Homework #12

Nonlinear dynamics and chaos

1. **Quasi-periodicity route to chaos for the periodically forced damped pendulum:** Use the Matlab program `pendulum.m` from the course home page for simulating a forced pendulum:

$$\ddot{\theta} + \gamma\dot{\theta} + g/L\sin\theta = k + f\cos(\omega_f t)$$

For each regime in the questions below, plot (1) a time series of $\dot{\theta}$ (explain why not of θ itself), (2) a 2d Poincare section based on sub-sampling the time series for $\theta(t)$ every period of the forcing, (3) a 3d continuous phase space picture (all three plots are included in the above program from the course home page). Explain why the Poincare section and the 3d reconstructed phase space looks the way they do for each of these regimes. Note that the Poincare section is obtained by subsampling the time series every period of the periodic forcing.

   (a) Let k=0.5. Increase f from zero and see how the pendulum responds. Find, specify the parameters and plot a quasi-periodic regime; a mode-locked regime; a chaotic regime. (Adjust the forcing frequency if necessary; if program does not work, consider making the time step smaller by changing the factor multiplying the forcing period in the expression for the time step.) What signifies each of these three regimes in the Poincare sections? why? In the 3d phase plot? In the time series plot? Explain your results by comparing them to the transition to chaos of the circle map.

   (b) Next, let k=0 and repeat the procedure of the previous question. What has changed. We might say that this case is a degenerate case of the quasi-periodic route to chaos, because it doesn't exhibit all the behavior we might have expected from study of the circle map. Which stage is missing? why?

   (c) **Optional challenge problem:** Try to crudely map the parameter space as function of the forcing frequency ω_f and the forcing amplitude a. Locate a few of the main arnold tongues, and a few points in the chaotic regime. Based on these few data points, schematically sketch some of the main Arnold tongues and the line marking the transition to chaos in this two dimensional parameter space.

2. Circle map: Plot θ_{n+1} as function of θ_n for the circle map when $K < 1$ and $K > 1$. Explain why it makes sense to say that the map becomes non invertible upon transition to chaos.

3. **Fractal dimension:** Calculate the dimension of the Koch curve.

4. **Fractal dimension:** Calculate the box dimension of the following object: take a square, divide it into 9 equal size squares, and remove the middle and side ones as shown in the plot. Repeat this for each of the smaller remaining squares (as shown in figure). Iterate an infinite number of times.
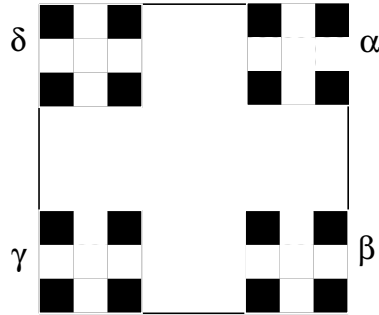
5. **Another fractal dimension:** (Strogatz 11.3.9, p. 419) Consider a 3d cube, divide it into 27 equal-size cubes, remove the center cube as well as the center cube from each of the faces. This is equivalent to drilling three square holes through the centers of the three faces of the cube. Iterate an infinite number of times. Now the question: (i) find the fractal dimension of the final object

6. **Optional challenge problem:** find the fractal dimension in the case where the procedure of the last question is followed for an arbitrary N-dimensional cube.

7. **Yet another fractal dimension:** Find the box dimension of the object in Fig 3.18 page 106 in Ott's book, also shown here: