

Homework #9  
Nonlinear dynamics and chaos

1. Calculate a numerical approximation for

$$\lim_{n \rightarrow \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

for the following two maps

$$x_{n+1} = r \sin(\pi x_n), \quad 0 \leq r \leq 1 \quad (\text{the sine map}) \quad (1)$$

$$x_{n+1} = r - x_n^4 \quad (2)$$

by iterating the maps for different  $r$ -values and finding the  $r$  values at which period doubling(s) occur. Compare the results to  $\delta$  for the logistic map and explain.

2. (Strogatz 10.7.3,4) some simple renormalization-related issues:

- (a) Show that if  $g(x)$  is a fixed point of the doubling transformation, that is,

$$g(x) = -\alpha g \left[ g \left( \frac{x}{-\alpha} \right) \right] \equiv T[g],$$

so is  $\mu g(x/\mu)$ .

- (b) show that  $g(x)$  crosses the line  $y = \pm x$  an infinite number of times by showing that if  $x^*$  is a fixed point of  $g(x)$ , so is  $-\alpha x^*$ .
- (c) Calculate an approximation to the universal  $\alpha$  for the period doubling route to chaos. Start with the map  $f(x, r) = r - x^2$ , assume a two-term expansion for the universal function:

$$g(x) = 1 + c_2 x^2.$$

and calculate  $c_2$  and  $\alpha$  that approximately satisfy the functional equation for  $g(x)$ .

3. Show that

$$g_{i-1}(x) = (-\alpha) g_i \left[ g_i \left( -\frac{x}{\alpha} \right) \right] (\equiv T[g_i(x)]).$$

Explain each stage in your derivation (see Schuster. . .).

4. Read, understand, and reproduce the approach of Strogatz “renormalization for pedestrians” pages 384-387 in order to analytically calculate an approximate to both  $\delta$  and  $\alpha$  for a quadratic maximum map. Skip example 10.7.2, but do example 10.7.3.

5. **Analytical Challenge question:** First the easier part: Find  $\alpha$  for quartic functions (such as  $x_{n+1} = r - x_n^4$ ) using the approach of question 2c  
Next: a challenge in the best sense of the word (i.e. I have not tried this myself, and I don't know that it is possible). Follow the approach of Strogatz "renormalization for pedestrians" on page 384-387 in order to analytically calculate both  $\delta$  and  $\alpha$  for a quartic maximum function (such as  $x_{n+1} = r - x_n^4$ ). Compare your analytically derived results to the numerical approximation for  $\delta$  from question 1.
6. **Numerical Challenge question:** Period doubling in the Lorenz model: try to find a period doubling sequence in the Lorenz model, as function of  $r$ . Dorian shows in his section notes (on homepage) one such sequence for  $r$  decreasing from 100, can you find others? Plot period 1,2,4 and higher if you can. You may have to zoom using Matlab to differentiate the higher order periods from each other. You may want to use the book "The Lorenz equations: bifurcations, chaos and strange attractors" by Colin Sparrow by scanning the pictures there as a guideline for which  $r$  values to try. An important note: the precise  $r$  values at which a certain behavior occurs depends on the numerical time stepping scheme one uses. The `lorenz1.m` program from the course home page uses a simple scheme that may result in different bifurcation values for  $r$  than that in the above book, so beware that one may need to deviate from the values given in the book in order to obtain the same behaviors. Another comment is that you need to plot the trajectory only after the transients have died out. That is, change the values of `beg` and `end1` in the plotting section of the Matlab program, such that plotting starts only after a significant integration time (e.g. `T=200.;` `beg=NP*9/10;` `end1=NP`). Finally, the finite thickness of the curves you plot sometimes indicate that what seems like a single trajectory, is found under zooming in Matlab to represent a more complex set of trajectories. This could also be a numerical artifact and you may have to ignore this. You may want to compare your results to those found using `lorenz2.m` which uses a more elaborate time scheme from Matlab.