CS125 **Lecture 12** **Fall 2014**

12.1 Nondeterminism

The idea of "nondeterministic" computations is to allow our algorithms to make "guesses", and only require that they accept when the guesses are "correct". For example, a simple nondeterministic polynomial-time algorithm to decide whether a number N is composite would nondeterministically guess a factorization L,M of the number, and then verify that $L \cdot M = N$. (It turns out that there is also a deterministic polynomial-time algorithm for deciding compositeness, discovered in 2002, but it is much more complicated.)

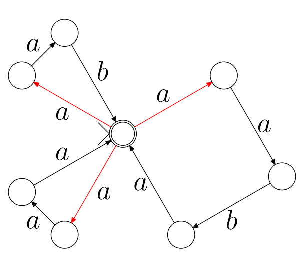
Nondeterminism is not a realistic or physical computational resource, but turns out to be very useful for capturing many computational problems of interest and better-understanding realistic deterministic models of computation. Just like introducing the imaginary number $i = \sqrt{-1}$ turns out to be very useful in answering questions about the ordinary real numbers.

12.2 Nondeterministic Finite Automata

A language for which it is hard to design a DFA:

$$L = \{aab, aaba, aaa\}^* = \{x_1 x_2 \cdots x_k : k \geq 0 \text{ and each } x_i \in \{aab, aaba, aaa\}\}.$$

But it is easy to imagine a "device" to recognize this language if there sometimes can be several possible transitions!



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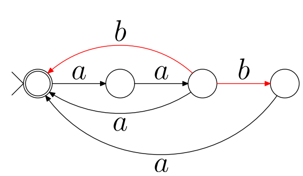

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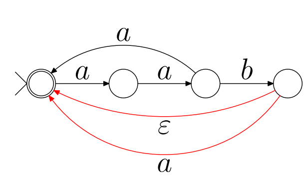


Def: An $\boxed{\text{NFA}}$ is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q, Σ, q_0, F are as for DFAs
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow P(Q)$.

When in state p reading symbol σ, can go to any state q in the set $\delta(p,\sigma)$.

- there may be more than one such q, or
- there may be none (in case $\delta(p,\sigma) = \emptyset$).

Can "jump" from p to any state in $\delta(p,\varepsilon)$ without moving the input head.

Computations by an NFA

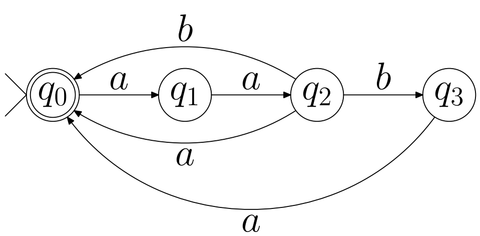
$N = (Q,\Sigma,\delta,q_0,F)$ accepts $w \in \Sigma^*$ if we can write $w = y_1 y_2 \cdots y_m$ where each $y_i \in \Sigma \cup \{\varepsilon\}$ and there exist $r_0,\ldots,r_m \in Q$ such that

1. $r_0 = q_0$,
2. $r_{i+1} \in \delta(r_i, y_{i+1})$ for each $i = 0,\ldots,m-1$, and
3. $r_m \in F$.

Nondeterminism: Given N and w, the states $r_0,\ldots,r_m$ are not necessarily determined.

Example of an NFA

N :



$N = (\{q_0,q_1,q_2,q_3\},\{a,b\},\delta,q_0,\{q_0\})$, where δ is given by:

	a	b	ε
q_0	$\{q_1\}$	$\emptyset$	$\emptyset$
q_1	$\{q_2\}$	$\emptyset$	$\emptyset$
q_2	$\{q_0\}$	$\{q_0,q_3\}$	$\emptyset$
q_3	$\{q_0\}$	$\emptyset$	$\emptyset$

Tree of computations

Tree of computations of NFA N on string *aabaab*:

An NFA N accepts w if there is at least one accepting computation path on input w, so we could check all computation paths to determine whether N accepts w. But the number of paths may grow exponentially with the length of w! Can the exponential search be avoided?

NFAs vs. DFAs

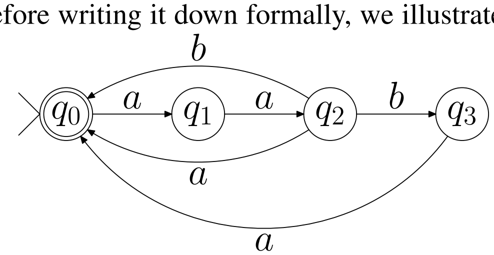
NFAs <u>seem</u> more "powerful" than DFAs. Are they?

Theorem 12.1 *For every NFA N, there exists a DFA M such that* $L(M) = L(N)$.

Proof by Construction: Given any NFA N, we construct a DFA M such that $L(M) = L(N)$. The idea is to have the DFA M keep track of the *set* of states that N could be in after having read the input string so far.

Before writing it down formally, we illustrate with an example. Recall our NFA N for $L = \{aab, aaba, aaa\}^*$.

N :



N starts in state q_0 so we will construct a DFA M starting in state $\{q_0\}$:

Formal Description of the Subset Construction

Given an NFA $N = (Q, \Sigma, \delta, q_0, F)$, we construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$ where

$$
\begin{aligned}
Q' &= P(Q) \\
q_0' &= E(\{q_0\}) \\
F' &= \{R \subseteq Q : R \cap F \neq \emptyset\} \text{ (that is, } R \in Q'\text{)} \\
\delta'(R, \sigma) &= E(\{q \in Q : q \in \delta(r, \sigma) \text{ for some } r \in R\}) \\
&= \bigcup_{r \in R} E(\delta(r, \sigma)),
\end{aligned}
$$

where for a set $S \subset Q$, $E(S)$ is the set of states that can be reached starting from a state in S and following 0 or more ε transitions.

It can be shown by induction on $|w|$ that for every string w, running M on input w ends in the state $\{q \in Q :$ some computation of N on input w ends in state $q\}$.

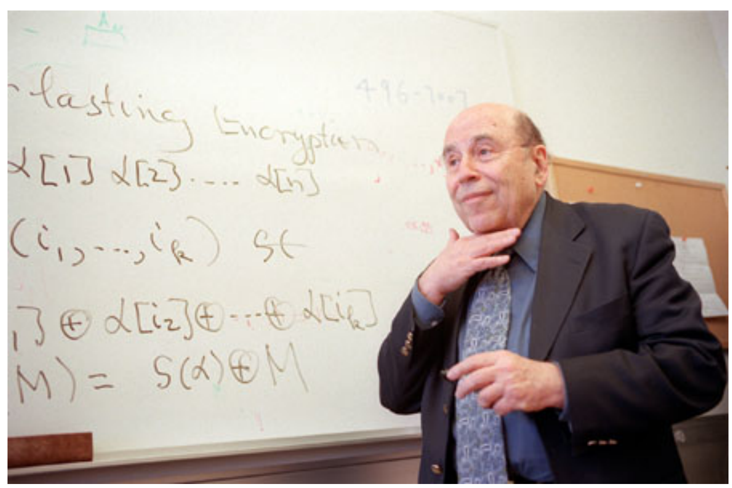
Rabin & Scott, "Finite Automata and Their Decision Problems," 1959

1976 – Michael O. Rabin ***See the ACM Author Profile in the Digital Library***

Citation
For their joint paper "Finite Automata and Their Decision Problem," which introduced the idea of nondeterministic machines, which has proved to be an enormously valuable concept. Their (Scott & Rabin) classic paper has been a continuous source of inspiration for subsequent work in this field.

Biographical Information

Michael O. Rabin (born 1931 in Breslau, Germany) is a noted computer scientist and a recipient of the Turing Award, the most prestigious award in the field.



Using NFAs for Pattern Recognition

NFAs can express quite complicated pattern-recognition problems. Indeed, it is easy to construct an NFA N that accepts exactly the strings generated by any given *regular expression*, such as

$$R = ((a \cup b \cup c \cup \cdots \cup z)^*(foo \cup bar)(a \cup b \cup c \cup \cdots \cup z)^*(foo \cup bar))^* \cup (a \cup b \cup c \cup \cdots \cup z)^*.$$

This regular expression R generates the set $L(R)$ of strings over alphabet $\Sigma = \{a, b, \ldots, z\}$ that have an even number of non-overlapping occurrences of "foo" or "bar". We can easily convert R (or any regular expression, for that matter) into an NFA N such $L(N) = L(R)$:

[The converse is also true (but harder to prove): for every NFA N, one can construct a regular expression R such that $L(R) = L(N)$. So DFAs, NFAs, and Regular Expressions all describe exactly the same set of languages! If you're interested in the full proof, see the recommended text by Sipser.]

So to decide whether a given string $w \in \Sigma^*$ matches a given regular expression R, we can convert R to an NFA N, convert N to a DFA M, and then run M on R.

Q: What's the problem with this approach? How can we do better?

Theorem 12.2 *Given an NFA $N = (Q, \Sigma, \delta, q_0, F)$ and a string w, we can decide whether $w \in L(N)$ in time $O(|Q|^2 \cdot |w|)$.*

Proof: