CS125	Lecture 11	Fall 2016

11.1 Finite Automata

Motivation:

- TMs without a tape: maybe we can at least fully understand such a simple model?
- Algorithms (e.g. string matching)
- Computing with very limited memory
- Formal verification of distributed protocols,
- Hardware and circuit design

Example: Home Stereo

- P = power button (ON/OFF)
- S = source button (CD/Radio/TV), only works when stereo is ON, but source remembered when stereo is OFF.
- Starts OFF, in CD mode.
- A computational problem: does a given a sequence of button presses $w \in \{P,S\}^*$ leave the system with the radio on?

The Home Stereo DFA

Formal Definition of a DFA

- A DFA M is a 5-Tuple $(Q, \Sigma, \delta, q_0, F)$

  Q : Finite set of states

  Σ : Alphabet

  δ : "Transition function", Q x $\Sigma \to Q$

  q_0: Start state, $q_0 \in Q$

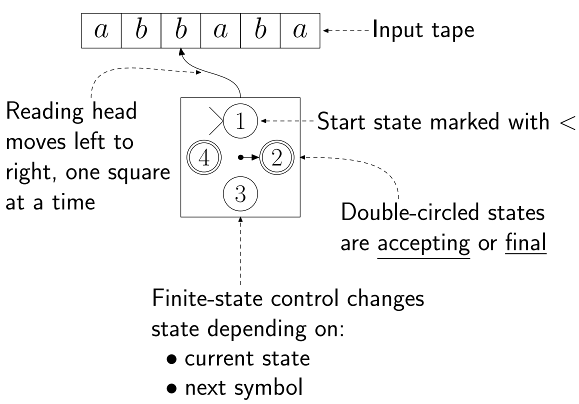
  F : Accept (or final) states, $F \subseteq Q$

- If $\delta(p, \sigma) = q$,

  then if M is in state p and reads symbol $\sigma \in \Sigma$

  then M enters state q (while moving to next input symbol)

Another Visualization

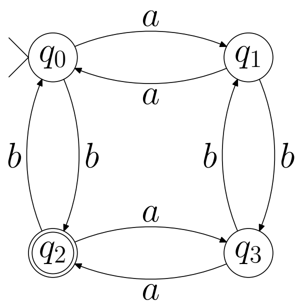


M accepts string x if

- After starting M in the start[initial] state with head on first square,
- when all of x has been read,
- M winds up in a final state.

Example

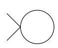
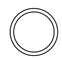
Bounded Counting: A DFA that recognizes $\{x : x \text{ has an even \# of } a\text{'s and an odd \# of } b\text{'s}\}$



Transition function δ:

	a	b
q_0	q_1	q_2
q_1	q_0	q_3
q_2	q_3	q_0
q_3	q_2	q_1.

i.e. $\delta(q_0, a) = q_1$, etc.

 = start state  = final state

$Q = \{q_0, q_1, q_2, q_3\}$ $\Sigma = \{a, b\}$ $F = \{q_2\}$

Formal Definition of Computation

$M = (Q, \Sigma, \delta, q_0, F)$ accepts $w = w_1 w_2 \cdots w_n \in \Sigma^*$ (where each $w_i \in \Sigma$) if there exist $r_0, \ldots, r_n \in Q$ such that

1. $r_0 = q_0$,
2. $\delta(r_i, w_{i+1}) = r_{i+1}$ for each $i = 0, \ldots, n-1$, and
3. $r_n \in F$.

The language recognized (or accepted) by M, denoted $L(M)$, is the set of all strings accepted by M.

Another Example, To Do On Your Own

- Pattern Recognition: A DFA that accepts $\{ x : x \text{ has } aab \text{ as a substring}\}$.

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Using DFAs for Pattern Recognition

Problem: given a *pattern* $w \in \Sigma^*$ of length m and a string $x \in \Sigma^*$ of length n, decide whether w is a substring of x.

Algorithm:

1. Construct a DFA M that accepts $L_w = \{x \in \Sigma^* : w \text{ is a substring of } x\}$.
   - States are $Q = \{0, 1, \ldots, m\}$. State q represents:
   - Transitions: $\delta(q, \sigma) =$
   - Time to construct M (naively): $O(m^3 \cdot |\Sigma|)$.
2. Run M on x.
   - Time: $O(n)$

The running time can be improved to $O(m+n)$, using an appropriate implicit representation of the DFA. Widely used in practice! (Look up the Knuth-Morris-Pratt algorithm.)

Characterizing the Power of Finite Automata

Def: A language $L \subseteq \Sigma^*$ is *regular* iff there is a DFA M such that $L(M) = L$. REG denotes the class of regular languages.

The terminology "regular" comes from an equivalent characterization in terms of *regular expressions* (which we won't cover in lecture, but possibly will on a problem set). Note that $\mathrm{REG} \subseteq \mathrm{TIME}_{\mathrm{TM}}(n)$; it also can be shown that $\mathrm{REG} \subseteq \mathrm{CF}$. Unlike classes associated with universal models (like TMs and Word-RAMs), we have a fairly complete understanding of the class of regular languages. In particular,

Myhill-Nerode Theorem: A language $L \subseteq \Sigma^*$ is regular iff there are only finitely many equivalence classes under the following equivalence relation $\sim_L$ on Σ^*: $x \sim_L y$ iff for all strings $z \in \Sigma^*$, we have $xz \in L \Leftrightarrow yz \in L$. Moreover, the minimum number of states in a DFA for L is exactly the number of equivalence classes under $\sim_L$.

(**Exercises:** refresh your memory on the definition of equivalence relations and equivalence classes.)

Proof: $\Rightarrow$. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA such that $L(M) = L$. Note that if $x, y \in \Sigma^*$ drive M to the same state (starting from q_0), then for all $z \in \Sigma^*$, xz and yz drive M to the same state and hence both are in $L(M) = L$ or neither are in $L(M)$. Thus $x \sim_L y$. Hence the number of equivalence classes under $\sim_L$ is at most $|Q|$.

$\Leftarrow$. Suppose $\sim_L$ has finitely many equivalence classes, where we write $[x]_L$ for the equivalence class containing x. We construct a DFA $M = (Q, \Sigma, \delta, q_0, F)$ as follows:

- Q is the set of equivalence classes under $\sim_L$.
- $q_0 = [\varepsilon]_L$.
- $F = \{[x]_L : x \in L\}$.
- $\delta([x]_L, \sigma) = [x\sigma]_L$. (Note that this is well-defined: if $x \sim_L y$, then $x\sigma \sim_L y\sigma$, so the choice of the representative x of the equivalence class does not affect the result.)

By induction on $|x|$, it can be shown that running M on x leads to state $[x]_L$, and hence we accept exactly the strings in L. ■

Proving that languages are nonregular. To show that L is nonregular, we only need to exhibit an infinite set of strings that are all inequivalent under $\sim_L$. Some examples follow:

- $L = \{a^n b^n : n \geq 0\}$. Exercise: prove that $\varepsilon, a, a^2, a^3, a^4, \ldots$ are all pairwise inequivalent under $\sim_L$.

- $L = \{w \in \Sigma^* : |w| = 2^n \text{ for some } n \geq 0\}$. Claim: $\varepsilon, a, a^2, a^3, a^4, \ldots$ are all inequivalent under $\sim_L$. Suppose $a^i \sim_L a^j$ for some $i > j$. Let k be any power of 2 larger than i and j. Then $a^j \cdot a^{k-j} \in L$, so $a^i \cdot a^{k-j} \in L$ and hence $k + i - j$ is a power of 2. But $2k$ is the next larger power of 2 after k. $\Rightarrow\Leftarrow$.

- $L = \{w \in \Sigma^* : w = w^R\}$ (palindromes). Exercise: prove that $a, a^2b, a^3b, \ldots$ are pairwise inequivalent.