CS125	Lecture 14	Fall 2016

NP-Completeness

The World if P $\neq$ NP?

Q: If P $\neq$ NP, can we conclude anything about any specific problems?

Idea: Try to find a "hardest" NP language.

- Want $L \in$ NP such that $L \in$ P iff every NP language is in P.

Reducibility

Informally, we say that a computational problem A reduces to a computational problem B (written $A \leq B$) if A can be solved (efficiently) by solving B. Thus, an (efficient) algorithm for B implies an (efficient) algorithm for A.

We have already seen many examples:

- CONTEXT-FREE RECOGNITION $\leq$ MATRIX MULTIPLICATION (HW3)
- MAX-FLOW $\leq$ LINEAR PROGRAMMING
- MATCHING $\leq$ MAX-FLOW
- ZERO-SUM GAMES $\leq$ LINEAR PROGRAMMING
- $L_{\text{fact}} \leq$ FACTORING
- FACTORING $\leq L_{\text{fact}}$

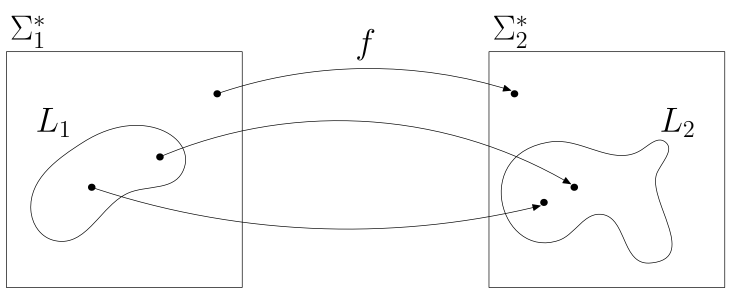
Here $L_{\text{fact}} = \{\langle N, m\rangle : N \text{ has a factor in } \{2,\ldots,m\}\}$. The last bullet follows since, to factor N, we can iteratively try to find one factor x then recurse on both x and N/x. To find a single factor, we can use a subroutine solving L_{fact} and binary search on m (recall to be efficient, our running time should be polylogarithmic in N, since the input length is $\lceil \log N \rceil$ bits to write down N). As the last bullet shows, reductions are useful not only for showing that problems can be solved efficiently, but also for giving evidence that problems are hard: under the widely believed conjecture that FACTORING has no polynomial-time algorithm, we can deduce that $L_{\text{fact}} \notin \text{P}$ (and hence $\text{P} \neq \text{NP}$). Hence "$A \leq B$" can be interpreted equivalently as saying "A is at least as easy as B" or "B is at least as hard as A".

Polynomial-Time Mapping Reductions

There are many forms of reducibility, and which one is most suitable depends on what kind of computational phenomena we are interested in studying. A very general notion is that of a *Turing reduction* (aka *oracle reduction*), where we say that $A \leq B$ if there is an algorithm that solves A given any "black box" that solves B. (For example, we add a Word-RAM instruction that will provide a solution to an instance of B written in memory in one time step. It's like programming with a library for which we have no idea how the the library functions themselves are implemented (or even if they can be implemented at all).) The polynomial-time analogue of Turing reductions are known as *Cook reductions*, and these are what we used in the reductions between FACTORING and L_{fact}.

However, for reductions between *languages*, it is often convenient to work with the following more restrictive notion of reduction (known as *polynomial-time mapping reductions* or *Karp reductions*):

Def: $L_1 \leq_P L_2$ iff there is a polynomial-time computable function $f : \Sigma_1^* \to \Sigma_2^*$ s.t. for every $x \in \Sigma_1^*$, $x \in L$ iff $f(x) \in L_2$.



- $x \in L_1 \Rightarrow f(x) \in L_2$
- $x \notin L_1 \Rightarrow f(x) \notin L_2$
- f computable in polynomial time

Proposition: If $L_1 \leq_P L_2$ and $L_2 \in \text{P}$, then $L_1 \in \text{P}$.

Proof:

Suppose that

- f is a reduction of L_1 to L_2 computable in time T_1, a polynomial.
- L_2 is decidable in time T_2, a polynomial.

To decide whether $x \in L_1$:

1. Compute $f(x)$. [takes time $T_1(|x|)$]
2. Decide whether $f(x) \in L_2$. [takes time $T_2(|f(x)|)$]

But we know that $|f(x)| \leq T_1(|x|)$, since the length of the output of a TM can't be longer than the time in which it runs.

Thus, $T_2(|f(x)|) \leq T_2(T_1(|x|))$.

So total time $\leq T_1(|x|) + T_2(T_1(|x|))$, a polynomial.

NP-Completeness

Def: L is NP-complete iff

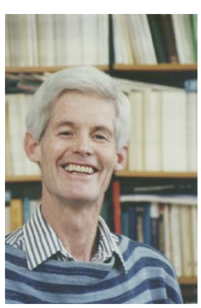
1. $L \in$ NP and
2. For every $L' \in$ NP, we have $L' \leq_P L$. ("L is NP-hard")

Prop: Let L be any NP-complete language. Then P = NP *if and only if* $L \in$ P.

Cook–Levin Theorem
(Stephen Cook 1971, Leonid Levin 1973)

Theorem: SAT (Boolean satisfiability) is NP-complete.

Proof: Need to show that every language in NP reduces to SAT (!) Proof next time.



More NP**-complete problems**

From now on we prove NP-completeness using:

Lemma: If we have the following

- L is in NP
- $L_0 \leq_P L$ for some NP-complete L_0

Then L is NP-complete.

Proof: Since by hypothesis $L \in \text{NP}$, it suffices to show that every $L' \in \text{NP}$ reduces to L.

- $L' \leq_P L_0$ since L_0 is NP-complete;
- $L_0 \leq_P L$ by hypothesis; and so
- $L' \leq_P L$ by transitivity.

Thus, L is NP-complete.

3-SAT

Def: A Boolean formula is in 3-CNF if it is of the form $C_1 \wedge C_2 \wedge \ldots \wedge C_n$, where each clause C_i is a disjunction ("or") of 3 literals:

$$C_i = (C_{i1} \vee C_{i2} \vee C_{i3})$$

where each literal C_{ij} is either a variable x, or the negation of a variable, $\neg x$ (sometimes written $\overline{x}$).

e.g. $(x \vee y \vee z) \wedge (\neg x \vee \neg u \vee w) \wedge (u \vee u \vee u)$

3-SAT is the set of satisfiable 3-CNF formulas.

Theorem: 3-SAT is NP-complete

Proof: We show that SAT $\leq_P$ 3-SAT.

1. Given an arbitrary Boolean formula, e.g.

$$F = (\underset{1}{\neg}((x \underset{2}{\vee} \underset{3}{\neg} y) \underset{4}{\wedge} (z \underset{5}{\vee} w)) \underset{6}{\vee} \underset{7}{\neg} x).$$

2. Number the operators.

3. Select a new variable a_i for each operator.

   The variable a_i is supposed to mean "the subformula rooted at operator i is true."

4. Write a formula F_1 stating the relation between each subformula and its children subformulas.

   For example, where

$$F = (\underset{1}{\neg}((x \underset{2}{\vee} \underset{3}{\neg} y) \underset{4}{\wedge} (z \underset{5}{\vee} w)) \underset{6}{\vee} \underset{7}{\neg} x),$$

$$F_1 = \left(\begin{array}{rcccc} & (a_3 \equiv \neg y) & \wedge & (a_7 \equiv \neg x) \\ \wedge & (a_2 \equiv x \vee a_3) & \wedge & (a_1 \equiv \neg a_4) \\ \wedge & (a_5 \equiv z \vee w) & \wedge & (a_6 \equiv a_1 \vee a_7) \\ \wedge & (a_4 \equiv a_2 \wedge a_5) & & \end{array} \right)$$

5. Let k be the number of the main operator/subformula of F.

   (Note: $k = 6$ in the example)

   Claim: $a_k \wedge F_1$ is satisfiable iff F is satisfiable.

6. Write F_1 in 3-CNF to obtain F_2.

   Fact: Every function $f : \{0,1\}^k \to \{0,1\}$ can be written as a k-CNF and as a k-DNF (OR of ANDs). [albeit with possibly 2^k clauses]

   Proof: Write the truth table for f. To obtain a k-DNF, for each row of the table for which $f(x) = 1$, we obtain a clause which ANDs all the literals in that row. We then OR these together over all such x. To obtain a k-CNF, we first build a k-DNF as in the last sentence for the function $\neg f$. This is the OR of many clauses: $C_1 \vee \ldots \vee C_m$. Each C_i is an AND of k literals. We then use De Morgan's laws to obtain $\neg(\neg f)$, which yields $\overline{C_1 \vee \ldots \vee C_m} = \overline{C_1} \wedge \ldots \wedge \overline{C_m}$, which is a k-CNF.

7. Output of the reduction: $a_k \wedge F_2$.

Execise: Note the above ingredients give us a CNF in which each clause has *at most* 3 literals. Some may have just 1 or 2. Show how to extend such clauses to have *exactly* 3 literals, from 3 distinct variables (hint: add new dummy variables and more clauses).

In contrast, 2-SAT $\in$ P

Method (resolution):

1. If x and $\neg x$ are both clauses, then not satisfiable

   e.g. $(x) \wedge (z \vee y) \wedge (\neg x)$

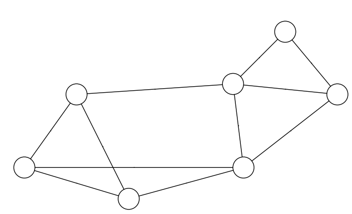
2. If $(x \vee y) \wedge (\neg y \vee z)$ are both clauses, add clause $(x \vee z)$ (which is implied).
3. Repeat. If no contradiction emerges $\Rightarrow$ satisfiable.

$O(n^2)$ repetitions of step 2 since only 2 literals/clause.

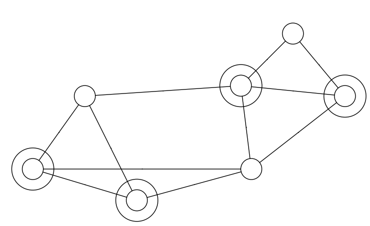
Proof of correctness: omitted

VERTEX COVER (VC)

- Instance:
  - a graph, e.g.



  - a number k (e.g. 4)
- Question: Is there a set of k vertices that "cover" the graph, i.e., include at least one endpoint of every edge?

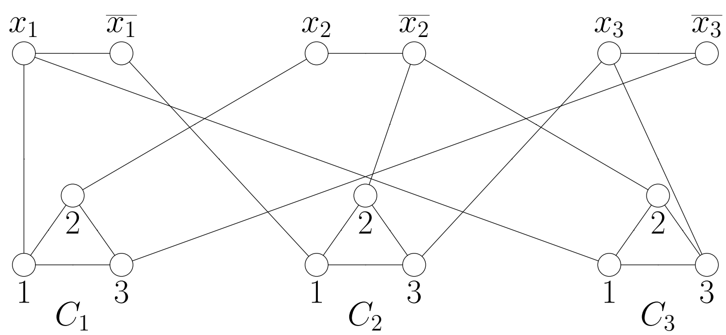


VC is NP-complete

- VC is in NP:
- 3-SAT $\leq_P$ VC:
  - Let F be a 3-CNF formula with clauses $C_1 \ldots, C_m$, variables $x_1, \ldots, x_n$.
  - We construct a graph G_F and a number N_F such that:

    G_F has a size N_F vertex cover iff F is satisfiable

    E.g. $F = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3)$

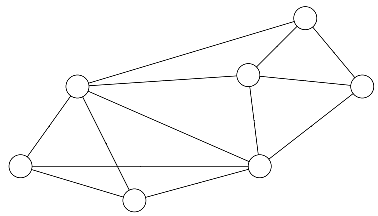


  - G_F = one dumbbell for each variable, one triangle for each clause, and corner j of triangle i is connected to the vertex representing the jth literal in C_i.
  - $N_F = 2m + n = 2$ (# clauses) $+$ (# variables).

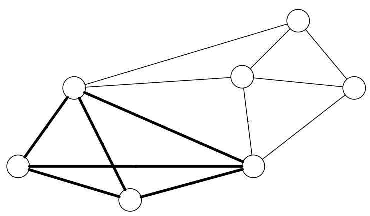
    $\Rightarrow$ 1 vertex from each dumbbell and 2 from each triangle.
  - **Exercise:** Show that F is satisfiable iff there is a cover of size N_F.

CLIQUE

- <u>Instance</u>:

- a graph, e.g. 
- a number k (e.g. 4)

• <u>Question</u>: Is there a clique of size k, i.e., a set of k vertices such that there is an edge between each pair?



• Easy to see that CLIQUE $\in$ NP.

VC $\leq_P$ CLIQUE

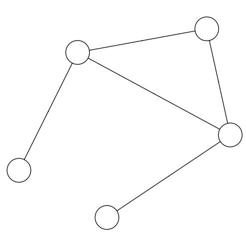
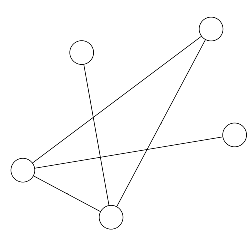
If G is any graph, let G^c be the graph with the same vertices such that:

there is an edge between x and y in G^c

iff

there is <u>no</u> edge between x and y in G

e.g.

$G =$  $G^c =$ 

• **Claim:** G has a k-cover iff G^c has an $(n-k)$-clique, where n is the number of vertices in G.

(So the mapping $(G,k) \mapsto (G^c, n-k)$ is a reduction of VC to CLIQUE.)

INTEGER LINEAR PROGRAMMING

An <u>integer linear program</u> is

- A set of variables $x_1,\dots,x_n$ which must take integer values.
- A set of linear inequalities:

  $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \le c_i$ $[i = 1,\dots,m]$

e.g. $x_1 - 2x_2 + x_4 \le 7$

$x_1 \ge 0$ $[-x_1 \le 0]$

$x_4 + x_1 \le 3$

ILP = the set of integer linear programs for which there are values for the variables that simultaneously satisfy all the inequalities.

ILP is NP-complete

ILP $\in$ NP. (Not obvious! Need a little math to prove it. The reason is that an integer solution might have really big integers – we need to make sure they only need a polynomial number of bits. Proof omitted.)

ILP is NP-hard: by reduction from 3-SAT (3-SAT $\le_P$ ILP). Given 3-CNF Formula F, construct following ILP P as follows.

If the variables are $x_1,\dots,x_n$, then we have the constraints $0 \le x_1,\dots,x_n \le 1$. Also, if there are m clauses, we have constraints $c_1,\dots,c_m \ge 1$, one for each clause. We also have a separate constraint for each clause. If the ith clause is, for exampe, $x_{i_1} \vee \bar{x}_{i_2} \vee x_{i_3}$, then we have a constraint $c_i \le x_{i_1} + (1 - x_{i_2}) + x_{i_3}$.

Recall: LINEAR PROGRAMMING where the variables can take *real* values is known to be in P.

More NP-complete/NP-hard Problems

- HAMILTONIAN CIRCUIT (and hence TRAVELLING SALESMAN PROBLEM) (see Sipser text for related problems)

- Scheduling
- Circuit Minimization
- Short Proof
- Nash Equilibrium with Maximum Payoff
- Protein Folding
- $\vdots$
- See book by Garey & Johnson for hundreds more.