in the current state. The plastic strain increment is not directly coupled to the Cauchy stress in the tensor version. In the tensor version, the direction of the plastic strain increment, $\dot{\varepsilon}_{ij}^{P}$, is not constrained by any pre-determined quantity in the current state. Instead, $Q_{ij}$ is co-directional to $\dot{\varepsilon}_{ij}^{P}$ and $\tau_{ijk}$ is co-directional to $\dot{\varepsilon}_{ij,k}^{P}$, and both of the higher order stresses are not known in the current state but are determined only by the solution to the incremental problem. Implications of this difference will emerge in the numerical solutions to the constrained film problem.

The *deformation theory* for the tensor version is still defined by the functional $\Phi$ in (3.1) with $\varepsilon_P = \sqrt{2\varepsilon_{ij}^{P}\varepsilon_{ij}^{P}/3}$ and $E_P$ defined in (1.1). Now, however, $\varepsilon_P^*$ is defined by (2.5). For proportional plastic straining the *flow theory* again coincides with the deformation theory.

## 5. Pure power-law solutions

A pure power-law material has uniaxial behavior specified by $\sigma(\varepsilon_P) = \sigma_0 \varepsilon_P{}^N$ with plastic work, $U_P(\varepsilon_P) = \sigma_0 \varepsilon_P{}^{N+1}/(N+1)$. Pure power-law solutions to the deformation theory neglect elasticity such that (3.1) becomes

$$\Phi(\mathbf{u}, \varepsilon_P) = \int_V U_P(E_P)\, dV - \int_{S_T} \left(T_i u_i + t\varepsilon_P\right) dS \qquad (5.1)$$

Any solution, $\varepsilon_P$, minimizes $\Phi$ subject to $\varepsilon_{ij}^P = (u_{i,j} + u_{j,i})/2$ (with $u_{j,j} = 0$) and satisfaction of conditions on $S_u$.

Consider *traction-prescribed* boundary value problems for the scalar theory with load parameter $\lambda$ such that $\left(T_i, t\right) = \lambda\left(T_i^0, t^0\right)$ on $S_T$ with $\left(T_i^0, t^0\right)$ as fixed spatial distributions and $\left(u_i, \varepsilon_P\right) = 0$ on $S_u$. It is straightforward to show that solutions to the scalar theory have the form

$$\left(u_{,i}, \varepsilon_P\right) = \lambda^{1/N}\left(u_{,i}^0, \varepsilon_P^0\right), \ \left(s_{ij}, Q, \tau_i\right) = \lambda\left(s_{ij}^0, Q^0, \tau_i^0\right) \qquad (5.2)$$

where the quantities with superscript “0” are functions of position but independent of $\lambda$. Similarly, for *displacement-prescribed* boundary value problems with $\left(u_i, \varepsilon_P\right) = \lambda\left(u_i^0, \varepsilon_P^0\right)$ on $S_u$ and $\left(T_i, t\right) = 0$ on $S_T$, the form of the solution is

$$\left(u_{,i}, \varepsilon_P\right) = \lambda\left(u_{,i}^0, \varepsilon_P^0\right), \ \left(s_{ij}, Q, \tau_i\right) = \lambda^N\left(s_{ij}^0, Q^0, \tau_i^0\right) \qquad (5.3)$$

For both types of boundary conditions, proportional straining occurs at every point in the body if $\lambda$ is increased monotonically. Consequently, any deformation theory solution is a solution to the corresponding flow theory. This is the extension of Ilyushin’s Theorem to strain gradient plasticity. It holds for the tensor version as well. It is worth noting that any such solution is also a solution to the flow theory proposed by Fleck & Hutchinson (2001); the possibility of negative plastic dissipation is not an issue for such solutions.

## 6. Plane strain compression or separation of thin films—pure-power-law solutions

A slab of thin film is depicted in the insert in Fig. 2 with height, $h$, and width, $L$. The slab is bonded to rigid platens on the top and bottom; the lower platen is fixed while the vertical displacement of the upper slab is denoted by $\Delta$. The sides ($x_1 = \pm L/2$) are traction-free. It is well known that the bonding constraint of the platens gives rise to high stress triaxiality in the central region of the film accompanied by horizontal flow to accommodate to the relative platen motion. The velocity of material elements paralleling the interfaces is approximately parabolic producing a linear variation of shear strain across the layer. Thus, as in film bending or wire torsion, compression or separation of the film intrinsically produces gradients of strain. Conventional theory for a rigid-perfectly plastic solid ( $N = 0$ ) predicts that the magnitude of the average normal stress, $\bar{n}$ , to yield a constrained layer in tension or compression increases with aspect ratio of the film according to (Hill, 1950)

$$\frac{\bar{n}}{\sigma_Y} \cong 1 + \frac{1}{2\sqrt{3}} \frac{L}{h} \tag{6.1}$$

where $\bar{n} = P/L$ with $P$ as the total force/depth. The purpose of this section is to illuminate how plasticity size effects related to the film thickness further elevate the average normal stress. [Place Fig. 2 around here]

Compression or separation of a slab of metal film that is well bonded to stiff platens may be a useful configuration for the experimental determination of material length scales and/or for assessing higher order boundary conditions, especially if companion test data are available for the slab in shear. Moreover, the effective yield strength of the metal layer under normal separation exhibits a thickness effect that has direct relevance to metallic bonding. Plane strain compression or separation of a rectangular slab of film also illustrates the simplicity and utility of power-law solutions. Full elastic-plastic solutions based on the flow theory will be presented in *Section 7* to bring out the interaction between elasticity and plasticity.

Two boundary conditions will be considered on the surfaces where the slab is bonded to the platens: (i) *unconstrained plastic flow* with no constraint on $\varepsilon_P$ (or on $\varepsilon_{12}^P$ in the tensor version), and (ii) *constrained plastic flow* with $\varepsilon_P = 0$ (or $\varepsilon_{12}^P = 0$ in the tensor version). The second boundary condition models dislocations blocked at a well bonded interface joining a metal to a stiff elastic solid. In a continuum description, the plastic strain must vanish at the bonded surface if slip planes parallel to the surface are not available. Conditions intermediate to these limiting conditions can be defined (e.g.

Fleck and Willis, 2009a) to model internal surfaces between different materials or other types of boundaries, but these will not be pursued here.

The uniaxial tensile response of the metal is taken as $\sigma(\varepsilon_P) = \sigma_0 \varepsilon_P{}^N$ or, equivalently, as $\sigma(\varepsilon_P) = \sigma_Y (\varepsilon_P / \varepsilon_Y)^N$ with $\sigma_0 = \sigma_Y / \varepsilon_Y{}^N$. The plastic work is $U_P(\varepsilon_P) = \sigma_Y \varepsilon_Y (\varepsilon_P / \varepsilon_Y)^{N+1} /(N+1)$. Let $x = x_1 / h$ and $y = x_2 / h$, and consider the displacement field

$$
\begin{aligned}
u_1 / h &= -c_1 x\left(y(1-y)\right) - c_2 x\left(y(1-y)\right)^2 - c_3 x\left(y(1-y)\right)^3 \\
u_2 / h &= c_1\left(\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right) + c_2\left(\tfrac{1}{3}y^3 - \tfrac{1}{2}y^4 + \tfrac{1}{5}y^5\right) \\
&\quad + c_3\left(\tfrac{1}{4}y^4 - \tfrac{3}{5}y^5 + \tfrac{1}{2}y^6 - \tfrac{1}{7}y^7\right)
\end{aligned}
\tag{6.2}
$$

with

$$\Delta / h = u_2(h)/h = \tfrac{1}{6}c_1 + \tfrac{1}{30}c_2 + \tfrac{1}{140}c_3 \tag{6.3}$$

With $\{c_i\}$ as free coefficients, the field generates an admissible plastic strain field, $\varepsilon_{ij}^P$, that satisfies $u_{i,i} = 0$ and the boundary conditions on the top and bottom surfaces, subject to the additional condition that $c_1 = 0$ for the constrained flow condition (ii). The effective plastic strain is $\varepsilon_P = \sqrt{4\left(\varepsilon_{11}^{P\,2} + \varepsilon_{12}^{P\,2}\right)/3}$ and the non-zero strain gradients from (6.2) are $\varepsilon_{12,2}^P$ and $\varepsilon_{11,2}^P = 2\varepsilon_{12,1}^P$. For the gradient measure for the scalar version, (2.2),

$$\varepsilon_P^{*2} = \frac{4}{9\varepsilon_P{}^2}\left(\varepsilon_{12}^{P\,2}\varepsilon_{11,2}^P{}^2 + 4\left(\varepsilon_{11}^P\varepsilon_{11,2}^P + \varepsilon_{12}^P\varepsilon_{12,2}^P\right)^2\right) \tag{6.4}$$

while the measure for the tensor version, (2.3), is

$$\varepsilon_P^{*2} = \frac{4}{3}\left(\varepsilon_{11,2}^P{}^2 + \varepsilon_{12,2}^P{}^2\right) \tag{6.5}$$

The solution process is as follows. Using (6.4) or (6.5) together with the generalized effective strain, $E_P$, in (1.1), one obtains an expression for the energy functional, $\Phi$, in (5.1) (the contribution from $S_T$ vanishes). It depends on the free parameters, $\{c_i\}$, and on the parameters prescribing the problem according to

$$\Phi = \sigma_Y \varepsilon_Y \frac{hL}{N+1}\left(\frac{1}{\varepsilon_Y}\frac{\Delta}{h}\right)^{N+1} f\left(c_i, \frac{L}{h}, N, \frac{\ell}{h}, \mu\right) \tag{6.6}$$

Numerical integration is employed to evaluate the integral in (5.1) for any $\{c_i\}$. The minimum of $\Phi$ with respect to the $\{c_i\}$ is also determined numerically, subject to (6.3), with $c_1 = 0$ for case (ii). The results presented below have been computed using (6.2) with two free coefficients for each of the two cases, providing $f$ with an accuracy of no less than about one percent. As an illustration, for case (i), let the reference be results obtained using all three coefficients. Then, $f$ is accurate to within 5% if only $c_1$ is used, while it is accurate, typically, to within a small fraction of a percent if $c_1$ and $c_2$ are used.

From $Pd\Delta = d\Phi$ and (6.6), the average normal stress, $\bar{n} = P/L$, for separating the platens is the power law

$$\frac{\bar{n}}{\sigma_Y} = \left(\frac{1}{\varepsilon_Y}\frac{\Delta}{h}\right)^N f \qquad (6.7)$$

where $f$ is the value obtained in the minimization of $\Phi$. This expression also holds for compression with $\bar{n}$ as the compressive stress and $\Delta$ as the compression.

The results presented in Fig. 2 display the most basic aspects of size dependence related to films having a finite aspect ratio. In the limit of (6.7) for a rigid-perfectly plastic material ($N = 0$), the average normal stress, $\bar{n}/\sigma_Y = f$, required to separate (or compress) the platens is independent of $\Delta$. In Fig. 2, the dependence of $\bar{n}/\sigma_Y = f$ on $L/h$ shown for conventional plasticity was computed using the scheme described above with $\ell = 0$ and no constraints on $\varepsilon_P$; it is in close agreement with the approximate formula (6.1). Included in the figure are results computed for the scalar measure (6.4) of effective plastic strain with $\mu = 2$ and $\ell/h = 1/4$ for both unconstrained and constrained plastic flow at the film/platen interfaces. Film size effects become significant when the film thickness is reduced to several times the material length parameter. Moreover, plastic flow constraint at the film/platen interfaces further enhances the average flow strength. The trends in Fig. 2 imply a significant elevation of the average flow strength stress due to strain gradient effects that superimpose on those associated the film aspect ratio. Several aspects of these trends will be further explored in this section and the next.

The power-law relation between $\bar{n}$ and $\Delta$ from (6.7) is shown in Fig. 3 for $N = 0.2$ and $L/h = 1$ for the scalar version. Due to the homogeneous composition (1) of the effective plastic strain, the relative elevation in strength due to the size effect for pure power materials is independent of strain, as evident from (6.7). For these examples, the effect of plastic flow constraint at the platens is roughly equivalent to a doubling of $\ell / h$ relative to the case of unconstrained plastic flow. For the unconstrained film the flow strength, $\bar{n}$, is elevated by a factor of two when the film thickness is reduced to about $2\ell$. [Place Fig. 3 around here]

Fig. 4 compares the responses predicted by for the scalar and tensor versions for two dimensionless length parameters for $L / h = 3$. The normalizations of the plastic strain gradient measure, $\varepsilon_P^*$, for the scalar and tensor versions defined in (2.2) and (2.3) coincide for shearing with a gradient (e.g., with $\varepsilon_{12,2}^P = \varepsilon_{21,2}^P$ as the only non-zero gradient) but they do not coincide for the present problem. For this problem, the tensor measure of $\varepsilon_P^*$ is approximately twice the scalar measure as reflected by the greater strengthening enhancement of tensor version.[2] [Place Fig. 4 around here]

Material length parameters must be calibrated against experimental data. As the discussion above illustrates, the numerical value of a length so obtained will depend on the choice of plasticity theory, possibly with extreme differences (Evans & Hutchinson 2009). The material length parameter will also depend on the choice of $\mu$ in (1.1). As noted earlier, $\mu$ should not be regarded as a material parameter. Most theoretical formulations have $\mu = 2$, explicitly or implicitly, while the case for a linear dependence on strain gradients ($\mu = 1$) has been argued by Nix & Gao (1998) and Evans & Hutchinson (2009). Strength elevation as dependent on $\mu$ for fixed $\ell / h$ is illustrated in Fig. 5 for the scalar version. The implication of this figure, consistent with related studies (Fleck & Hutchinson, 1997; Idiart et al. 2009; Idiart & Fleck, 2009), is that a length parameter calibrated against experimental data for a model having $\mu = 1$ will be somewhat smaller than that for a model with $\mu = 2$. [Fig. 5 around here]

[2] The tensor contribution, $\ell \varepsilon_P^*$, to the effective plastic strain defined in (2.3) is the same as that of the 3-parameter scalar contribution of Fleck and Hutchinson (2001) and Fleck and Willis (2009a) with $\ell_1 = \sqrt{2/3}\,\ell$, $\ell_2 = \sqrt{1/6}\,\ell$ and $\ell_3 = \ell / 2$.

## 7. Plane strain compression or separation of a film – numerical flow theory solutions

### (a) Numerical method

Numerical solutions based on the 1-parameter flow theory are carried out incrementally using a finite element procedure based on the minimum principles devised by Fleck & Willis (2009a,b). Details will be restricted to the scalar version. The uniaxial stress-strain relationship is taken as the Ramberg-Osgood curve

$$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{\sigma_y}\right)^{1/N} \tag{7.1}$$

For the present problem this will ensure that the entire block of material behaves plastically throughout the history of deformation sidestepping the issue of a condition for initial plastic yielding. Moreover, no special consideration needs to be given to disjoint active plastic zones; $\dot{\varepsilon}_P$ can be expressed as a product of a multiplier, $\Lambda$, and a trial-field, $\dot{\varepsilon}_P(x_i) = \Lambda \hat{\varepsilon}_P(x_i)$.

For each increment the solution is carried out in two steps:

Step 1. Based on a plastic trial-field, $\hat{\varepsilon}_P(x_i)$, and a specified load increment, the displacement rate, $\dot{u}_i(x_i)$ is obtained together with the plastic multiplier, $\Lambda$, by minimizing $J$ in (3.10).

Step 2. The plastic trial-field, $\hat{\varepsilon}_P(x_i)$, is obtained by minimizing $H$ in (3.7).

*Step 1:*

Stationarity of $J$ leads to

$$\int_V \left\{ L_{ijkl}\left(\dot{\varepsilon}_{ij} - \Lambda\hat{\varepsilon}_P m_{ij}\right)\left(\delta\dot{\varepsilon}_{kl} - \delta\Lambda\hat{\varepsilon}_P m_{kl}\right) + h(E_P)\hat{E}_P{}^2\Lambda\delta\Lambda \right\} dV = \int_{S_T}\left(\dot{T}_i\delta\dot{u}_i + \dot{t}\delta\Lambda\hat{\varepsilon}_P\right)dS$$

which can be separated into the following two systems of equations

$$\int_V L_{ijkl}\left(\dot{\varepsilon}_{ij} - \Lambda\hat{\varepsilon}_P m_{ij}\right)\delta\dot{\varepsilon}_{kl}\, dV = \int_{S_T} \dot{T}_i\delta\dot{u}_i dS \tag{7.2}$$

$$\int_V \left\{ -L_{ijkl}\left(\dot{\varepsilon}_{ij} - \Lambda\hat{\varepsilon}_P m_{ij}\right)\hat{\varepsilon}_P m_{kl}\delta\Lambda + h(E_P)\hat{E}_P{}^2\Lambda\delta\Lambda \right\} dV = \int_{S_T} \dot{t}\delta\Lambda\hat{\varepsilon}_P dS \tag{7.3}$$

Using standard finite element interpolation with quadratic shape functions according to

$$\dot{u}_i = \sum_{N=1}^{8} N_i^N \dot{U}^N, \quad \dot{\varepsilon}_{ij} = \sum_{N=1}^{8} B_{ij}^N \dot{U}^N ,$$

or in matrix notation

$$\dot{\mathbf{u}} = \mathbf{N}\dot{\mathbf{U}}, \qquad \dot{\boldsymbol{\varepsilon}} = \mathbf{B}\dot{\mathbf{U}} ,$$

we obtain the discretized equations

$$\int_V \mathbf{B}^T \mathbf{L}\mathbf{B} dV \cdot \dot{\mathbf{U}} - \int_V \mathbf{B}^T \mathbf{L}\left(\hat{\varepsilon}_P \mathbf{m}\right) dV \cdot \Lambda = \int_{S_T} \dot{\mathbf{T}}\mathbf{N}^T dS \tag{7.4}$$

$$-\int_V \left(\hat{\varepsilon}_P \mathbf{m}\right)^T \mathbf{L}\mathbf{B} dV \cdot \dot{\mathbf{U}} + \int_V \left(\left(\hat{\varepsilon}_P \mathbf{m}\right)^T \mathbf{L}\left(\hat{\varepsilon}_P \mathbf{m}\right) + h \cdot \left(\hat{E}^P\right)^2\right) dV \cdot \Lambda = \int_{S_T} \dot{\mathbf{t}}\hat{\varepsilon}_P dS \tag{7.5}$$

From these the incremental nodal displacement vector, $\dot{\mathbf{U}}$, and the plastic multiplier, $\Lambda$, are obtained. In these expressions, $\mathbf{L}$ and $\mathbf{m}$, are matrix forms of the tensors $L_{ijkl}$ and $m_{ij}$, respectively.

Initially, a zero plastic trial-field is assumed. At later load increments, the plastic trial-field is solved for in step 2.

*Step 2:*

Stationarity of $H$ results in

$$\int_V \left(Q\delta\dot{\varepsilon}_P + \tau_i \delta\dot{\varepsilon}_{P,i}\right) dV = \int_V \sigma_e \delta\dot{\varepsilon}_P dV + \int_{S_T} t^0 \delta\dot{\varepsilon}_P dS \tag{7.6}$$

An iterative procedure is used to solve for $\hat{\varepsilon}_P(x_i)$. Within a 2D planar setting $\hat{\varepsilon}_P$ is interpolated using bilinear shape functions. Hence, the trial-field and its spatial derivatives can be expressed as

$$\left(\hat{\varepsilon}_P, \hat{\varepsilon}_{P,1}, \hat{\varepsilon}_{P,2}\right)^T = \sum_{N=1}^{4} M_I^N \hat{\varepsilon}_P^N = \mathbf{M}\hat{\boldsymbol{\varepsilon}}_P$$

with

$$\mathbf{M} = \begin{bmatrix} M_1 & M_2 & M_3 & M_4 \\ M_{1,1} & M_{2,1} & M_{3,1} & M_{4,1} \\ M_{1,2} & M_{2,2} & M_{3,2} & M_{4,2} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_v \\ \mathbf{M}_{v,1} \\ \mathbf{M}_{v,2} \end{bmatrix}$$

Introducing this in the variational statement above together with the constitutive equations, as well as requiring it to hold for all admissible variations in the plastic trial-field, we obtain the discretized system of equations

$$\int_V \frac{\sigma\left(E_P\right)}{\hat{E}_P} \mathbf{M}^T \mathbf{A}\mathbf{M} dV \cdot \hat{\boldsymbol{\varepsilon}}_P = \int_V \sigma_e \mathbf{M}_v^T dV + \int_{S_T} t^0 \mathbf{M}_v^T dS \tag{7.7}$$

Here, for the scalar family of theories,

$$\mathbf{A} = \mathrm{diag}\left(\left(\frac{\hat{\varepsilon}_P}{\hat{E}_P}\right)^{\mu-2}, l^2\left(\frac{l\hat{\varepsilon}_P^*}{\hat{E}_P}\right)^{\mu-2}, l^2\left(\frac{l\hat{\varepsilon}_P^*}{\hat{E}_P}\right)^{\mu-2}\right) \tag{7.8}$$

which for the case of $\mu = 2$ reduces to $\mathbf{A} = \mathrm{diag}\left(1, l^2, l^2\right)$.

This system of equations is solved iteratively for $\hat{\boldsymbol{\varepsilon}}_P$. An initial guess, $\hat{\boldsymbol{\varepsilon}}_P^0$, is taken as the solution from the former increment, except at the first increment when $\hat{\boldsymbol{\varepsilon}}_P^0$ is taken to be unity in all nodes, except on any boundary nodes where there is constraint. The system of equations is solved iteratively ($k = 1, 2, 3, \ldots$) according to

$$\int_V \frac{\sigma\left(E_P\right)}{\hat{E}_P^{k-1} / \left\|\hat{E}_P^{k-1}\right\|_\infty} \mathbf{M}^T \mathbf{A} \mathbf{M} dV \cdot \hat{\boldsymbol{\varepsilon}}_P^k = \int_V \sigma_e \mathbf{M}_v^T dV + \int_{S_T} t^0 \mathbf{M}_v^T dS \tag{7.9}$$

Here, $\left\|\hat{E}_P^{k-1}\right\|_\infty$ denotes the max-norm of the gradient enhanced effective plastic strain from the previous increment.

**(b) Flow theory predictions**

Normalized load-separation curves are presented in Fig. 6 for short film slabs ($L/h = 1$) of an elastic-plastic material with $\varepsilon_Y = \sigma_Y / E = 0.01$, $\nu = 0.49$ and $N = 0.2$ for conventional $J_2$ flow theory and for the scalar version of the flow theory with two values of $\ell / h$. The lower set of curves is for the case in which $\varepsilon_P = 0$ is enforced at the film/platen interfaces and the upper set is for the case where plastic flow is unconstrained at the interfaces. Included in each figure are the predictions based on the corresponding deformation theory for the pure power-law material with elasticity neglected. [Fig. 6 around here]

The transition from predominantly elastic to nearly fully plastic behavior occurs for rather small normal displacements ($\Delta / \varepsilon_Y h \approx 4$) for the constrained case. For the constrained case considered in Fig. 6B, the response curves for the deformation theory and the incremental theory agree reasonably well at large deformation levels, for both the conventional predictions and the gradient dependent predictions with $\ell / h \approx 1/8$ and

$\ell / h \approx 1/4$. In the initial range of $\Delta$, the difference between the predictions of the two theories is due to the fact that the deformation theory solutions neglect elasticity.

For the unconstrained case in Fig. 6A the predictions from the deformation theory and the flow theory do not agree when gradient effects are important ($\ell / h > 0$), with significantly softer responses predicted for the flow theory. The discrepancy is associated with the fact that the plastic strain rates in certain regions of the slab are much larger according to the flow theory than the deformation theory. In fact, over large parts of the domain the plastic strain rates are so large in the flow theory that the Mises effective stress decreases after initially increasing. As the slab is further deformed the effective stress approaches zero in two regions in the slab (c.f., Fig. 7). Hence, even though plastic deformation occurs, the Mises stress decreases steadily as a result of a deviatoric stress relaxation. Eventually, this would lead to an indeterminate plastic flow direction, as the Mises stress and the stress-deviator vanish. To circumvent this problem in the present calculations the direction of plastic flow as specified by $m_{ij}$ is 'frozen' when Mises stress drops below 10% of the initial yield stress. The domains where this method has been implemented are shown at the overall deformation level $\Delta / \varepsilon_Y h = 15$ with $\ell / h \approx 1/8$, for the unconstrained case in Fig. 7A and the constrained case in Fig. 7B. The regions over which the effective stress is well below $\sigma_Y$ are much larger. For both cases, these regions are connected to the center of the film/platen interface or located at the free surface. The regions where the Mises stress has approached zero are appreciably larger for the unconstrained case, and this is even further exaggerated for increasing values of $\ell$. [Fig. 7 around here]

Also included in Figure 7, are contour lines of the plastic trial-field evaluated at $\Delta / \varepsilon_Y h = 15$. It is observed for the constrained case in Fig. 6B that the trial-field goes toward zero at the constrained boundary, and that the contour lines are perpendicular to the all other parts of the exterior of the computational domain in accordance with the natural boundary condition. Correspondingly, in Fig. 7A for the unconstrained case, the natural boundary condition is fulfilled at the entire exterior boundary of the computational domain. These trial fields provide insight as to why the deformation theory solutions and the flow theory solutions agree reasonably well for the constrained case but not for the unconstrained case—the constraint on plastic flow at the film/platen interfaces shrinks significantly the regions where the deviatoric stress is relaxed, whereas this is not the case for the unconstrained analysis.

Similar comparisons of pure-power deformation solutions and full elastic-plastic flow theory solutions for the tensor version are presented in Fig. 8. Flow theory solutions are presented for both the rate-independent formulation and the visco-plastic, rate-dependent formulation (Fleck & Willis, 2009b; Gudmundson, 2004; Gurtin & Anand, 2005), with either full constraint on plastic flow at the platens ($\varepsilon_{12}^{P} = \varepsilon_{11}^{P} = \varepsilon_{22}^{P} = \varepsilon_{33}^{P} = 0$) or no constraint on plastic flow ($t_{12} = t_{11} = t_{22} = t_{33} = 0$). The visco-plastic solutions are obtained using a method very similar to that used for the time-independent solutions. However, for the visco-plastic formulation, Minimum Principle II in Fleck & Willis (2009b) delivers the entire plastic strain rate field (no unknown multiplicative factor is needed), which means that the corresponding Minimum Principle I is used to solve for the displacement field alone. While the displacement interpolation is the same as that used for the scalar time-independent solutions, the plastic strain components are interpolated individually in the tensor version using bilinear shape functions, while exploiting plastic incompressibility to define one normal component in terms of the two others (Niordson and Legarth, 2010). For plane problems this results in 3 plastic strain degrees of freedom for each node as compared to just one for the scalar case. A standard visco-plastic power-law has been employed using a visco-plastic exponent of 0.01 in combination with an overall loading rate on the order of the reference strain rate, thus minimizing viscous effects in the solutions obtained. [Fig. 8 around here]

For the tensor versions, there is reasonable agreement between deformation and flow theory solutions in the fully plastic region for both the constrained and unconstrained cases. It is still true that the effective stress approaches zero as $\Delta$ increases over a non-negligible region of the slab for both the rate-independent and rate-dependent flow theories even though the material continues to deform plastically (plots are not shown). The components of the Cauchy deviator stress can even switch signs. However, unlike the scalar version, this behavior does not present a numerical problem because the plastic strain rate is not constrained to be co-directional with the deviator stress in the tensor version.

## 8. Concluding remarks

Constrained plastic compression or extension of thin metal films intrinsically involves strain gradients which, in turn, give rise to appreciable size-dependent strengthening when the film thickness is reduced below 3 to 5 times the material length parameter employed in the present formulation. Strengthening effects have clear implications for metal bonding layers, including increased resistance to plastic flow and higher stresses which are likely to affect ductility.

In this paper both deformation and flow theory versions of strain gradient plasticity have been employed. In rate-independent form, the flow theories used here are the new formulations proposed by Fleck & Willis (2009a,b). For the thin slab problem considered here, difficulties arise with the scalar version of the flow theory wherein the strengthening effect due to the strain gradients appears to be significantly underestimated for the case in which plastic flow is not constrained at the loading platens. Although the source of the difficulties is not entirely clear, it seems likely that it stems in part from the fact that the theory predicts regions having a decreasing Cauchy deviator stress with the Mises stress going to zero even as the plastic deformation increases. This presents numerical difficulties due to the fact that the plastic strain rate direction becomes indeterminate in the scalar formulation. That continuing plastic flow is predicted to occur at small, or even zero, Mises stress may also be problematic from a physical standpoint. The Mises stress also approaches zero in regions of the slab in the tensor version of the flow theory. This does not create numerical difficulties for this class of theories because the plastic strain rate is not tied to the Cauchy deviator stress. The apparently anomalous behavior in which the Mises stress becomes small even in the presence of appreciable plastic straining has also been noted in simpler problems such as in pure bending by Idiart et al. (2009).

The behavior noted above can arise because plastic straining is largely decoupled from the Cauchy stresses in these new flow theory formulations. Similar behavior occurs for solutions we have carried out based on the visco-plastic tensor version reported for the film slabs in Fig. 8. Plastic straining is even less tightly coupled to the Cauchy stresses in the visco-plastic formulations than in the rate-independent versions. These same features pertain to the general class of visco-plastic theories proposed by Gudmundson (2004) and Gurtin and Anand (2005). Thus, for some problems this class of flow theories predicts that plastic straining can occur in regions of the body in which the elastic lattice strains are essentially zero. Such regions are predicted to be as large as one or more times the material length parameter, $\ell$, i.e. on the order of microns in size. It remains to be seen if such behavior is observed experimentally and whether such predictions are truly problematic from a physical standpoint.

## Appendix: Formulations based on the strain measure proposed by Nix and Gao

Nix & Gao (1998) proposed an alternative way to combine the effective plastic strain, $\varepsilon_P$, and the magnitude of its gradient, $\varepsilon_P^*$, to model the effect of geometrically necessary dislocations. Evans & Hutchinson (2009) discussed differences in trends based the Nix-Gao composition from those based on (1.1), highlighting the physical basis for each as well as their advantages and disadvantages. Here, a generalized effective strain measure will be defined based on the Nix-Gao composition. Then, that measure will be used to postulate strain gradient theories following the procedure outlined earlier using the deformation theory as the starting template. It should be noted that Nix & Gao (1998) did not propose a complete strain gradient plasticity theory—they suggested the manner in which the flow stress is enhanced by the gradient contribution. Later, Huang et al. (2000) proposed a higher order theory based on the Nix-Gao composition; however, that proposal is quite different from what is given below. Indeed, the version suggested by Huang et al. introduces a second material length parameter tied to the higher order stresses and unrelated to that of Nix and Gao used below. In most instances, applications of the of the Nix-Gao composition have been restricted to lower order versions (Huang et al. 2004).

With $\sigma_e$ as the effective flow stress, $\sigma_0$ as a reference yield stress, and $\sigma_e = \sigma_0 f(\varepsilon_P)$ as the relation between stress and plastic strain in uniaxial tension, Nix and Gao proposed that the flow stress is enhanced by the gradients according to

$$\sigma_e = \sigma_0 \sqrt{f(\varepsilon_P)^2 + \ell \varepsilon_P^*} \qquad \text{(A1)}$$

This incorporates the gradient measure of the geometrically necessary dislocations in accord with the Taylor hardening law with a linear dependence on $\ell\varepsilon_P^*$ as the classical limit is approached, and it reduces to the uniaxial relation when the gradients can be ignored. For uniaxial behavior with monotonically increasing stress, i.e. $df/d\varepsilon_P > 0$, let $f^{-1}(x)$ denote the inverse of $f(x)$, and define an generalized effective plastic strain by

$$E_P = f^{-1}\left(\sqrt{f(\varepsilon_P)^2 + \ell\varepsilon_P^*}\right) \qquad \text{(A2)}$$

With this definition, $\sigma_e = \sigma_0 f(E_P)$ is identical to (A1) of Nix and Gao. Specifically, for a power-law in uniaxial tension, $\sigma_e = \sigma_0 \varepsilon_P{}^N$, (A2) is

$$E_P = \left(\varepsilon_P{}^{2N} + \ell\varepsilon_P^*\right)^{1/(2N)} \qquad \text{(A3)}$$

The subsequent discussion will be restricted to (A3).

With $E_P$ in (A3), the deformation theory based on (3.1) is fully defined. For the one-parameter gradient measure (2.2), one has

$$Q = \frac{\partial U_P}{\partial \varepsilon_P} = \sigma(E_P)\left(\frac{\varepsilon_P}{E_P}\right)^{2N-1}, \quad \tau_i = \frac{\partial U_P}{\partial \varepsilon_{P,i}} = \sigma(E_P)\frac{\ell}{2N}\left(\frac{1}{E_P}\right)^{2N-1}\frac{\varepsilon_{P,i}}{\varepsilon_P^*} \qquad \text{(A4)}$$

The definition of the flow theory is not as straightforward. Unlike the measure in (1.1), (A3) is not homogeneous in $\varepsilon_P$ and $\varepsilon_P^*$, except if $N = 1/2$. Nevertheless, a generalized plastic strain-rate can be defined that coincides with (A3) for proportional straining. With $\dot{\varepsilon}_P = \sqrt{2\dot{\varepsilon}_{ij}^P \dot{\varepsilon}_{ij}^P / 3}$, $\dot{\varepsilon}_P^*$ defined by (2.7), and total values defined by (2.10), it is easily seen that a measure with the desired property is

$$\dot{E}_P = \frac{1}{E_P{}^{2N-1}}\left(\varepsilon_P{}^{2N-1}\dot{\varepsilon}_P + \frac{\ell}{2N}\dot{\varepsilon}_P^*\right) \qquad \text{(A5)}$$

If $N = 1/2$, (A5) coincides with the present measure in (2.9) for $\mu = 1$ and it is otherwise similar in the sense that it is a linear composition of the rate contributions. The constitutive relations for the flow theory are taken as

$$Q = \sigma(E_P)\left(\frac{\varepsilon_P}{E_P}\right)^{2N-1} \quad \text{and} \quad \tau_i = \frac{\sigma(E_P)\ell}{2N}\left(\frac{1}{E_P}\right)^{2N-1}\frac{\dot{\varepsilon}_{P,i}}{\dot{\varepsilon}_P^*} \qquad \text{(A6)}$$

such that the plastic dissipation rate is again $Q\dot{\varepsilon}_P + \tau_i \dot{\varepsilon}_{P,i} = \sigma(E_P)\dot{E}_P$. The field equation for $\dot{\varepsilon}_P$ is obtained from the higher order equilibrium equation (2.12), and it is the Euler equation associated with Minimum Principle I defined, as before, by (3.7). One can also show that

$$\dot{Q}\dot{\varepsilon}_P + \dot{\tau}_i \dot{\varepsilon}_{P,i} = h(E_P)\dot{E}_P{}^2 + (1-2N)\frac{\sigma(E_P)}{E_P}\left(\dot{E}_P{}^2 - \left(\frac{\varepsilon_P}{E_P}\right)^{2N-1}\dot{\varepsilon}_P{}^2\right) \qquad \text{(A7)}$$

Thus, the functional associated with Minimum Principle II in (3.10) must be modified by replacing $h(E_P)\dot{E}_P{}^2$ by the right hand side of (A7).

It is interesting to note that the deformation and flow theories proposed above based on the effective strain measure of Nix & Gao (1998) coincide with the corresponding theories based on the Fleck-Hutchinson measure when $\mu = 1$ and $N = 1/2$. The nice properties of solutions for pure power-law materials noted in *Section 5* require a measure that is homogeneous in the plastic strain and its gradient. Therefore, the simple solution structure and its consequences in Section 5 do not hold for the theories based on the Nix-Gao measure unless $N = 1/2$.

**Acknowledgement**: This work is supported by the Danish Research Council for Technology and Production Sciences in a project entitled Plasticity Across the Scales.

**References**


Aifantis, E C. 1984 On the microstructural origin of certain inelastic models. J. Engng. Mater. Tech. 106, 326-330.

Evans, A.G. & Hutchinson, J.W. 2009 A critical assessment of the theories of strain gradient plasticity, Acta Mater. 57, 1675-1688.

Fleck, N.A. & Hutchinson, J.W. 1997 Strain gradient plasticity, Adv. Appl. Mech. 33, 295-361.

Fleck, N.A. & Hutchinson, J.W. 2001 A reformulation of strain gradient plasticity, J. Mech. Phys. Solids 49, 2245–2271.

Fleck, N.A. & Willis, J.R. 2009a A mathematical basis for strain gradient plasticity theory. Part I: scalar plastic multiplier, J. Mech. Phys. Solids 57; 161–177.

Fleck, N.A. & Willis, J.W. 2009b. A mathematical basis for strain gradient plasticity theory. Part II: tensorial plastic multiplier. J. Mech. Phys. Solids **57**; 1045–1057.

Fredriksson, P. Gudmundson, P. 2005 Size-dependent yield strength of thin films, Int. J. Plasticity 21, 1834–1854.

Gao H, Huang Y, Nix W D, & Hutchinson J W. 1997 Mechanism based strain gradient plasticity: I—theory. J. Mech. Phys. Solids 47, 1239–1263.

Gudmundson, P. 2004 A unified treatment of strain gradient plasticity, J. Mech. Phys. Solids 52, 1379–1406.

Gurtin, M.E. & Anand, L. 2005 A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. Part I: Small deformations. J. Mech. Phys. Solids 53, 1624-1649.

Gurtin, M.E. & Anand, L. 2009 Thermodynamics applied to gradient theories involving accumulated plastic strain. The theories of Aifantis and Fleck and Hutchinson and their generalizations. J. Mech. Phys. Solids 57, 405-421.

Hill, R. 1950 *The Mathematical Theory of Plasticity.* Clarendon Press, Oxford.

Huang Y, Gao H, Nix W D, Hutchinson J W. 2000 Mechanism based strain gradient plasticity: II—analysis. J. Mech. Phys. Solids;48, 99–128.

Huang, Y., Qu S., Hwang, K.C., Li, M., Gao, H. 2004 A conventional theory of mechanism-based strain gradient plasticity. Int. J. Plasticity 20, 753-782.

Idiart, M.I., Deshpande, V.S., Fleck, N.A. & Willis, J.R. 2009 Size effects in bending of thin foils, to appear in Int. J. Eng. Sci. 47, 1251-1264.

Idiart, M.I. & Fleck, N.A. 2010 Size effects in the torsion of thin metal wires, Model. Simul. Mater. Sci. Engrg. 18, 015009,1-10.

Muhlhaus, H-B., Aifantis, E.C. 1991 A variational principle for gradient plasticity. Int. J. Solids Struct. 28, 845–857.

Niordson, C.F. & Legarth B.N. 2010 Strain gradient effects on cyclic plasticity. J. Mech. Phys. Solids, 58, 542-557.

Nix, W.D. & Gao, H. 1998 Indentation size effects in crystalline materials: a law for strain gradient plasticity. J. Mech. Phys. Solids 46, 411-425.

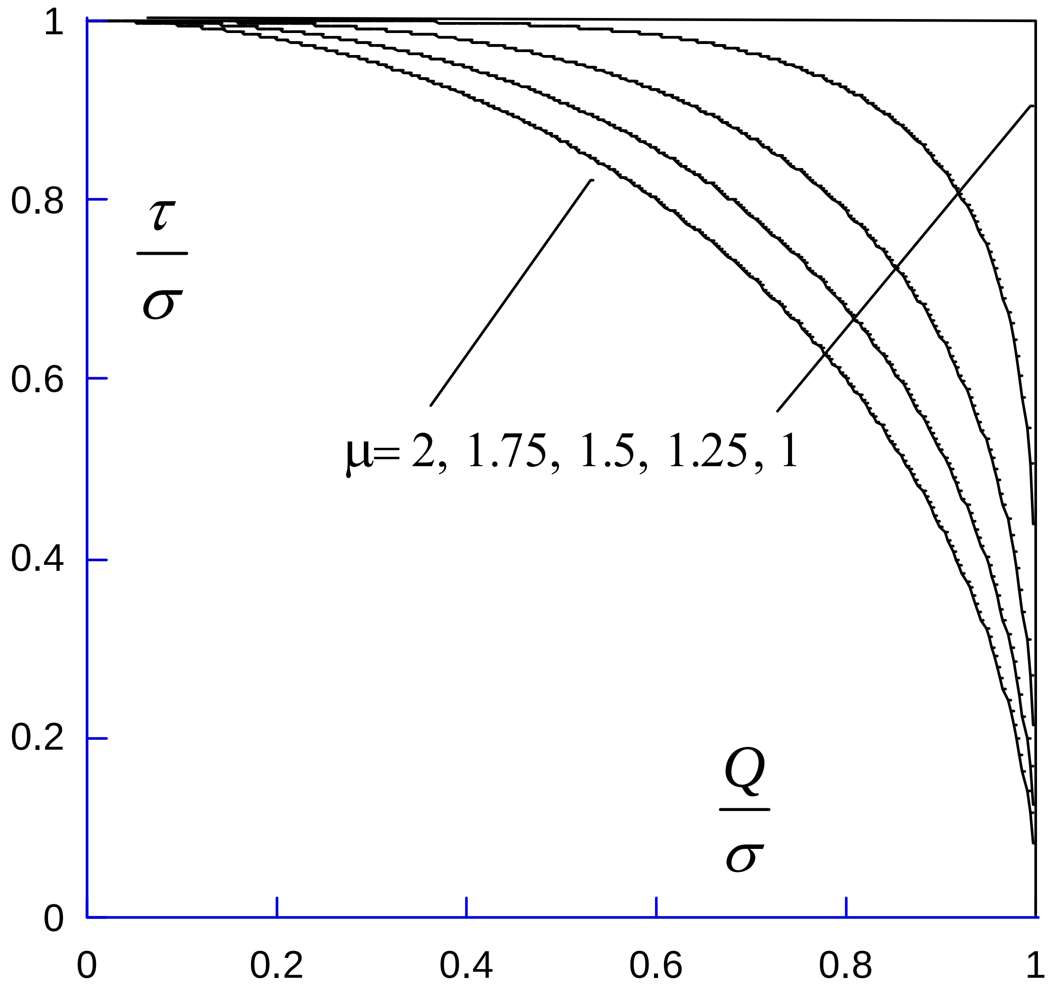



Fig. 1 Yield surfaces for 1-parameter family of scalar theories.

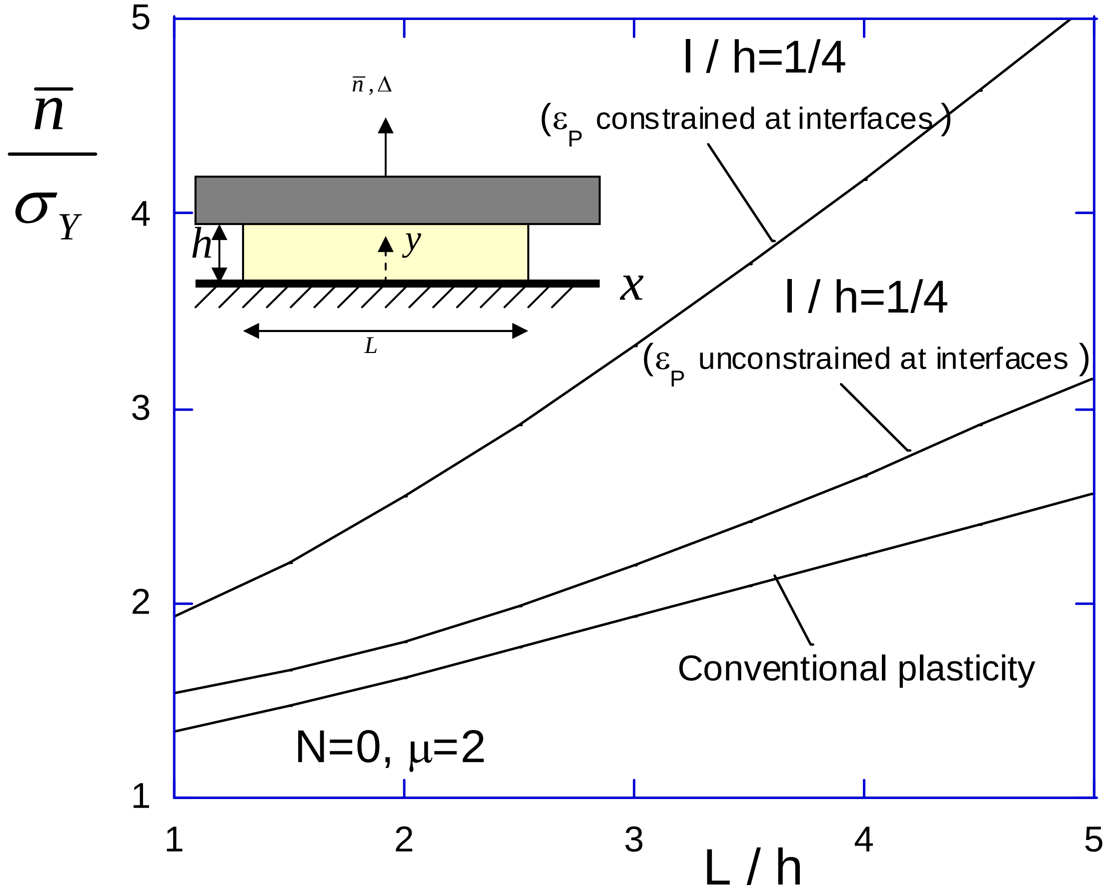



Fig. 2 Effect of film aspect ratio on the average normal stress required to yield a rigid-perfectly plastic material ( $N = 0$ ). The curves for the strain gradient material are based on the scalar model with $\mu = 2$ and dimensionless length parameter $\ell / h = 1/4$. The upper curve assumes plastic flow is blocked at the platen/film interfaces ( $\varepsilon_P = 0$ ) while the lower two curves are determined with no such restriction on plastic flow.

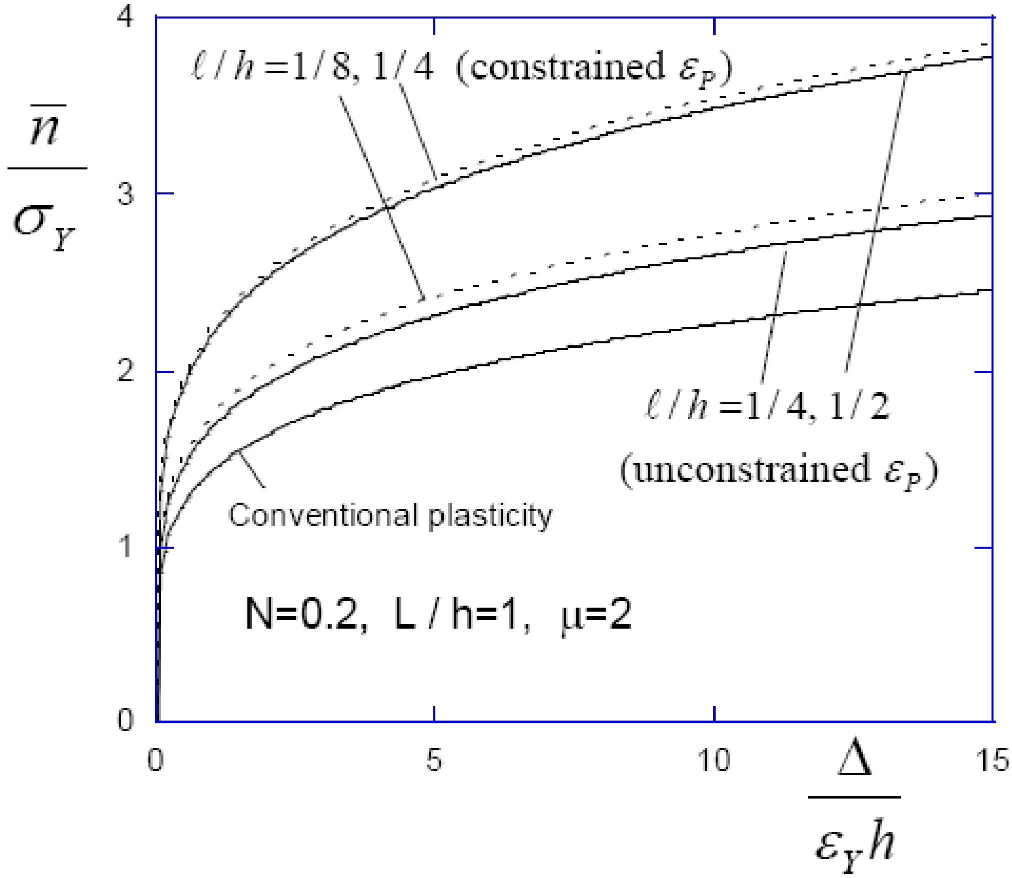



Fig. 3 Pure-power relation between average normal stress and platen separation for a film slab with $L/h = 1$ based on the scalar measure of effective plastic strain. The relation applies in tension or compression with appropriate choice of signs. The solid curves are for films with no constraint on plastic strain at the film/platen interfaces while the plastic strain at the interfaces is constrained to be zero for the dashed curves.

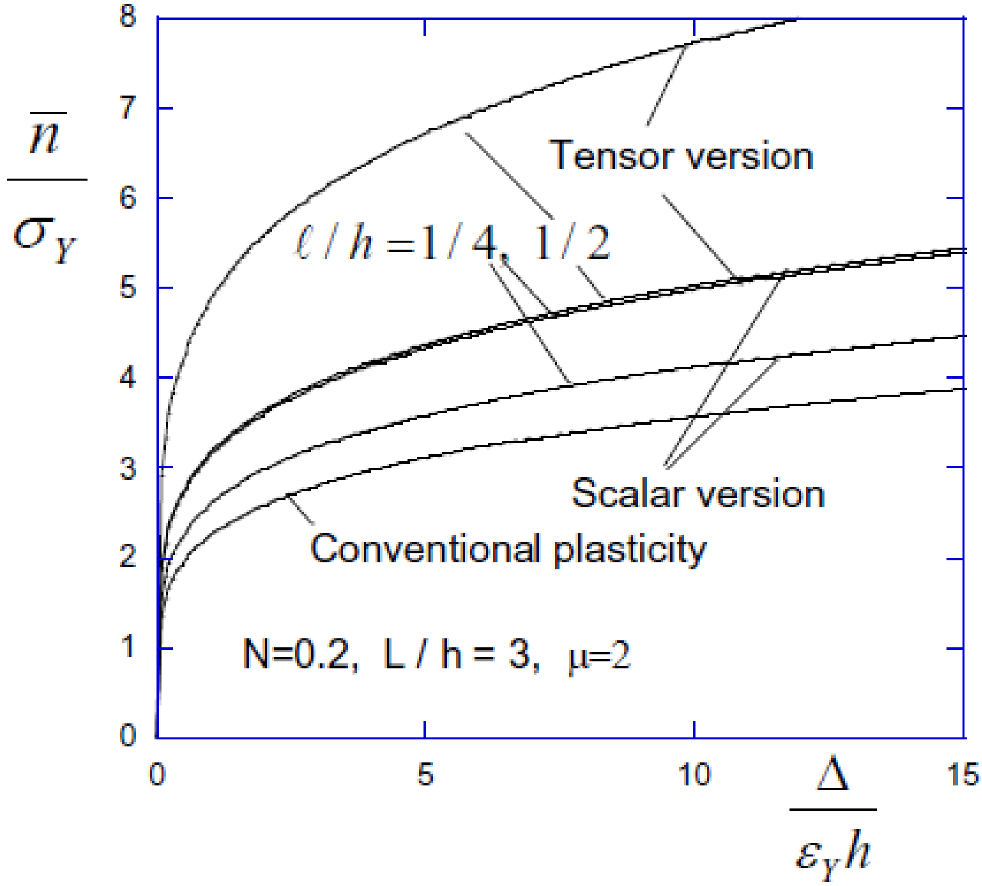



Fig. 4 Comparison between pure-power law solutions based on scalar version and tensor version. Plastic flow at the film/platen interfaces is not constrained.

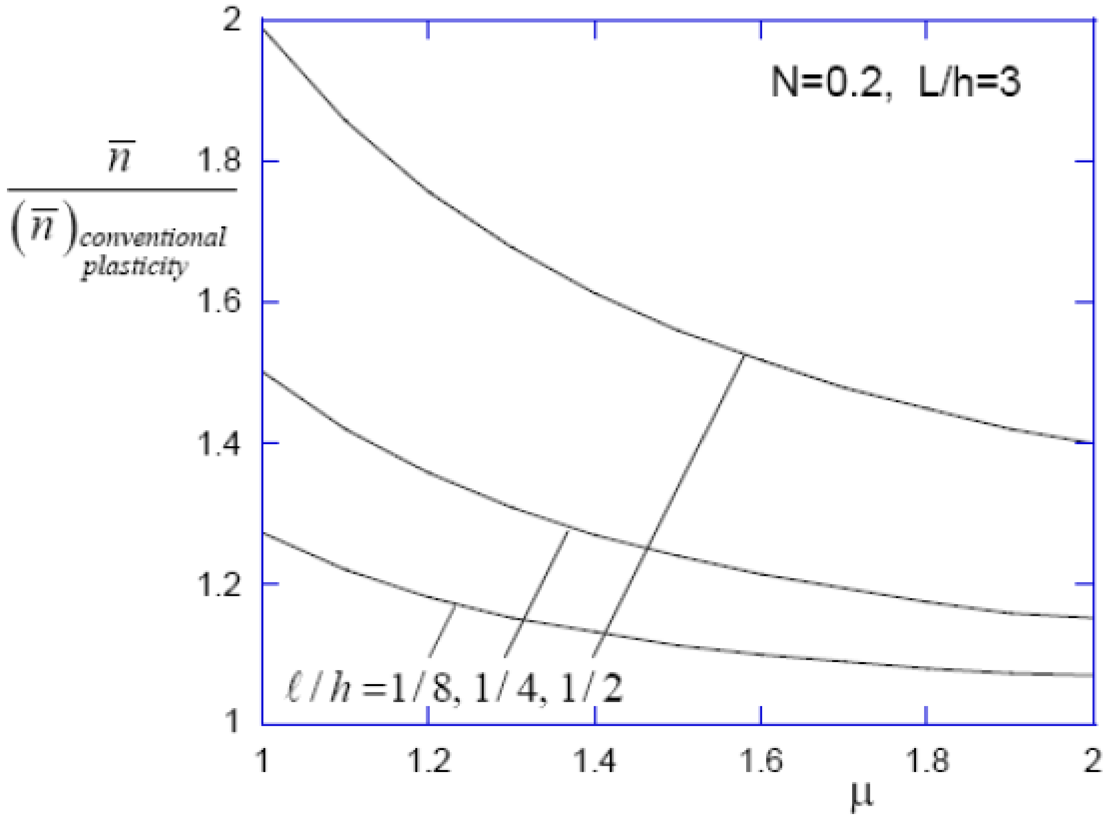
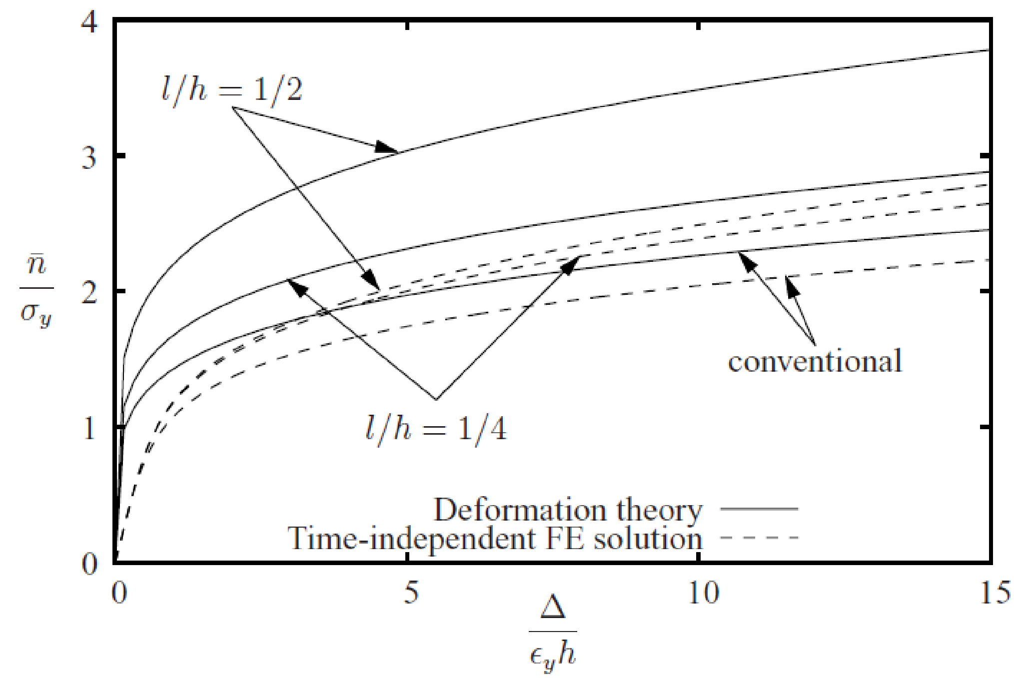
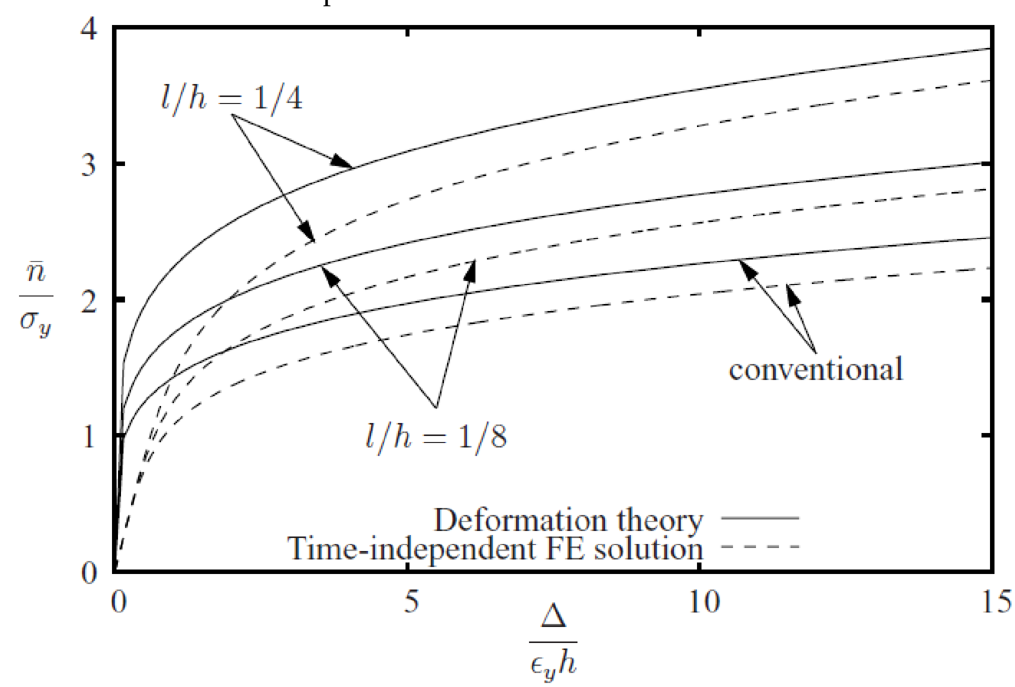



Fig. 5 Elevation of average normal stress above that for conventional plasticity (at identical platen separations) due to strain gradient effects as dependent on the parameter $\mu$ characterizing the composition of the effective plastic strain measure. These results are derived for the pure-power law material based on the scalar model. Plastic flow at the film/platen interfaces is not constrained.




A) Unconstrained flow at the platens.




B) Constrained flow at the platens.

Fig. 6 Normalized load-separation response computed using the scalar flow theory for an elastic-plastic solid specified in the text (dashed curves) compared with predictions from the deformation theory for a pure power-law material with elasticity neglected (solid curves). All cases have $L/h = 1$, $N = 0.2$ and $\mu = 2$.

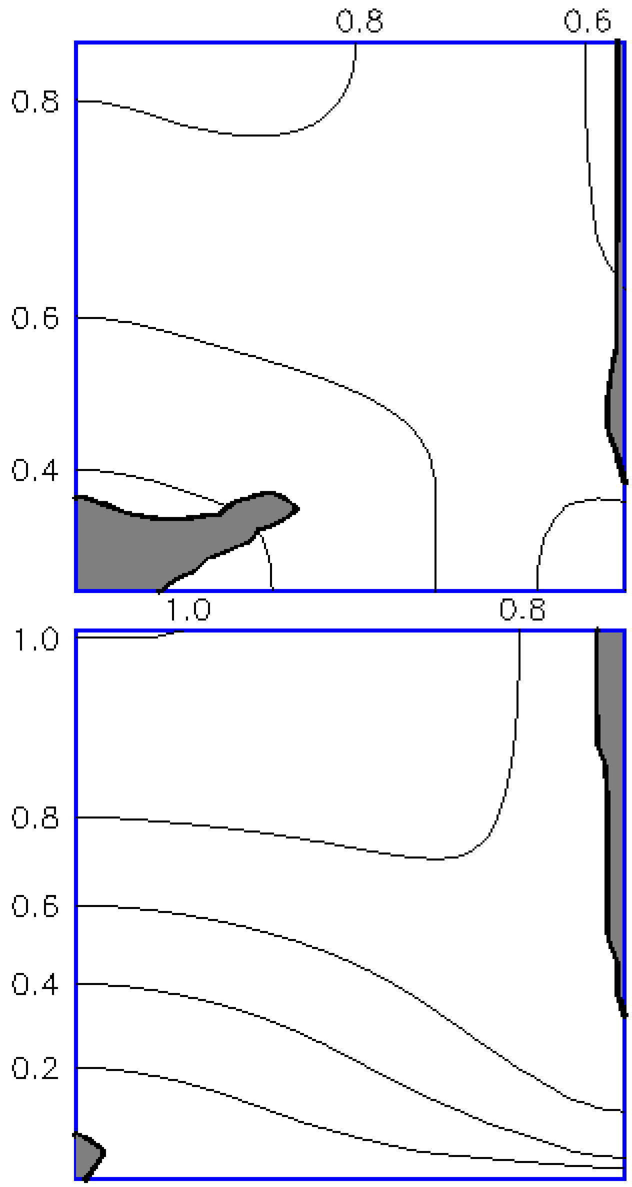
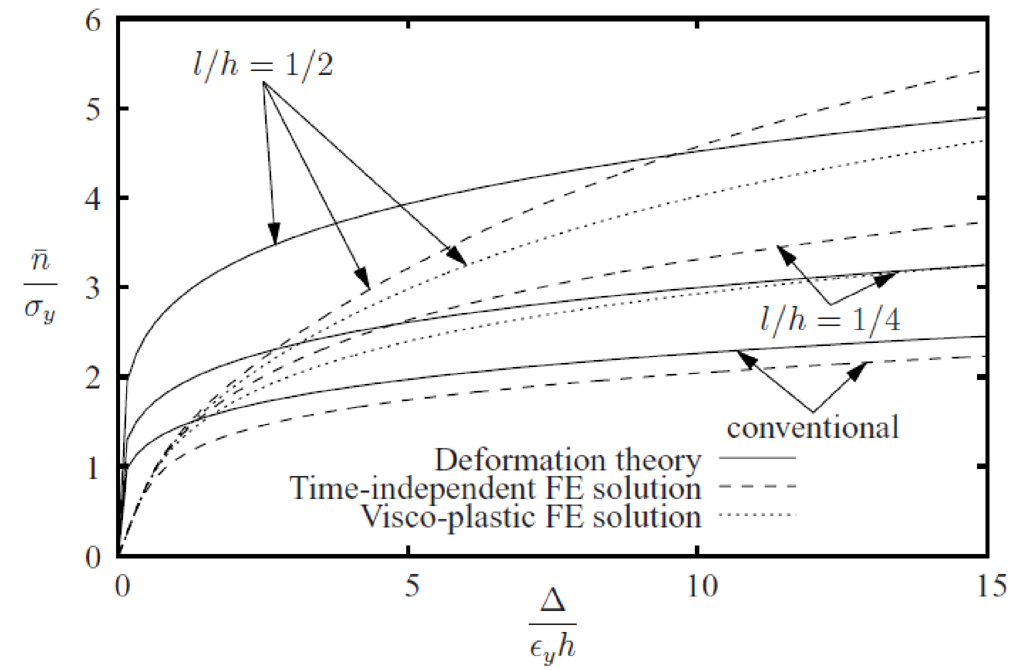
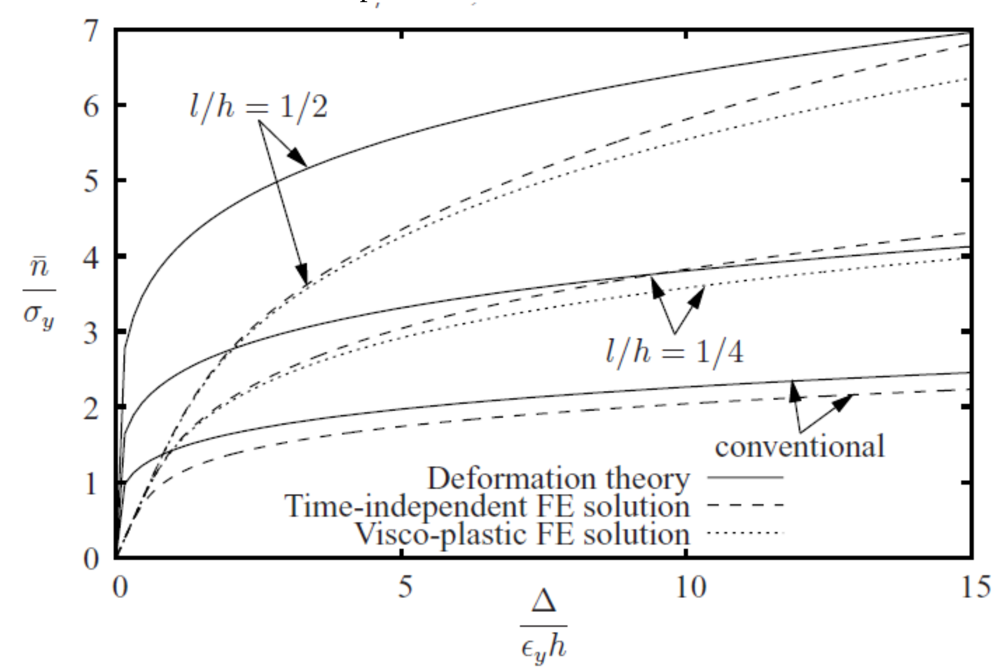



Fig. 7 Regions (in grey) within which the effective stress, $\sigma_e$, has approached zero during continued plastic loading at $\Delta/\varepsilon_Y h = 15$ with $\ell/h = 1/8$. Upper figure for case in which plastic strain is unconstrained at the film/platen interfaces and lower figure for the constrained case, both for the scalar version with $L/h = 1$, $N = 0.2$ and $\mu = 2$. The rectangular region shown is one-quarter the entire cross-section of the film slab, extending upwards from the lower platen to the horizontal centerline and rightward from the vertical centerline to the right edge. The contour lines describe constant values of the trial function, $\hat{\varepsilon}_P$, whose maximum value has been normalized to be unity. Plastic loading occurs through out the slab.




A) Unconstrained flow at the platens.




B) Constrained flow at the platens.

Fig. 8 Normalized load-separation response computed using the tensor version of the flow theory for an elastic-plastic solid specified in the text (dashed and dotted curves) compared with predictions from the corresponding deformation theory for a pure power-law material with elasticity neglected (solid curves). The dashed curves are for the rate-independent formulation while the dotted curves are obtained using the viscous-plastic formulation. All cases have $L/h=1$, $N=0.2$ and $\mu=2$.