Homework #4
Nonlinear dynamics and chaos

1. **Circle map experimentation and Phase locking:** Consider the circle map,

$$\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_n) \qquad mod\ 1 \tag{1}$$

a Matlab program for the circle map is on the course home page.

   (a) Find a period 2 solution ($p/q = 1/2$) for $K > 1/2$, $\Omega \neq 1/2$.

      i. Show that the solution does not depend on the initial conditions θ_0 by iterating the map from 2 different initial conditions, converging to the same period-2 solution.

      ii. Find another ($p/q = 1/2$) solution for different K, Ω, with again $K > 1/2$, $\Omega \neq 1/2$, show that it is different from the previous one although it has the same period

   (b) Find a $p/q = 0/1$ solution for $\Omega \neq 0$, $K > 1/2$. Describe the behavior of this solution as function of n. Do the same for a $p/q = 1/1$ solution for $\Omega \neq 1$, $K > 1/2$.

   (c) Find a solution $p/q = 3/4$ for $K > 1/2$.

   (d) Try a few cases with $K > 1$. What happens?

2. **Linearized 2d systems:**

Classify the stability of the fixed points of the following systems by solving for their eigenvalues/ vectors and plotting the vector field in the phase plane. If the eigenvectors are real, plot them in phase space. Can use the quiver function of Matlab for the plot.

$$\dot{x} = x - y; \qquad \dot{y} = x + y \tag{2}$$

$$\dot{x} = 5x + 2y; \qquad \dot{y} = -17x - 5y \tag{3}$$

$$\dot{x} = 5x + 10y; \qquad \dot{y} = -x - y \tag{4}$$

3. **Nonlinear 2d systems:**

Find the fixed points, classify them, sketch neighboring trajectories, and try to fill in the rest of the phase space portrait:

$$\dot{x} = x - y; \qquad \dot{y} = x^2 - 4 \tag{5}$$

$$\dot{x} = \sin y; \qquad \dot{y} = \cos x \tag{6}$$

$$\dot{x} = xy - 1; \qquad \dot{y} = x - y^3 \tag{7}$$

$$\dot{x} = xy; \qquad \dot{y} = x^2 - y \qquad \text{beware, linearization fails here. why?} \tag{8}$$

4. **Analytic determination of Arnold tongues in circle map near $K = 0$:** Consider the circle map

$$\theta_{n+1} = F(\theta_n) = \theta_n + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_n) \qquad (\text{mod } 1). \tag{9}$$

It can be shown that the rotation (winding) number, defined as the limit $w = \lim_{n\to\infty}(\theta_n - \theta_o)/n$ (where θ_n is not taken mod 1 for the purpose of calculating the winding number) is p/q if and only if

$$F^q(\theta) - (\theta + p) = 0. \tag{10}$$

First, test this numerically for some two different values of (p,q). (**Optional (yet easy):** prove that if condition (10) is satisfied, the winding number is indeed p/q). Next, use this relation to show that the edges of the Arnold tongues $q = 1$ and $p = 0$ or $p = 1$ are at $\Omega = K/(2\pi)$ (for $p = 0$) and $\Omega = 1 - K/(2\pi)$ (for $p = 1$). Hint: use (9) and (10) to find the condition satisfied by θ_n within the appropriate tongue, and then deduce a condition for the edge of the tongue.

5. **Challenge/ Extra credit:** Using the same approach as the previous question, show that the boundary of the Arnold tongues $p = 1$ and $q = 2$ for small K is given by $\Omega = \frac{1}{2} \pm \frac{K^2}{8\pi}$. Hint: write $\Omega = 1/2 + \varepsilon$ and expand the relevant equations in terms of the small parameters K and ε.