Homework #5

Nonlinear dynamics and chaos

1. A bead of mass $m$ sliding on a rotating circulation loop of radius $R$ is described by

$$\ddot{\theta} + 2\mu\dot{\theta} + (g/R)\sin\theta - \omega^2 \sin\theta\cos\theta = 0$$

Here, $\theta$ describes the angular position of the bead on the hoop, $g$ is acceleration due to gravity, $\mu$ is a measure of the friction experienced by the bead, and $\omega$ is the angular velocity of the hoop.

   (a) for $\mu = 0$, determine the fixed points (equilibrium positions) of the system, and sketch the two dimensional phase portrait in each of the following cases: (i) $\omega < g/R$; (ii) $\omega^2 = g/R$; (iii) $\omega^2 > g/R$. You may want to use a modified version of the Matlab program `my_quiver.m` from the course home page.

   (b) for $\mu > 0$, choose $\omega$ as a control parameter, and examine the different bifurcations of fixed points that occur as $\omega$ is increased from zero. Construct an appropriate bifurcation diagram.

2. Reversible system on a cylinder: Do problem 6.6.8 from Strogatz, page 191.

3. Find fixed points, draw vector field around them, and calculate their index for the system:

$$\dot{x} = xy; \qquad \dot{y} = x + y$$

4. Do the following systems have a limit cycle solution?

   (a) (Construct a Lyapunov function...):

$$\dot{x} = y - x^3; \qquad \dot{y} = -x - y^3$$

   (b) Consider the system:

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

   Defining $\mathbf{x} \equiv (x,y)$, this system can be written $\dot{\mathbf{x}} = (f(x,y), g(x,y))$. If for some V we can write $\dot{\mathbf{x}} = -\nabla V$, then this is called a gradient system. Show that $\partial f/\partial y - \partial g/\partial x = 0$ if and only if this is a gradient system.

Using the above, does the following system have a limit cycle solution:

$$\dot{x} = y + 2xy; \qquad \dot{y} = x + x^2 - y^2$$