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APM147
12/9/04

Problem Set 11 Solutions

1. See Ott, page 95. This argument appears to only apply for integer dimensions.

2.

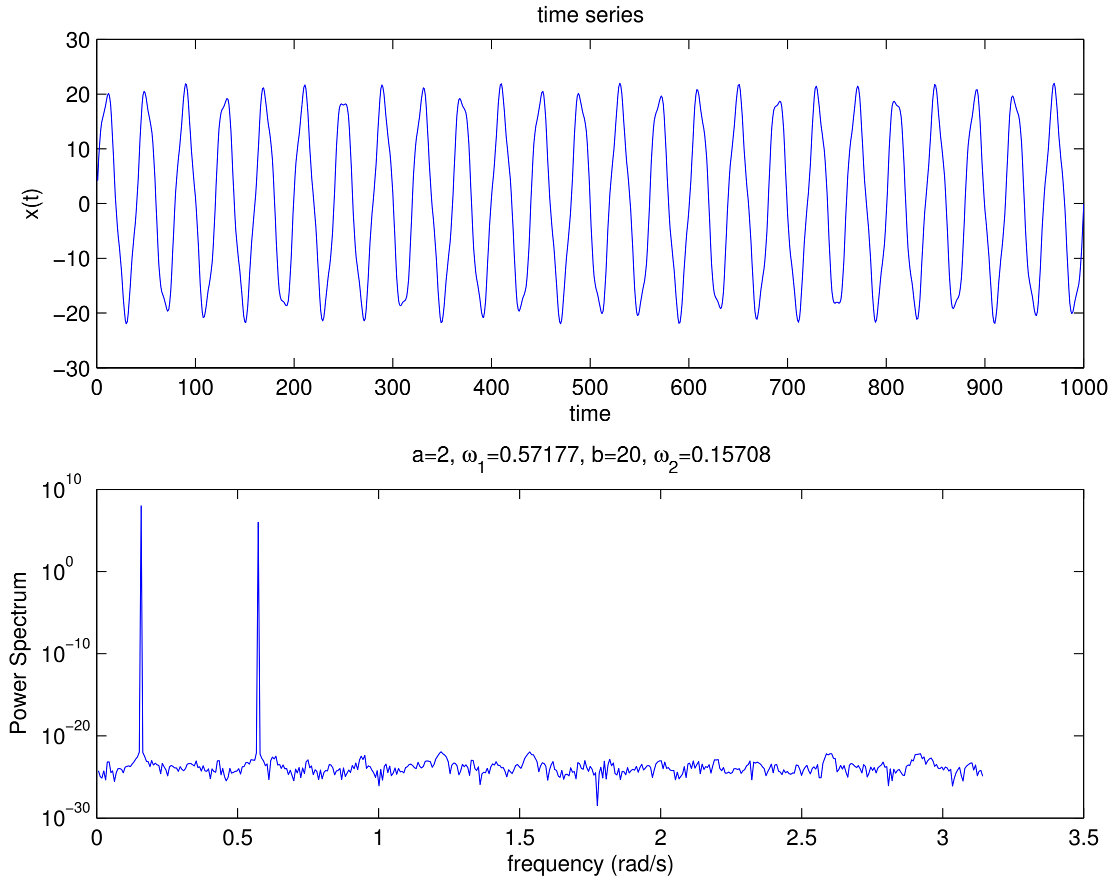

Figure 1: The height of the peaks in the power spectrum is proportional to the square of the amplitude (a,b). For instance, in this plot b/a=10 while the peak in the power spectrum at ω_1 is about 10^6 and at ω_2 is about 10^8 (you can zoom in to see this), giving a ratio of the peaks of 100.

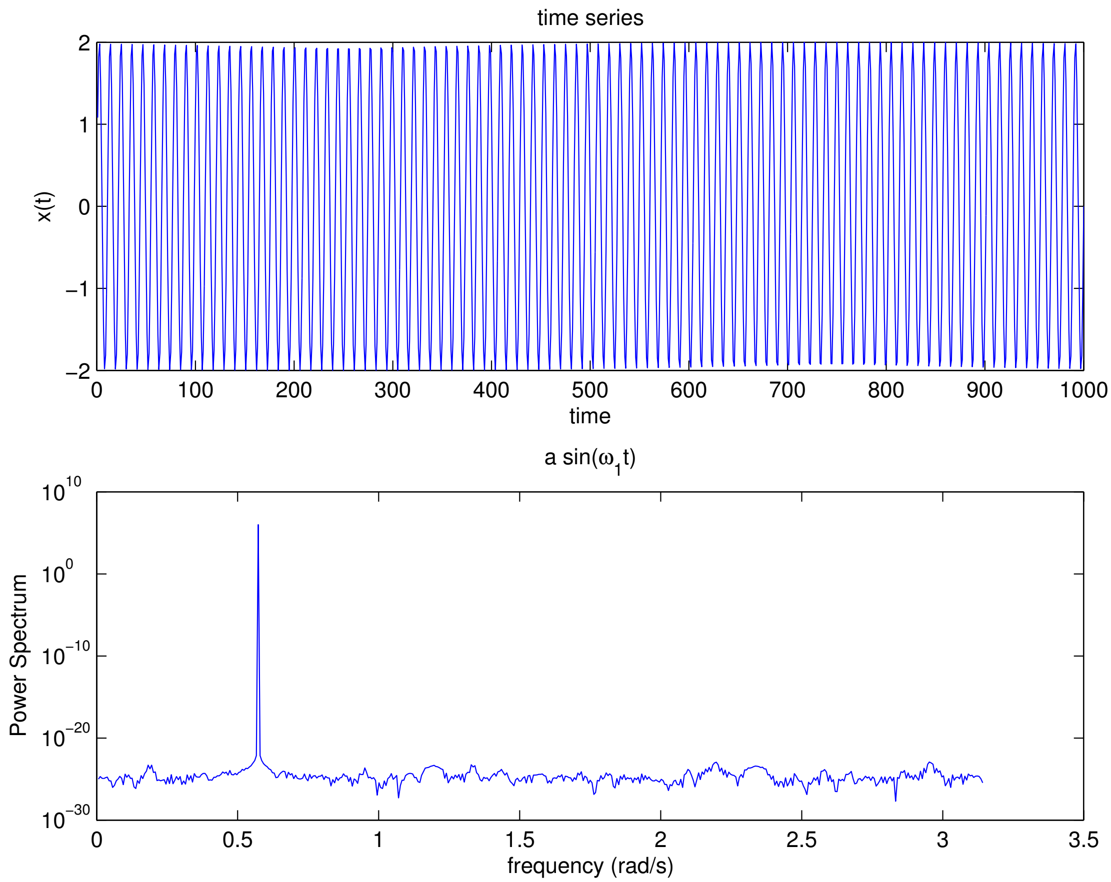

Figure 2: Here there is only one peak at the frequency of oscillation.

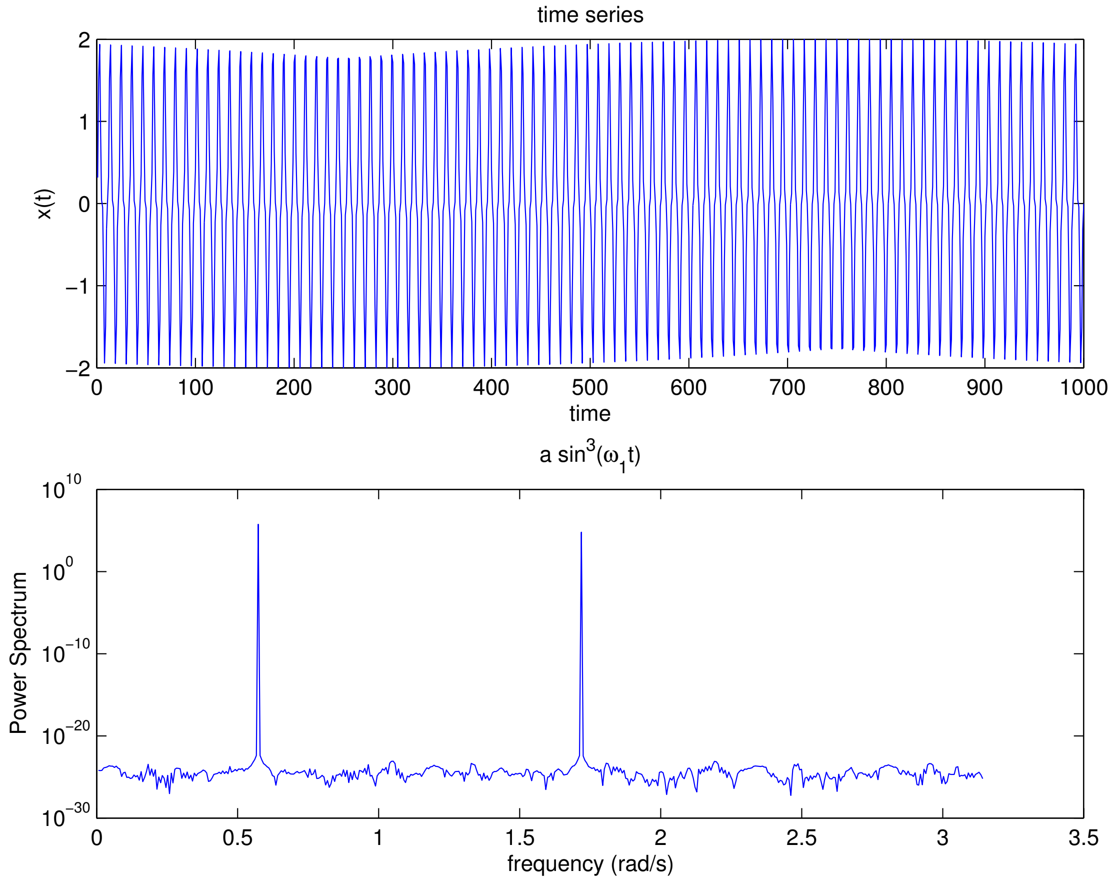

Figure 3: $sin^3(\omega_1 t) = \frac{3}{4}sin(\omega_1 t) - \frac{1}{4}sin(3\omega_1 t)$ so we expect a peak 1/9 as large at a frequency three times the original frequency.

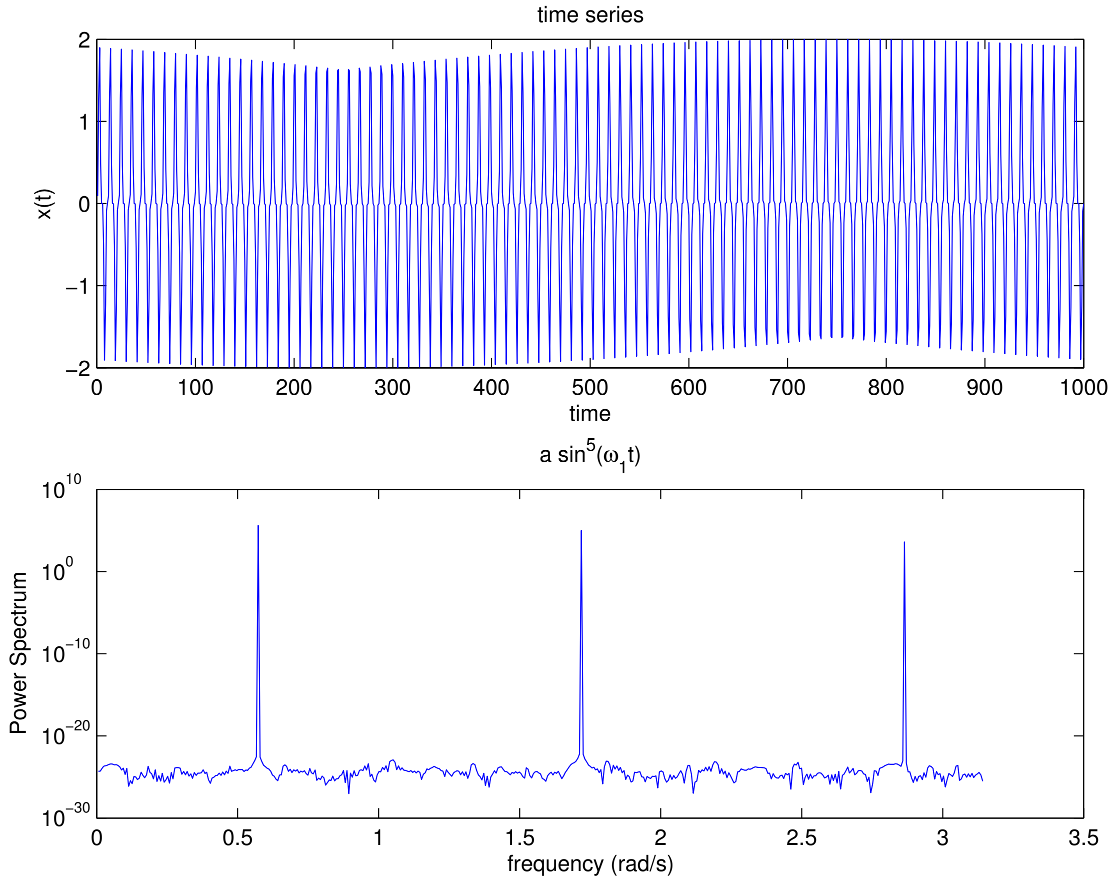

Figure 4: $sin^5(\omega_1 t) = \frac{5}{8}sin(\omega_1 t) - \frac{5}{16}sin(3\omega_1 t) + \frac{1}{16}sin(5\omega_1 t)$ so we expect a peak 1/4 as large at a frequency three times the original frequency and a peak 1/100 as large at a frequency five times the original frequency.

3. Start with the map

$$f(x_n) = x_{n+1} \equiv \varepsilon + x_n + x_n^2$$

Consider f(f(x_n)). x_n is small so drop terms O(x_n^3):

$$\begin{aligned} f(f(x_n)) &= \varepsilon + (\varepsilon + x_n + x_n^2) + (\varepsilon + x_n + x_n^2)^2 \\ &= (2\varepsilon + \varepsilon^2) + (1+2\varepsilon)x_n + (2+2\varepsilon)x_n^2 + O(x_n^3) \end{aligned} \quad (1)$$

$\varepsilon \ll 1$ so drop terms a factor of ε less than something they are summed with:

$$\begin{aligned} f(f(x_n)) &\approx 2\varepsilon + x_n + 2x_n^2 \\ 2f(f(x_n)) &\approx 4\varepsilon + 2x_n + (2x_n)^2 \\ F(X_n) &\approx \Gamma + X_n + X_n^2 \end{aligned} \quad (2)$$

Here F=$\frac{1}{2}f^2$, $\Gamma = 4\varepsilon$, $X_n = 2x_n$. So to lowest order f^2 is similar to f and the map is self-similar.

4. We can describe Type III intermittency with the map:

$$x_{n+1} = -(1+\varepsilon)x_n - x_n^3 \quad (3)$$

Here $\varepsilon \equiv \mu - \mu_c$ for some bifurcation parameter μ with critical value μ_c and $\varepsilon, x_n \ll 1$. We expect this map to alternate during as it transitions from its laminar period to intermittent chaos, so it is appropriate for Type III intermittency. Since the map alternates like this, we can get a rough estimate of the derivative of x by: $\frac{dx}{dn} \approx x_{n+2} - x_n$. This should tell us how the envelope is changing. Using this we get and dropping terms of O(x_n^4):

$$\frac{dx}{dn} \approx [(1+\varepsilon)^2 - 1]x + (1+\varepsilon)[1+(1+\varepsilon)^2]x^3 \quad (4)$$

Take ε to be small, so we have:

$$\frac{dx}{dn} \approx 2\varepsilon x + 2x^3 \quad (5)$$

If we take x_0 to be the value the chaotic region seeds the map at, then we get an estimate of the amount of time spent in the laminar region from:

$$\begin{aligned} N &\sim \frac{1}{2}\int_{x_0}^{\infty} \frac{1}{x}\frac{1}{\varepsilon + x^2} dx \\ &\sim \frac{1}{2\varepsilon}\int_{x_0}^{\infty} (\frac{1}{x} - \frac{x}{\varepsilon + x^2}) \\ &\sim \frac{1}{2\varepsilon} ln \mid \frac{x}{\sqrt{\varepsilon + x^2}} \mid |_{x_0}^{\infty} \\ &\sim \frac{1}{2\varepsilon} ln \mid \frac{\sqrt{\varepsilon + x_0^2}}{x_0} \mid \end{aligned} \quad (6)$$

We've been pretty hand-wavy up to now (with the $\frac{dx}{dn}$ argument and all), so we might as well continue in that spirit. The only way we can get $ln \mid \frac{\sqrt{\varepsilon + x_0^2}}{x_0} \mid$ to be a constant that's not zero or infinity and doesn't depend on ε is to take x_0 to be O($\varepsilon^{\frac{1}{2}}$),so to lowest order $x_0 = \alpha\varepsilon^{\frac{1}{2}}$. If we do this $ln \mid \frac{\sqrt{\varepsilon+x_0^2}}{x_0} \mid = ln \mid \frac{\sqrt{\alpha^2+1}}{\alpha} \mid$ so that N$\sim \frac{const.}{\varepsilon}$. This scaling makes some assumptions about the average seeding from the chaotic interval. A rigorous derivation of this result requires the use of a PDF and much math after this, see Schuster.