Dorian Abbot
APM147
10/28/04

Problem Set 5 Solutions

1. (a) We consider $-\pi \leq \theta \leq \pi$. The pendulum pointing down corresponds to θ=0 and the pendulum pointing up corresponds to $\theta = \pm\pi$. Define $\nu \equiv \dot{\theta}$. The system can be rewritten:

$$\dot{\theta} = \nu \tag{1}$$
$$\dot{\nu} = sin(\theta)[\omega^2 cos(\theta) - \frac{g}{R}] \tag{2}$$

Notice that this system is reversible. The fixed points of this system are at ν=0 and $\theta = 0,\ \pm\pi,\ \pm arcos(\frac{g}{R\omega^2})$. Calculate the Jacobian:

$$J = \begin{pmatrix} 0 & 1 \\ \omega^2(cos^2(\theta) - sin^2(\theta)) - \frac{g}{R}cos(\theta) & 0 \end{pmatrix}$$

The trace and determinant of the Jacobian are:

$$\tau = 0 \tag{3}$$
$$\Delta = \frac{g}{R}cos(\theta) + \omega^2(sin^2(\theta) - cos^2(\theta)) \tag{4}$$

Evaluate Δ at the fixed points:

$$\Delta|_{(0,0)} = \omega^2(\frac{g}{R\omega^2} - 1) \tag{5}$$
$$\Delta|_{(\pm\pi,0)} = -\omega^2(\frac{g}{R\omega^2} + 1) \tag{6}$$
$$\Delta|_{(\pm arcos(\frac{g}{R\omega^2}),0)} = \omega^2(1 - (\frac{g}{R\omega^2})^2) \tag{7}$$

i. $\frac{g}{R\omega^2} < 1$
The f.p. at (0,0) is a saddle. The f.p. at ($\pm\pi$,0) is a saddle. The f.p. at ($\pm arcos(\frac{g}{R\omega^2})$,0) are nonlinear centers.

ii. $\frac{g}{R\omega^2} = 1$
The f.p. at ($\pm\pi$,0) is a saddle. There is another f.p. at (0,0) which is a center (from plotting).

iii. $\frac{g}{R\omega^2} > 1$
The f.p. at (0,0) is a center. The f.p. at ($\pm\pi$,0) is a saddle. The f.p. at ($\pm arcos(\frac{g}{R\omega^2})$,0) don't exist.

(b) If we include friction our equations can be written:

$$\dot{\theta} = \nu \tag{8}$$
$$\dot{\nu} = sin(\theta)[\omega^2 cos(\theta) - \frac{g}{R}] - 2\mu\nu \tag{9}$$

The system is no longer reversible, so we don't expect to get centers. We get the same fixed points as before. The Jacobian is:

$$J = \begin{pmatrix} 0 & 1 \\ \omega^2(cos^2(\theta) - sin^2(\theta)) - \frac{g}{R}cos(\theta) & -2\mu \end{pmatrix}$$

The trace and determinant of the Jacobian are:

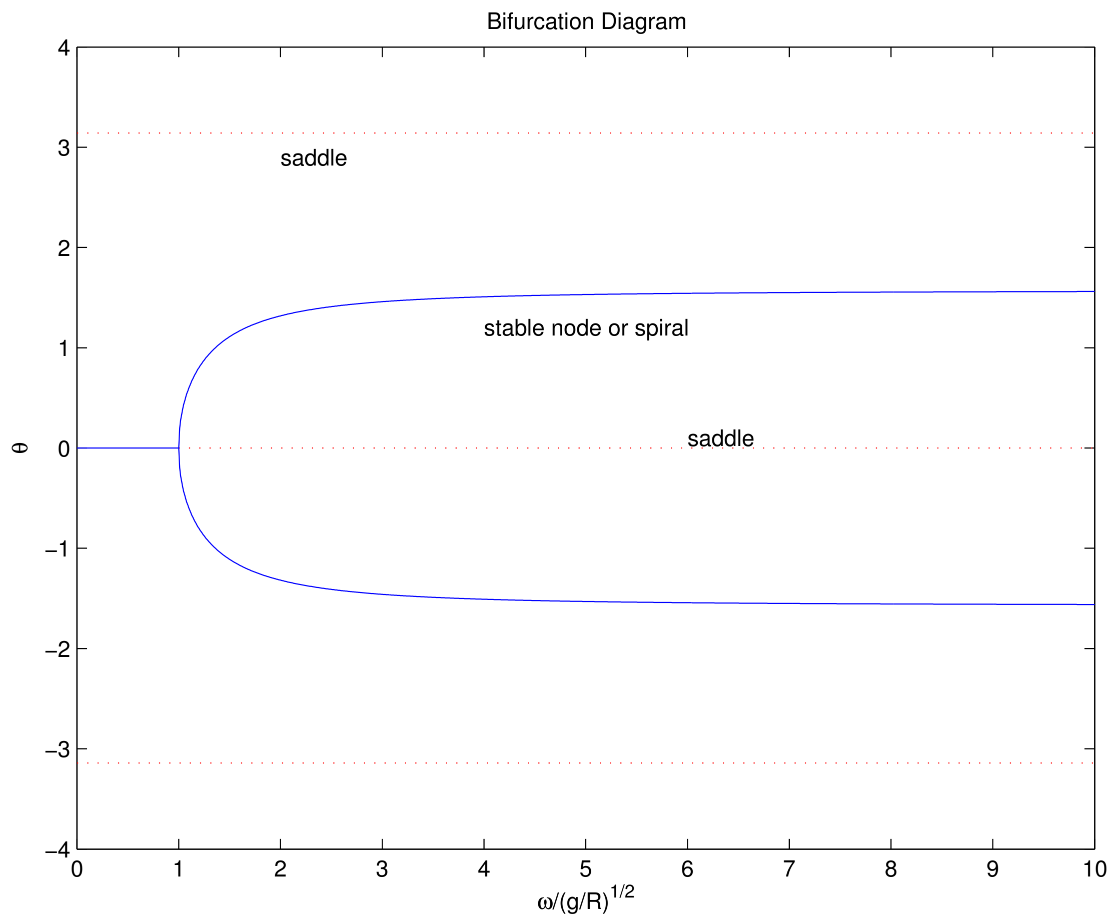
$$\begin{aligned} \tau &= -2\mu && (10) \\ \Delta &= \frac{g}{R}cos(\theta) + \omega^2(sin^2(\theta) - cos^2(\theta)) && (11) \end{aligned}$$

Now let's consider our cases again:

(a) $\frac{g}{R\omega^2} < 1$
The f.p. at (0,0) is a saddle. The f.p. at ($\pm\pi$,0) is a saddle. The f.p. at ($\pm arcos(\frac{g}{R\omega^2})$,0) are either stable nodes or stable spirals. Whether $\tau^2 - 4\Delta = \mu^2 - \omega^2 + \frac{g^2}{R^2\omega^4}$ is greater than or less than zero determines whether these points will be nodes or spirals.

(b) $\frac{g}{R\omega^2} = 1$
At this point a supercritical pitchfork bifurcation occurs.

(c) $\frac{g}{R\omega^2} > 1$
The f.p. at ($\pm\pi$,0) is a saddle. The f.p. at (0,0) is either a stable spiral or a stable node. Whether $\tau^2 - 4\Delta = \mu^2 - \frac{g}{R} + \omega^2$ is greater than or less than zero determines whether these points will be nodes or spirals. The f.p. at ($\pm arcos(\frac{g}{R\omega^2})$,0) don't exist.
Here is a bifurcation diagram:



Bifurcation Diagram
saddle
stable node or spiral
saddle
θ
ω/(g/R)^1/2

2. First notice that the paper cited was written by Harvard's very own Professor Howard Stone!

   (a) The system is invariant under the transformation $t \to -t$, $\phi \to -\phi$, $x \to x$, so it is reversible.

   (b) $\dot{x} = 0$ at $\phi = 0, -\pi$ and x=0,1 while $\dot{\phi} = 0$ at $x = 8(\frac{\sqrt{2}\beta}{cos(\phi)} - 1)$.
   Now consider the intersection of the nullclines. At $\phi = -\pi$ we can never get a f.p. with positive β and x. At $\phi = 0$ there will be a f.p. at x=8($\sqrt{2}\beta - 1$) for $\frac{1}{\sqrt{2}} < \beta < \frac{9}{8\sqrt{2}}$. At x=0 there will be fixed points at $\phi = \pm arccos(\sqrt{2}\beta)$ for $0 \leq \beta \leq \frac{1}{\sqrt{2}}$. At x=1 there will be fixed points at $\phi = \pm arccos(\frac{8\sqrt{2}\beta}{9})$ for $0 \leq \beta \leq \frac{9}{8\sqrt{2}}$.
   So for $\frac{1}{\sqrt{2}} < \beta < \frac{9}{8\sqrt{2}}$ there are three fixed points. The Jacobian of the system is:

$$J = \begin{pmatrix} \frac{\sqrt{2}}{4}(2x-1)sin(\phi) & \frac{\sqrt{2}}{4}x(x-1)cos(\phi) \\ -\frac{1}{16\sqrt{2}}cos(\phi) & \frac{sin(\phi)}{2\sqrt{2}}(\frac{x}{8}+1) \end{pmatrix}$$

   The trace and determinant of the Jacobian are:

$$\tau = \frac{17\sqrt{2}}{32}xsin(\phi) \tag{12}$$

$$\Delta = \frac{1}{8}(\frac{x}{8}+1)(2x-1)sin^2(\phi) + \frac{1}{64}x(x-1)cos^2(\phi) \tag{13}$$

   At (ϕ,x)=(0,8($\sqrt{2}\beta - 1$)), we have:

$$\tau = 0 \tag{14}$$

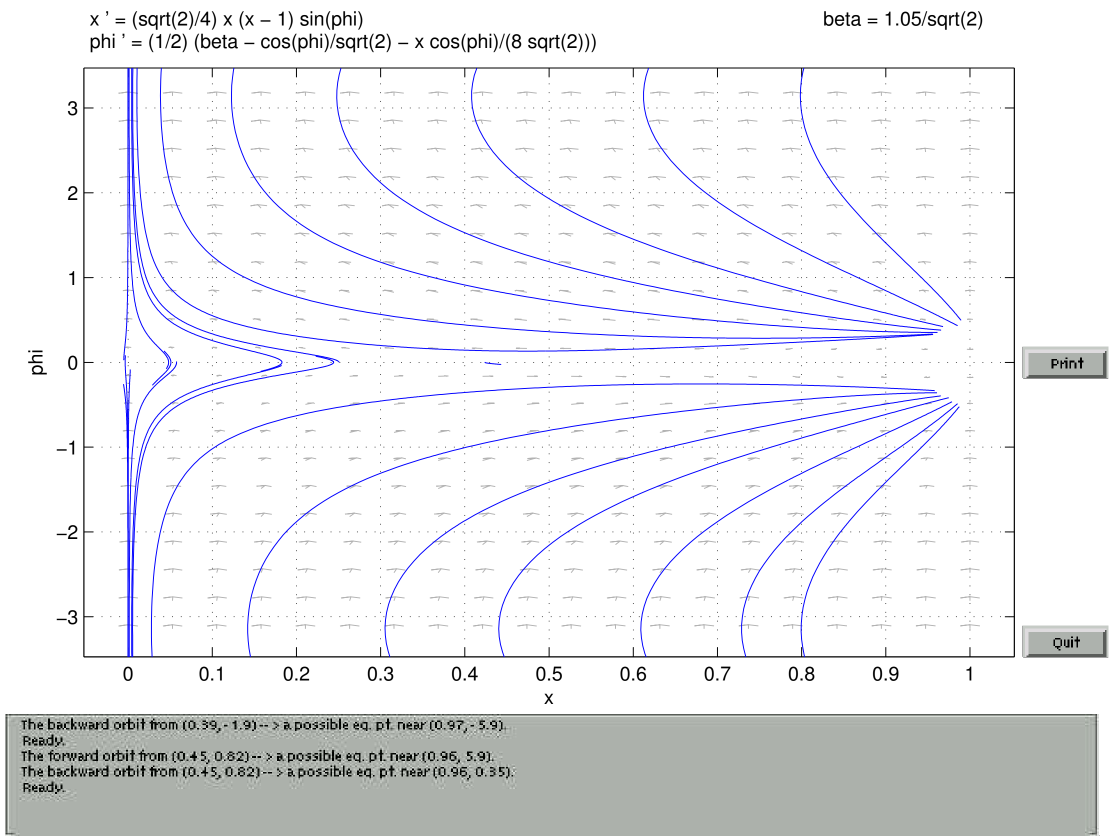
$$\Delta = (\sqrt{2}\beta - 1)(\sqrt{2}\beta - \frac{9}{8}) \tag{15}$$

   So for $\frac{1}{\sqrt{2}} < \beta < \frac{9}{8\sqrt{2}}$ (when this f.p. exists) Δ<0 and it is a saddle.
   At $(\phi, x) = (\pm arccos(\frac{8\sqrt{2}\beta}{9}), 1)$, we have:

$$\tau = \pm\frac{17\sqrt{2}}{32}sin(\phi) \tag{16}$$

$$\Delta = \frac{9}{64}sin^2(\phi) \tag{17}$$

   So $\tau^2 - 4\Delta = sin^2(\phi)((\frac{17\sqrt{2}}{32})^2 - \frac{9}{16}) = \frac{sin^2(\phi)}{512}$. So the f.p. at positive ϕ is an unstable node and the f.p. at negative ϕ is a stable node.
   At x=0, $\dot{x}$=0 and for our β range $\dot{\phi} \neq 0$, so a closed orbit winds around the cylinder. For x near one, the stable node at $(\phi, x) = (+arccos(\frac{8\sqrt{2}\beta}{9}, 1)$ is approached at long times. So there must be a homoclinic orbit that intersects with the saddle point and winds around the cylinder to separate these two domains.
   By the Poincare-Bendixon theorem there must be at least one closed orbit between x=0 and the homoclinic orbit. We expect to have non-isolated closed orbits in reversible nonlinear systems, so we expect to have an entire band of closed orbits in this region.
   Here is a phase portrait:


x ' = (sqrt(2)/4) x (x − 1) sin(phi)
phi ' = (1/2) (beta − cos(phi)/sqrt(2) − x cos(phi)/(8 sqrt(2)))
beta = 1.05/sqrt(2)
phi
3
2
1
0
−1
−2
−3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
Print
Quit
The backward orbit from (0.39, - 1.9) -- > a possible eq. pt. near (0.97, - 5.9).
Ready.
The forward orbit from (0.45, 0.82) -- > a possible eq. pt. near (0.96, 5.9).
The backward orbit from (0.45, 0.82) -- > a possible eq. pt. near (0.96, 0.35).
Ready.

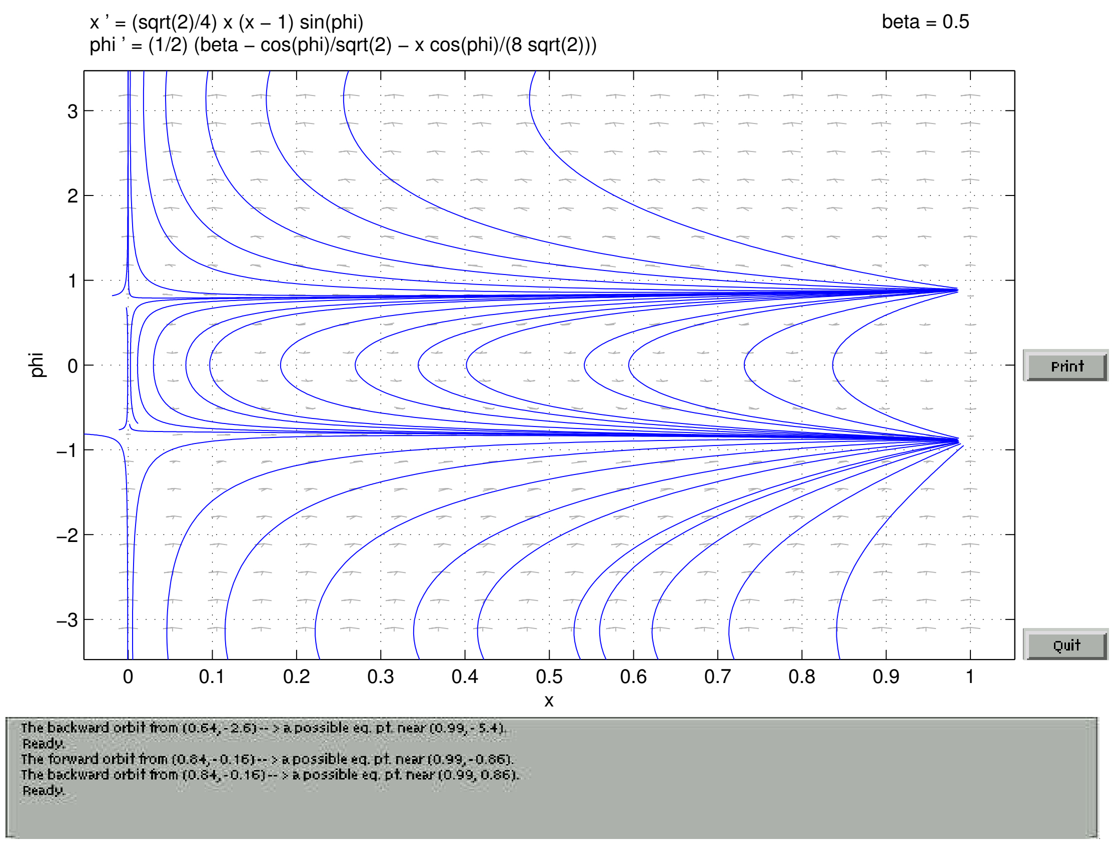
(c) Define $\beta \equiv \frac{1}{\sqrt{2}} + \alpha$. So $\beta \to \frac{1}{\sqrt{2}}$ from above corresponds to $\alpha \to 0$. The position of the saddle is $(\frac{8\alpha}{\sqrt{2}},0)$, so the saddle moves toward the line x=0 as $\alpha \to 0$, shrinking the region where the closed orbits exist. At $\alpha = 0$ ($\beta = \frac{1}{\sqrt{2}}$), the homoclinic orbit becomes the circle in phase space at x=0 and the band of closed orbits disappears.

(d) For $0 < \beta < \frac{1}{\sqrt{2}}$ we must consider the fixed points at $(\phi, x) = (\pm arccos(\sqrt{2}\beta), 0)$.

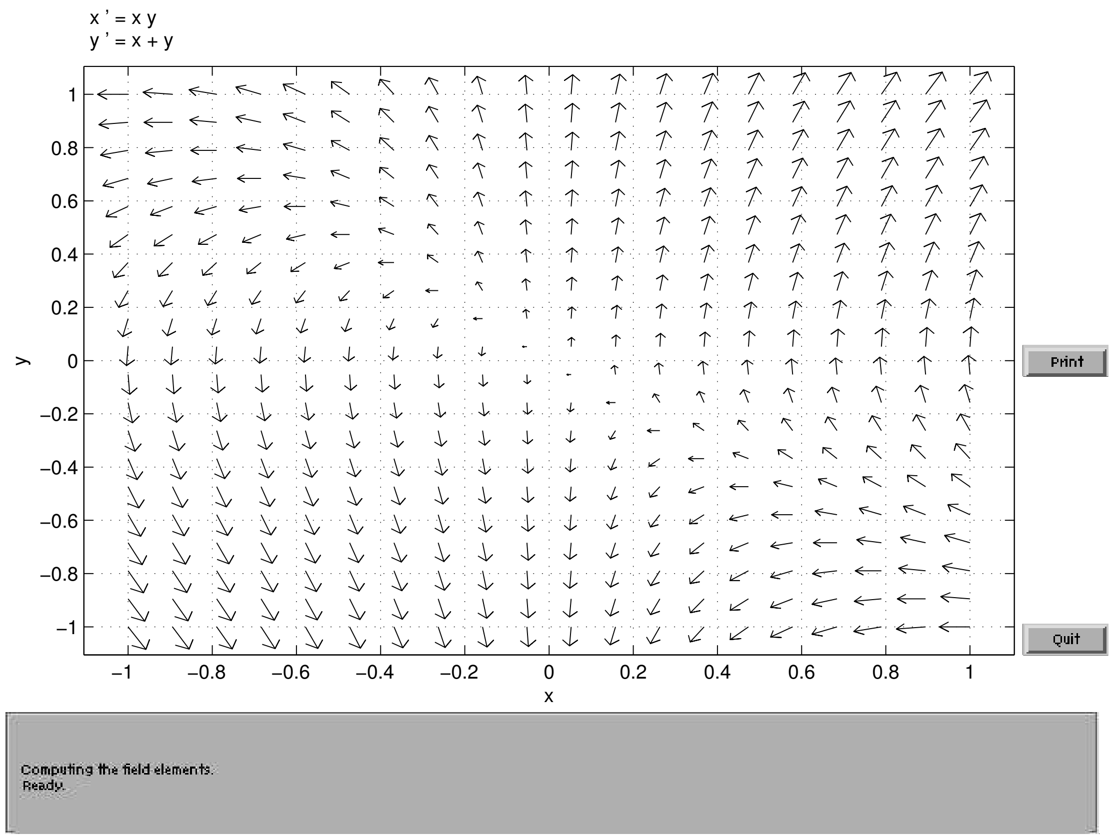
$$\tau = 0 \tag{18}$$

$$\Delta = -\frac{1}{8} sin^2(\phi) \tag{19}$$

So these points are saddles. Here is phase portrait:



3. The only fixed point occurs at the origin. Linearization tells us to expect a non-isolated fixed point, but by plotting the nullclines we can see that this is not the case. From the plot of the vector field below, we can see that the index around the fixed point at the origin is zero.



4. (a) The only fixed point is at the origin. We'd like to construct a Liapunov function, V($\mathbf{x}$), such that V($\mathbf{x}$)>0 and $\frac{dV}{dt}$<0 for all $\mathbf{x} \neq \mathbf{x}^*$. Consider V=$x^2+ay^2$ with a>0. The first condition is automatically satisfied. Let's investigate the second:

$$\begin{aligned} \dot{V} &= 2x\dot{x}+2ay\dot{y} && (20)\\ &= 2x(y-x^3)+2ay(-x-y^3) && (21)\\ &= 2(1-a)xy-2x^4-2ay^4 && (22)\\ & && (23) \end{aligned}$$

So the second condition is satisfied if we choose a=1. Our Liapunov function is V=x^2+y^2.

A system that has a Liapunov function cannot have a closed orbit. Assume such a system did have a closed orbit and consider the point $\mathbf{x_0}$ on it. After one circuit the system would have to return to $\mathbf{x_0}$ and we would have ΔV=0 (since V depends only on position). But we know that ΔV must also be given by $\Delta V = \int_0^T \dot{V}dt$ where T is the period of the orbit. Since the closed orbit cannot intersect with the origin, this tells us that ΔV is strictly less than zero - a contradiction!

So there can be no limit cycle for this system.

(b) If we have a gradient system, then f=-$\frac{\partial V}{\partial x}$ and g=-$\frac{\partial V}{\partial y}$, so as long as V is well-behaved we know:

$$\frac{\partial f}{\partial y}-\frac{\partial g}{\partial x} = -\frac{\partial^2 V}{\partial y \partial x}+\frac{\partial^2 V}{\partial x \partial y} = 0 \tag{24}$$

Now let's do a proof in the opposite direction. Start with $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ and integrate over x and y:

$$\begin{aligned} \int^x\int^y \frac{\partial f}{\partial y}dy'dx' &= \int^x\int^y \frac{\partial g}{\partial x}dy'dx' && (25)\\ \int^x f dx' &= \int^y g dy' && (26) \end{aligned}$$

So if we define $V(x,y) = -\int^x f dx' = -\int^y g dy$ we are guaranteed that f=-$\frac{\partial V}{\partial x}$ and g=-$\frac{\partial V}{\partial y}$, *i.e.* the system is a gradient system.

$\frac{\partial f}{\partial y}$=1+2x and $\frac{\partial g}{\partial x}$=1+2x so $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, the system is a gradient system.

Let's assume we have a limit cycle. If we take one trip around it and return to our starting point, we must have ΔV=0 for the trip (since V must be single-valued). However, if the cycle has period T, we can also write: $\Delta V = \int_0^T \dot{V}dt = \int_0^T(\nabla V \cdot \dot{\mathbf{x}})dt = -\int_0^T |\dot{\mathbf{x}}|^2 dt$. For a cycle to exist, we cannot have $\dot{\mathbf{x}} = 0$ (or else motion would stop) so we have shown that ΔV must be strictly less than zero. This is a contradiction, so gradient systems cannot have limit cycles.

Since this system is a gradient system, it can have no limit cycle.