Peeling, Healing, and Bursting in a Lubricated Elastic Sheet

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We consider the dynamics of an elastic sheet lubricated by the flow of a thin layer of fluid that separates it from a rigid wall. By considering long wavelength deformations of the sheet, we derive an evolution equation for its motion, accounting for the effects of elastic bending, viscous lubrication, and body forces. We then analyze various steady and unsteady problems for the sheet, such as peeling, healing, levitating, and bursting, using a combination of numerical simulation and dimensional analysis. On the macroscale, we corroborate our theory with a simple experiment, and, on the microscale, we analyze an oscillatory valve that can transform a continuous stream of fluid into a series of discrete pulses.

We have all had the experience of the runaway transparency in the midst of a seminar; one that slides off the projector by riding on a thin film of air before coming to rest as far away from the speaker as possible [1]. The basic mechanism responsible for this event is the lubricating effect of a thin fluid film [2–4]. This mundane situation is hardly unique and is paradigmatic of many industrial processes involving moving tapes, paper, and textiles [5] and in small scale biological and microelectromechanical systems (MEMS) [6–8]. In all of these situations, the competition between the elastic and fluid forces eventually leads to effects such as the surly behavior of the unruly transparency. These problems are analogs of free-surface flows in hydrodynamics that arise in many applications (see [9], and references therein) but are qualitatively different owing to the presence of the elastic sheet.

An experimental realization of this class of problems is exemplified in Fig. 1. A flexible sheet of plastic is clamped at the left slightly above a rigid floor; when glycerine is pumped in from the lower left, the plastic sheet lifts off and balloons as a peeling front advances to the right. Following a short transient, the front moves at constant velocity and the sheet eventually lifts off, completely supported by the fluid. In this Letter, we will focus on some of the simplest problems motivated by this example, both on the macroscale and microscale using an asymptotic description of the “elastohydrodynamics” of fluid-lubricated elastic sheets.

We start by considering the two-dimensional dynamics of a fluid-lubricated elastic sheet of thickness $b$ and length $L$ ($L \gg H_g \gg b$, where $H_g$ is a typical gap thickness), density $\rho_s$, Young’s modulus $E$, Poisson ratio $\nu$, and bending stiffness $B = E b^3/12(1 - \nu^2)$ in a geometry shown in Fig. 1. The intercalating incompressible fluid of density $\rho_f$ and viscosity $\mu$ satisfies the equations of momentum and mass conservation:

\begin{equation}
\rho_f(u_t + u \cdot \nabla u) = -\nabla p + \mu \nabla^2 u + \rho_f g,
\end{equation}

\begin{equation}
\nabla \cdot u = 0,
\end{equation}

where $p$ is pressure, $u = (u_t, u_y)$ is the fluid velocity, and $g = (0, -g)$ is gravity. Along the rigid floor $y = 0$, the fluid does not slip or penetrate the solid so that $u_{y|y=0} = 0$, while along the elastic sheet $y = h(x, t)$, the no-slip condition reads $u_{y|y=h(x,t)} = 0$ and the kinematic boundary condition reads $h_t + u_{y|h(x,t)} = v_{y|h}$. Here and elsewhere subscripts denote derivatives. Finally, continuity of traction normal to the center line of the elastic sheet requires that

\begin{equation}
\sigma_{xy} = \int_0^\infty \frac{\rho_f}{b} u_y^2 dy = \mu \frac{\partial h}{\partial x}.
\end{equation}

FIG. 1 (color online). A schematic of the system and an image of a simple experiment showing a propagating peeling front in a plastic shim on a layer of glycerine. Dark solid lines indicate experimental data; dashed lines are the result of solving (7) and (8) numerically. These plots show a snapshot in time as the plastic peels off the underlying substrate. Experimental parameters are $\rho_f = 1.2 \text{ g/cm}^3$, $\mu = 10 \text{ g/cm s}$, $Q = 3.3 \text{ cm}^2/\text{s}$, $E = 8.3 \times 10^{10} \text{ dynes/cm}^2$, $b = 0.1 \text{ mm}$, $\Delta \rho = 5 \text{ g/cm}^3$. This corresponds to $H_g = 2.0 \text{ cm}$, $L_g = 12.0 \text{ cm}$, and $G = 283.2$. (Received 8 April 2004; published 24 September 2004)
The horizontal length scale $L_g \equiv \left[ E b^2 / |f(h_0)| \right]^{1/3}$ is set by the competition between bending and the external body force, and the vertical length scale $H_g = [\mu Q L_g / |f(h_0)|]^{1/3}$ is set by the competition between the external body force and viscous lift, where $Q$ is the average flux in the gap. Furthermore, the horizontal velocity scale is $U = Q / H_g$, while the vertical velocity scale is $V = e U$, where $e = H_g / L_g \ll 1$. Using these scales, we define the dimensionless variables $\hat{x} = x / L_g$, $\hat{y} = y / H_g$, $\hat{u} = u / U$, $\hat{v} = v / e U$, $\hat{t} = t / L_g = Q t / L_g H_g$, and $\hat{p} = p / p_f = p / f$. Substituting the scaled variables into (1)–(3), with $f = \Delta p g b$, we get (on dropping the hats)

$$e^3 \text{Re}(u_t + u u_x + v u_y) = -p + e^2 u_{xx} + u_{yy},$$
$$e^4 \text{Re}(v_t + u v_x + v v_y) = -p + e^2 v_{xx} + e^2 v_{yy} - e G,$$
$$h_x = 0, v_y = 0$$

subject to the boundary conditions $u_t|_{y=0} = u_t|_{y=h} = v|_{y=0} = 0, v|_{y=h} = h_t$, and

$$h_t - \frac{e}{12} \left( \frac{h^3 h_{xxx}}{12(1 - \nu^2)} + G h^3 h_x + \frac{H^3_g}{\mu Q L_g} f_s \right) = 0.$$  

This equation, valid in the limit of a thin fluid-filled gap, is similar to those seen in the context of free-surface flows [9], except for the term arising from elasticity, and represents a tremendous simplification from the partial differential equations that describe the coupled motion of the sheet and fluid. To complete the formulation of the problem, we need an initial profile and six boundary conditions. Motivated by the experiment shown in Fig. 1, for a clamped-free sheet, the appropriate boundary conditions are

$$h = h_0$$
$$h_x = \theta_0$$
$$q(h, h_x, \ldots) = 1$$

at $x = 0$,

$$h_x = 0$$
$$h_{xxx} = 0$$
$$p = \frac{e G}{2} h$$

at $x = L / L_g$.

where $q(h, h_x, \ldots) = \int_0^h u d y$ is the fluid flux. The first three boundary conditions correspond to a prescribed height, slope, and fluid flux at the clamped end $x = 0$, while the last three correspond to the condition of zero force, zero torque, and a matched pressure at the free end.
Solving the system (7) (with \( f_x = 0 \)) and (8) numerically using a finite difference method, we find that the numerical solution matches the experimentally observed transient peeling profiles (Fig. 1) with no adjustable parameters. In Fig. 2(a) we show the evolution of the peeling traveling wave. Although these peeling waves are known to exist in the membrane-tension dominated regime [6], here they are dominated by bending and are thus qualitatively different. In the inlet region [roughly \( x = 0 \) to \( x = 4 \) in Fig. 2(a)] bending and hydrostatic forces balance each other so that \( E b^3 \Delta H / L_{\text{entry}} \sim \rho g \Delta H \), which yields a scaling law for entrance length

\[
L_{\text{entry}} \sim L_g(\mathcal{G})^{-1/4}.
\]

(9)

In the central region [roughly \( x = 4 \) to \( x = 35 \) in Fig. 2(a)], the membrane is relatively flat, and bending does not play a prominent role. The dominant balance is between viscous stresses and hydrostatic pressure so that \( \rho f_x h_{\text{lev}} / L_g \sim \mu \mathcal{Q} / h_{\text{lev}}^3 \), leading to a scaling law for the levitation height

\[
h_{\text{lev}} \sim H_g(\epsilon \mathcal{G})^{-1/4}.
\]

(10)

The scaling laws (9) and (10) are confirmed over a range of parameter values as shown in Figs. 2(b) and 2(c). Finally, in the outlet region, there is a small tail where the sheet is slightly curved to accommodate the free-end condition; the horizontal and vertical extent of this zone scale with \( L_g \) and \( H_g \), respectively, although there is a weak dependence on other parameters as well.

While these length scales characterize the steady levitating sheet, the transient behavior leading up to this involves a peeling front moving at a velocity, \( v_f \), which is constant as long as the front is sufficiently far from the inlet and the exit. At the front, the viscous power dissipated (per unit width) must be balanced by the work done against the hydrostatic load so that 

\[
\mu (v_f / h_{\text{lev}})^2 H_{\text{lev}} L_g \sim \rho f_x L_p (h_{\text{lev}} / L_g) v_f.
\]

In dimensionless terms, this yields

\[
\frac{v_f}{U} \sim \left( \frac{\mathcal{G}}{\epsilon} \right)^{1/2}.
\]

(11)

This scaling law is confirmed numerically as shown in Fig. 2(d). As expected, the scaling breaks down for small \( \mathcal{G} \) when hydrostatic forces no longer play a dominant role in the dynamics.

Having used this simple macroscopic setting as a testbed for our theory, experiment, and numerical simulations, we now turn to a microscale phenomenon motivated by fluid-actuated switches and valves in MEMS and microfluidics [8], where van der Waals forces can potentially play a role. We consider the geometry shown in Fig. 1, but now set the body force to be the disjoining pressure between the elastic sheet and the rigid surface with \( f = \Pi(h) = \frac{1}{6\pi}(A_\alpha / h^3 - A_\alpha / h^n) \) in (3), where \( A_\alpha \) and \( A_\alpha \) are the attractive and repulsive Hamaker constants, respectively; here \( n = 3 \) and \( m = 9 \) corresponding to the standard [6, 12] Lennard-Jones potential. Following the asymptotic reduction procedure that led to (7) for the macroscopic problem now yields

\[
h_x = \frac{\epsilon}{12} \left( \frac{h^3 h_{\text{lev}}}{12(1 - \mu^2)} + A_x \frac{h_x}{h} - R_x \frac{h_x}{h} \right) = 0,
\]

(12)

where \( A_x = \frac{3}{6\pi}(A_\alpha / \mu Q L_g) \) and \( R_x = \frac{9}{6\pi}(A_\alpha / \mu Q L_g H_g^3) \) are rescaled Hamaker constants, with the length scales \( H_g, L_g \) defined using \( f = \Pi(h_0) \) instead of \( f = \Delta \rho g b \). Solving (12) subject to the boundary conditions (8), with

FIG. 2. (a) The evolution of a peeling front obtained by solving (7) and (8). Successive profiles are shifted vertically (thus time increases from bottom to top in the plot). (b) The entry length \( L_{\text{entry}} \) as a function of the dimensionless hydrostatic pressure \( \mathcal{G} \); the line is the scaling law (9). (c) The levitation height \( h_{\text{lev}} \) as a function of \( \epsilon \mathcal{G} \); the line is the scaling law (10). (d) The velocity of the peeling front \( v_f \) as a function of \( \mathcal{G} / \epsilon \); the line is the scaling law (11).

FIG. 3. (a) Bursting events driven by attractive van der Waals forces. As in Fig. 2(a), successive profiles are shifted vertically. \( G = 0, \epsilon = 0.02, R = 0.1, A = 10^3, \) and \( H_g = 0.1 \). (b) Scaling law for the time between bursting events \( t_{\text{burst}} \) when attractive van der Waals forces dominate. Points represent data from numerical simulations of (7) and (8); the line corresponds to the scaling law (13).
[L_p(H_p/L_p)]v_p. Combining this with the condition that the time between bursting events \( t_{\text{burst}} \) is essentially a filling time, i.e., \( t_{\text{burst}}v_p = L \), we find that \( v_p/U \sim A(H_p^9/\epsilon H_p^9)^{1/5} \); hence

\[
t_{\text{burst}} \sim \left( \frac{L^5 \mu^5 R^{3/2}}{A^{5/3} \epsilon H_p L_p^6} \right)^{1/5}.
\]

This scaling law is confirmed in Fig. 3(b) over a certain range of parameter values but breaks down as \( A \) becomes small and the attractive forces become relatively weak. Choosing a typical value for the attractive Hamaker constant of \( A_a \sim 10^{-13} \) dyne cm, a flow rate of \( U \sim 1 \text{ cm/s} \), \( H_p \sim 100 \text{ nm} \), and \( H_p \sim 100 \text{ \mu m} \), gives a bursting time scale of \( t_{\text{burst}} \sim 0.1 \text{ s} \), suggesting that such a design might be experimentally feasible.

We conclude with a brief discussion of various generalizations of our ideas. These include the free motion of a falling sheet of paper, which requires an additional equation for the horizontal velocity of the center of mass of the sheet, the touch down of a sheet of paper (which admits a similarity solution), as well as generalizations to account for solid inertia to understand the flutter of the sheet in the context of voice and song production, all of which are subjects of current study.

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[10] The viscous shear stresses lead to a force of order \( \mu UL/H_p \) and a stretching strain of order \( \mu UL/H_p L_p^2 \). Comparing this with a typical bending strain \( bH/L_p^2 \) leads to a maximal length of the sheet \( L \sim bH^2/\mu U \) above which stretching effects cannot be ignored.