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APM147
11/18/04

Problem Set 8 Solutions

1. The equation for r contains the important dynamics in this system. There will be a f.p. at r=0 for all μ. It will be stable for μ<0 and unstable for μ>0. As $\mu \to$0 from above a stable limit cycle intersects with this f.p. This is a supercritical Hopf bifurcation.

   Limit cycles will occur in stable/unstable pairs for μ=sin(r). For -1<μ<1 there will be an infinite number of such pairs. For μ=1 there are an infinite number of saddle-node bifurcations of cycles at r=$(2n+\frac{1}{2})\pi$ while for μ=-1 there are an infinite number of saddle-node bifurcations of cycles at r=$(2n+\frac{3}{2})\pi$. Here n is an integer.

   For μ<1 all trajectories go toward the origin and for μ>1 all trajectories go toward r=∞.

2. The Lorenz system is:

$$\begin{aligned} \dot{x} &= \sigma(y-x) && (1) \\ \dot{y} &= rx-y-xz && (2) \\ \dot{z} &= xy-bz && (3) \end{aligned}$$

   One f.p. is at the origin, while the other two are denoted by C^+/C^- and are located at:

$$C^{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1) \quad (4)$$

   These f.p. only exist for r>1. The Jacobian of the system is:

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

   Evaluate at C^+. By symmetry we only need to consider one of the C f.p.

$$J|_{C^+} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{pmatrix}$$

   Now calculate the eigenvalues:

$$|J|_{C^+} - \lambda I| = \begin{vmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{vmatrix} = 0$$

   The characteristic equation can be reduced to:

$$\lambda^3 + (b+\sigma+1)\lambda^2 + (r+\sigma)b\lambda + 2b\sigma(r-1) = 0 \quad (5)$$

Here we look for a Hopf bifurcation, so at the bifurcation point we expect to have two eigenvalues of the form $\lambda_{\pm} = \pm i\omega$ for some nonzero ω. Substitute this ansatz into the characteristic equation:

$$\pm i\omega(-\omega^2 + (r_h + \sigma)b) + (2b\sigma(r_h - 1) - (b + \sigma + 1)\omega^2) = 0 \tag{6}$$

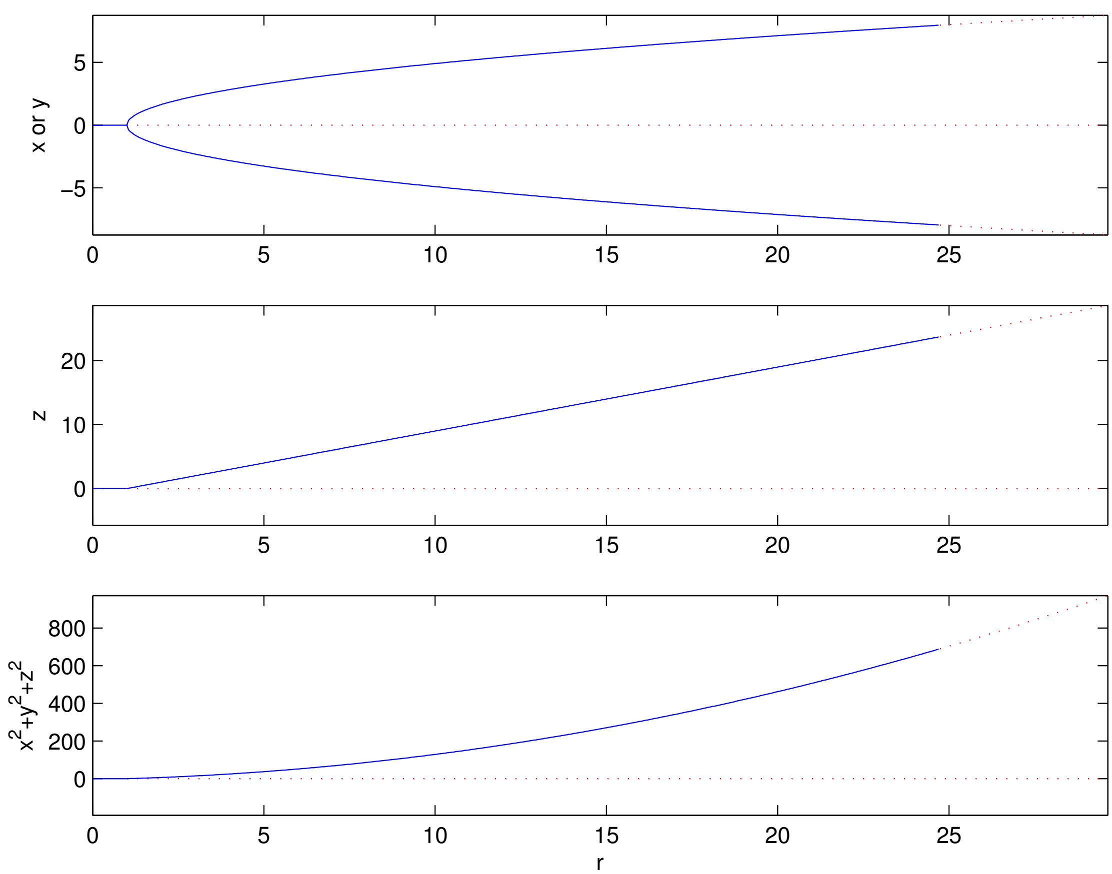
We require the real and imaginary parts of this equation to be satisfied. From the real part we see that:

$$\omega^2 = \frac{2b\sigma(r_h - 1)}{b + \sigma + 1} \tag{7}$$

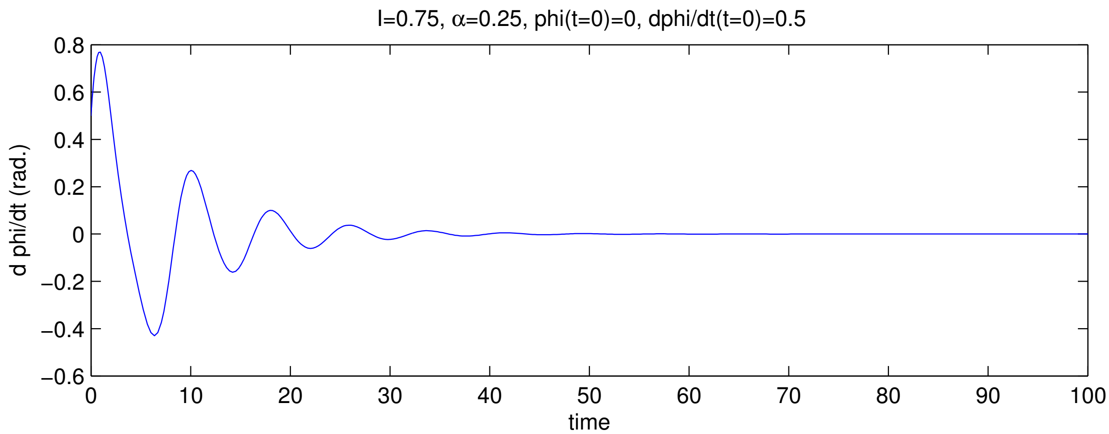
Which we can use with the imaginary part to see that:

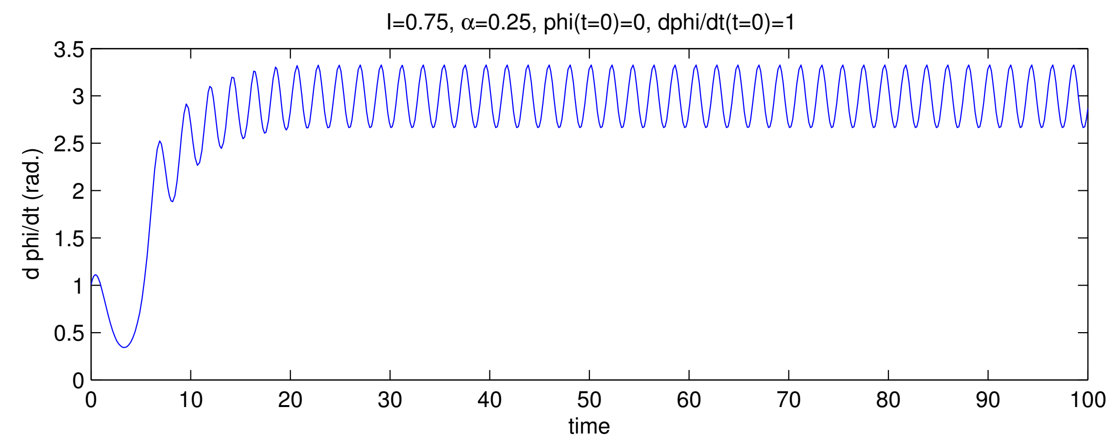
$$r_h = \sigma \frac{\sigma + b + 3}{\sigma - b - 1} \tag{8}$$

3. When we have a 1D bifurcation in a system with many dimensions, the scalar we choose to plot against the bifurcation parameter on a bifurcation diagram can be crucial to our understanding. In this case the plots against x and y would best help us see that a pitchfork bifurcation is happening.

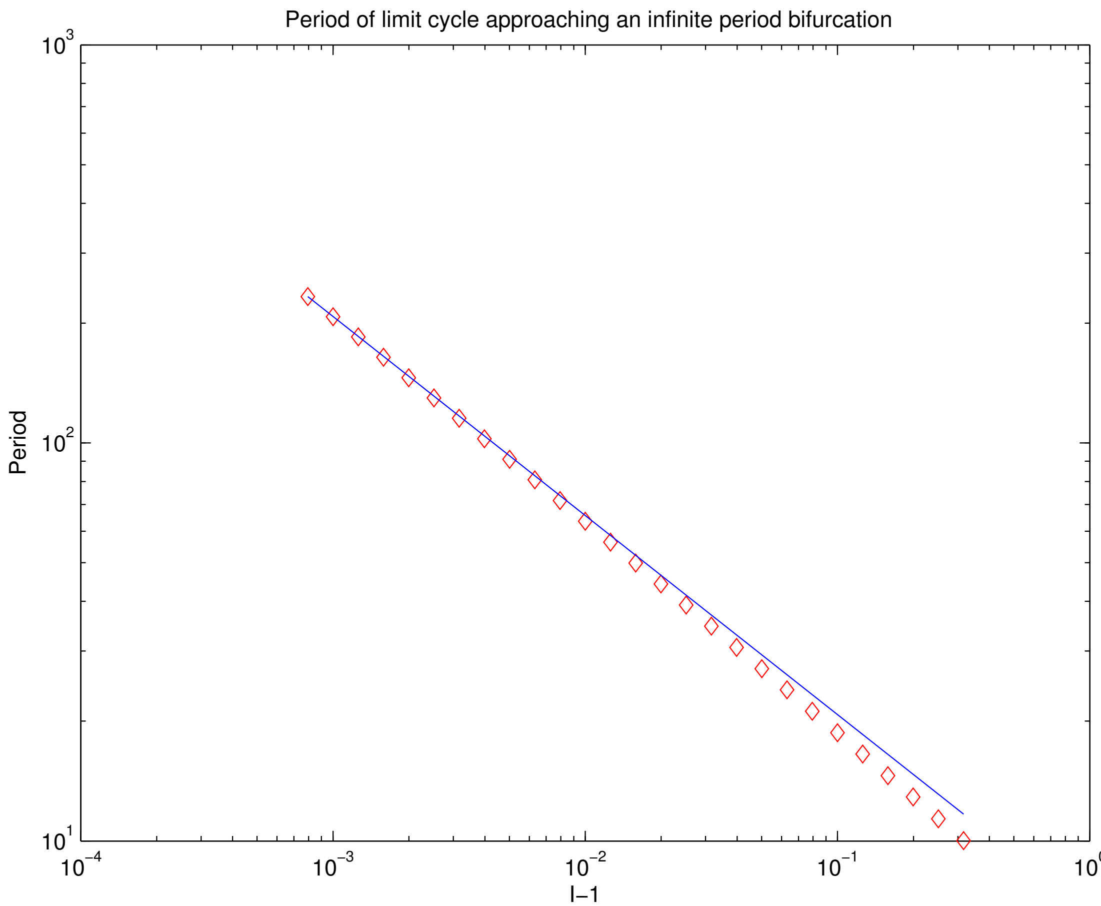


4. (a) For the parameters given, the system can approach a stable limit cycle or the fixed point at the bottom. The limit cycle represents the pendulum swinging around the in full circles, so we expect the system to go to the limit cycle if we start with larger $\frac{d\phi}{dt}$. This is what we find.

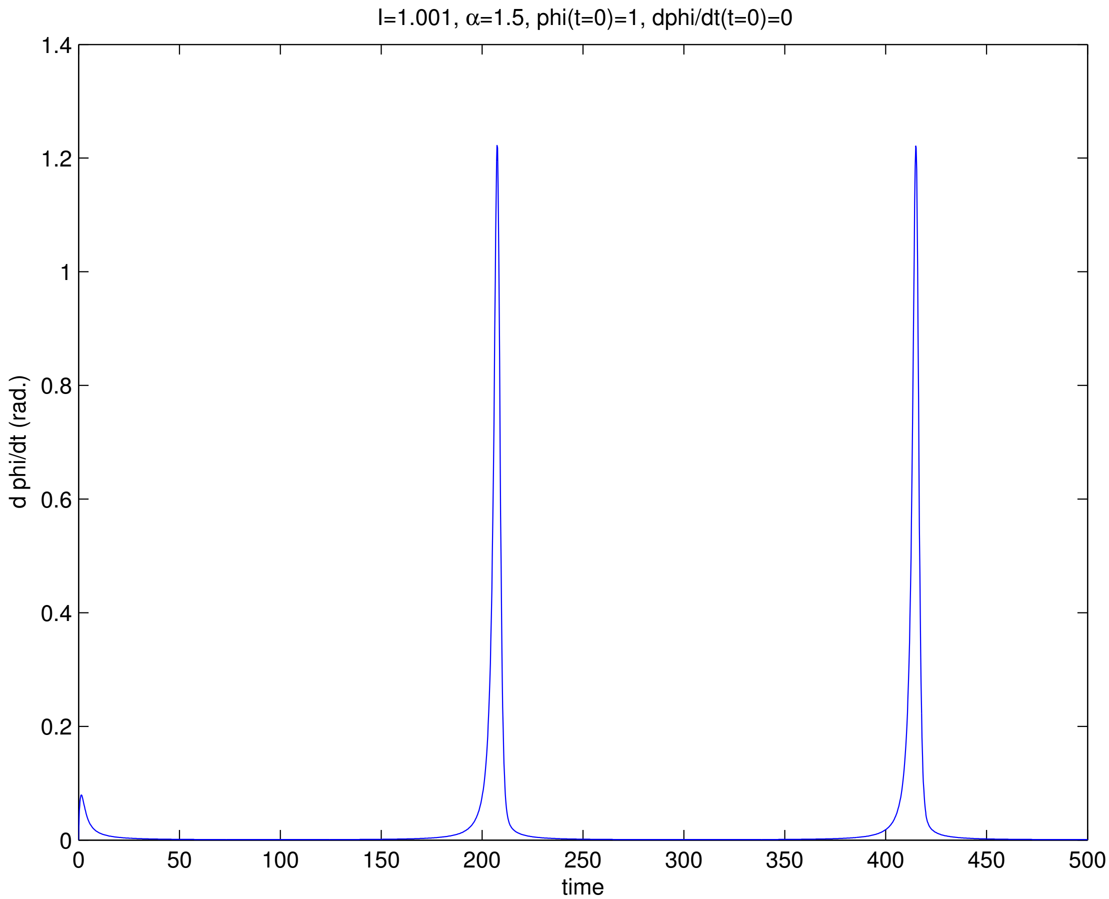




(b) The period is plotted as a function of I-1. Since an infinite-period bifurcation is approached, we expect the period to scale like $T \sim (I-1)^{-\frac{1}{2}}$. On a log-log plot this means we expect the data points to fall on a straight line with slope $-\frac{1}{2}$. We expect this formula to work better for smaller I-1, so the data will deviate from the line as I-1 gets larger. The figure seems to be consistent with our understanding.



Here the oscillations for I=1.001 are plotted. The system spends a lot of time in a certain region where flow is slow, then quickly moves through a fast region.

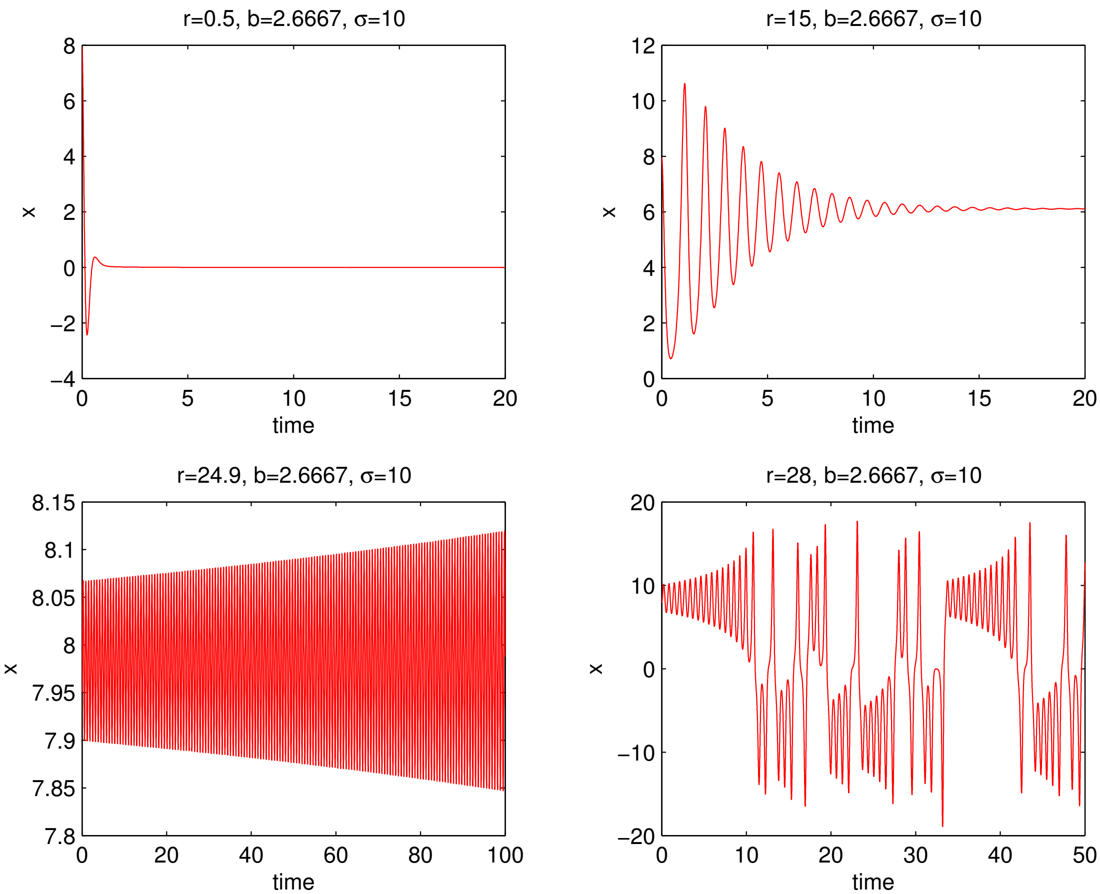


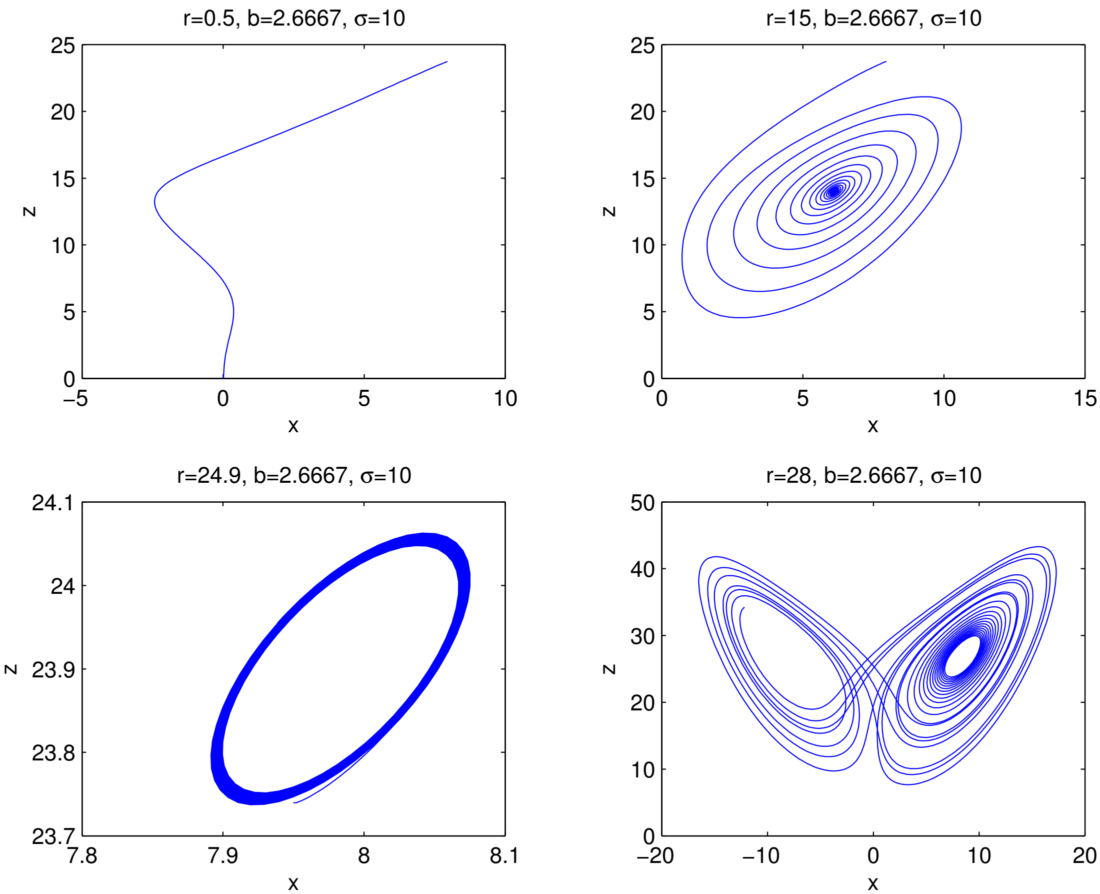
5. (a) The system approaches the stable f.p. at the origin.

   (b) The system approaches C^+.

   (c) We start near C^+. The system oscillates with a slowly expanding envelope as it moves away from C^+. We know to expect this oscillation since a Hopf bifurcation has just happened. Eventually the system will get away from C^+ and will exhibit chaotic behavior.

   (d) Chaos!




r=0.5, b=2.6667, σ=10
r=15, b=2.6667, σ=10
r=24.9, b=2.6667, σ=10
r=28, b=2.6667, σ=10
x
z
25
20
15
10
5
0
–5
24.1
24
23.9
23.8
23.7
7.8
7.9
8
8.1
50
40
30
20
–20
–10