Homework #3
Nonlinear dynamics and chaos

1. **Transformation to normal form:** (Strogatz 3.2.7). Follow the steps outlined in Strogatz exercise (3.2.6) to show that one can transform the equation $\dot{X} = RX - X^2 + a_n X^n + O(X^{n+1})$ where $n \geq 3$ to the form $\dot{x} = Rx - x^2 + O(x^{n+1})$ by a near identity transformation $x = X + b_n X^n + O(X^{n+1})$ for an appropriate choice of b_n. Note that the new form is closer to the normal form as its residual terms are smaller, and that this procedure may be repeated indefinitely to transform the equation arbitrarily close to the normal form. This is the basis for determining the bifurcation type from the first few terms in the Taylor expansion.

2. **Flows and bifurcations on a circle:** (Strogatz 4.3.3,4,6,8) For each of the following flows on a circle, draw the phase portrait as a function of the control parameter μ. Classify the bifurcations that occur as μ varies, and find all the bifurcation values of μ:
   (a) $\dot{\theta} = \mu \sin\theta - \sin 2\theta$
   (b) $\dot{\theta} = \mu + \cos\theta + cos 2\theta$
   (c) $\dot{\theta} = \mu + \sin\theta + \cos 2\theta$
   (d) $\dot{\theta} = \frac{\sin 2\theta}{1+\mu\sin\theta}$

3. **Numerics:** solve $\dot{x} = \sin x$ for 20 different initial conditions in the range $-4\pi < x(t=0) < 4\pi$, plot $x(t)$ for several of these cases for some reasonable range of t. Use the modified Euler scheme, can start with the sample program `modified_euler_1d.m` from course home page.

4. **Optional challenge problem:** Consider the following system

$$\dot{x} = x^3 + \delta x^2 - \mu x$$

   determine the fixed points of this system and study the bifurcations in the (x,μ) plane for zero and nonzero values of δ. Show that the pitchfork bifurcation at $(0,0)$ for $\delta = 0$ becomes a transcritical bifurcation for small δ. Find in particular what happens at $(-\frac{1}{2}\delta, -\frac{1}{4}\delta^2)$. Sketch the bifurcation diagram in the (x,μ) plane.