APM 203, Fall 2005 Homework #1 SOLUTIONS

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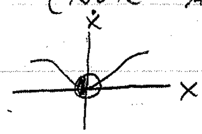
① i. fixed points: $0 = f(x^*) = 1 - e^{-x^{*2}} \Rightarrow x^* = 0$

stability: $f'(x^*) = 2x e^{-x^{*2}} = 0$

linearized stability fails; use graphical approach

(Note: Analytically, $f(x^* + \delta x) = f(\delta x) \simeq 1 - (1 - \delta x^2) = \delta x^2$, so)



$\boxed{x^* = 0 \text{ is semi-stable}}$

ii. fixed points: $0 = f(x^*) = x^*(a - x^{*2}) \Rightarrow x^* = \{0, \pm\sqrt{a}\}$

stability: $f'(x^*) = a - 3x^{*2}$

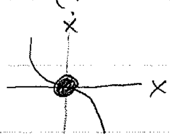
$\underline{a < 0}$: $f'(0) = a < 0 \Rightarrow \boxed{x^* = 0 \text{ is stable}}$

$\underline{a > 0}$: $f'(0) = a > 0 \Rightarrow \boxed{x^* = 0 \text{ is unstable}}$

$f'(\pm\sqrt{a}) = -2a < 0 \Rightarrow \boxed{x^* = \pm\sqrt{a} \text{ is stable}}$

$\underline{a = 0}$: $f'(0) = 0 \Rightarrow$ linearized stability fails; use graphical approach

(Note: Analytically, $f(x^* + \delta x) = -\delta x^3$, so)



$\boxed{x^* = 0 \text{ is stable}}$

Note: supercritical pitchfork bifurcation occurs at $a = 0$, $x^* = 0$

iii. fixed points: $0 = f(x^*) = x^*(1 - x^*)(2 - x^*) = \{0, 1, 2\}$

stability: $f'(x^*) = 3x^{*2} - 6x + 2$

$f'(0) = 2 \Rightarrow \boxed{x^* = 0 \text{ is unstable}}$

$f'(1) = -2 \Rightarrow \boxed{x^* = 1 \text{ is stable}}$

$f'(2) = 2 \Rightarrow \boxed{x^* = 2 \text{ is unstable}}$

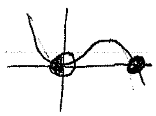
iv. fixed points: $0 = f(x^*) = x^{*2}(6-x^*) \Rightarrow x^* = \{0, 6\}$

stability: $f'(x^*) = 12x - 3x^2$

$f'(6) = -36 \Rightarrow \boxed{x^* = 6 \text{ is stable}}$

$f'(0) = 0 \Rightarrow$ linearized stability fails; use graphical approach

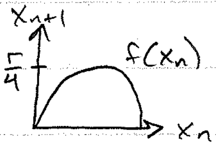
(Note: Analytically, $f(\delta x) = 6\delta x^2 + O(\delta x^3)$, so)



$\boxed{x^* = 0 \text{ is semi-stable}}$

v. fixed points: $0 = f(x^*) = \ln(x^*) \Rightarrow x^* = 1$

stability: $f'(x^*) = \frac{1}{x^*} = 1 \Rightarrow \boxed{x^* = 1 \text{ is unstable}}$

(2) a) 

$$x_{n+1} = f(x_n) = r x_n (1 - x_n), \quad 0 \le x \le 1$$

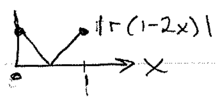
First, we need to place some restrictions on r so that all $x_n \in [0,1]$ map to an $x_{n+1} \in [0,1]$.

$r \ge 0$: otherwise $x_n \in [0,1]$ map to $x_{n+1} < 0$.

$\max\{ f([0,1]) \} = f(\frac{1}{2}) = \frac{r}{4} \Rightarrow r \le 4$

↳ Require $\boxed{0 \le r \le 4}$

dissipative if $|f'(x)| < 1$ everywhere on interval.

$|r(1-2x)| < 1$, and since 

(max at 0 & 1)

$|r(1 - 2 \cdot 0)| = |r(1-2)| = |r| < 1$

so map is dissipative for $\boxed{0 \le r < 1}$

b) Here are a few different approaches:

method 1: series. Since $0 \le x \le 1$, $(1-x) \le 1$, so $x_{n+1} \le r x_n$

Since $r < 1$, $x_{n+1} < x_n \Rightarrow \boxed{x_n \to 0}$ as $n \to \infty$. Note that for large n (small x_n), $x_{n+1} \approx r x_n$, so $x_{N+n} \approx x_N r^n$.

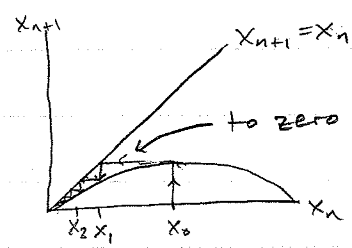
method 2: fixed points. $x^* = f(x^*) = r x^* (1 - x^*)$

So $x^* = \{0, 1 - \frac{1}{r}\}$, and $(1 - \frac{1}{r}) < 0$ since $r < 1$.

Stability? $|f'(x^*)| = r(1 - 2x^*) = r < 1$

So there's only 1 f.p., $x^* = 0$, and it's stable $\Rightarrow \boxed{x_n \to 0}$ as $n \to \infty$

method 3. cobweb plot



c) Analytically

$x_{n+1}^* = x_n^* = r x_n^* (1 - x_n^*) \Rightarrow x_n^* = \{0, 1 - \frac{1}{r}\}$

$|f'(x^*)| < 1$ for stability

$|f'(x^*)| = r|1 - 2x^*| = \{r, |2-r|\}$

$0 \le r < 1$: Only $x^* = 0$ is in $[0,1]$.

$|f'(0)| = r < 1 \Rightarrow x^* = 0$ is stable

$r = 1$: $|f'(0)| = 1$, so linearized stability fails.

$x_* + \delta x_{n+1} = \delta x_{n+1} = f(\delta x_n) = \delta x_n (1 - \delta x_n) = \delta x_n - \delta x_n^2 < \delta x_n$

$\hookrightarrow \delta x_{n+1} < \delta x_n$ and $\delta x > 0$ since $x \in [0,1] \Rightarrow x^* = 0$ is (marginally) stable

$1 < r < 3$: $|f'(0)| = r > 1 \Rightarrow x^* = 0$ is unstable

$|f'(1 - \frac{1}{r})| = |2 - r| < 1 \Rightarrow x^* = 1 - \frac{1}{r}$ is stable

$r = 3$: $|f'(0)| = r > 1 \Rightarrow x^* = 0$ is unstable

$|f'(1 - \frac{1}{r})| = |2 - r| = 1$, so linearized stability fails.

$\delta x_{n+1} = f(x^* + \delta x_n) - x^* = -\delta x_n (1 + 3\delta x_n)$ so map oscillates about $x^* = 1 - \frac{1}{r}$ $(\delta x_{n+1} \doteq -\delta x_n)$. Iterate again.

$\delta x_{n+2} = f(f(x^* + \delta x_n)) - x^* = \delta x (1 - 18\delta x^2) + O(\delta x^4)$

Since $(1 - 18\delta x^2) < 1$, $x^* = 1 - \frac{1}{r}$ is (marginally) stable

$3 < r \le 4$: $|f'(0)| = r > 1 \Rightarrow x^* = 0$ is unstable

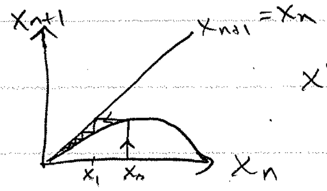
$|f'(1 - \frac{1}{r})| = |2 - r| > 1 \Rightarrow x^* = 1 - \frac{1}{r}$ is unstable

Summary

range	$x^* = 0$	$x^* = 1 - \frac{1}{r}$
$0 \le r < 1$	stable	—
$r = 1$	marginally stable	$[1 - \frac{1}{r} = 0]$
$1 < r < 3$	unstable	stable
$r = 3$	unstable	marginally stable
$3 < r \le 4$	unstable	unstable

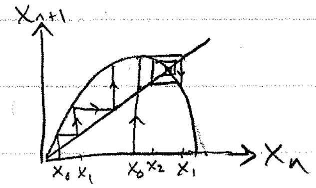
c) Graphically

$0 \le r \le 1$



$x^* = 0$ is stable

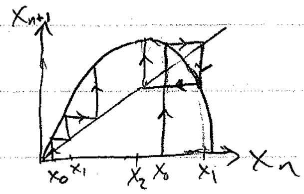
$1 < r \le 3$



$x^* = 0$ is unstable

$x^* = 1 - \frac{1}{r}$ is stable

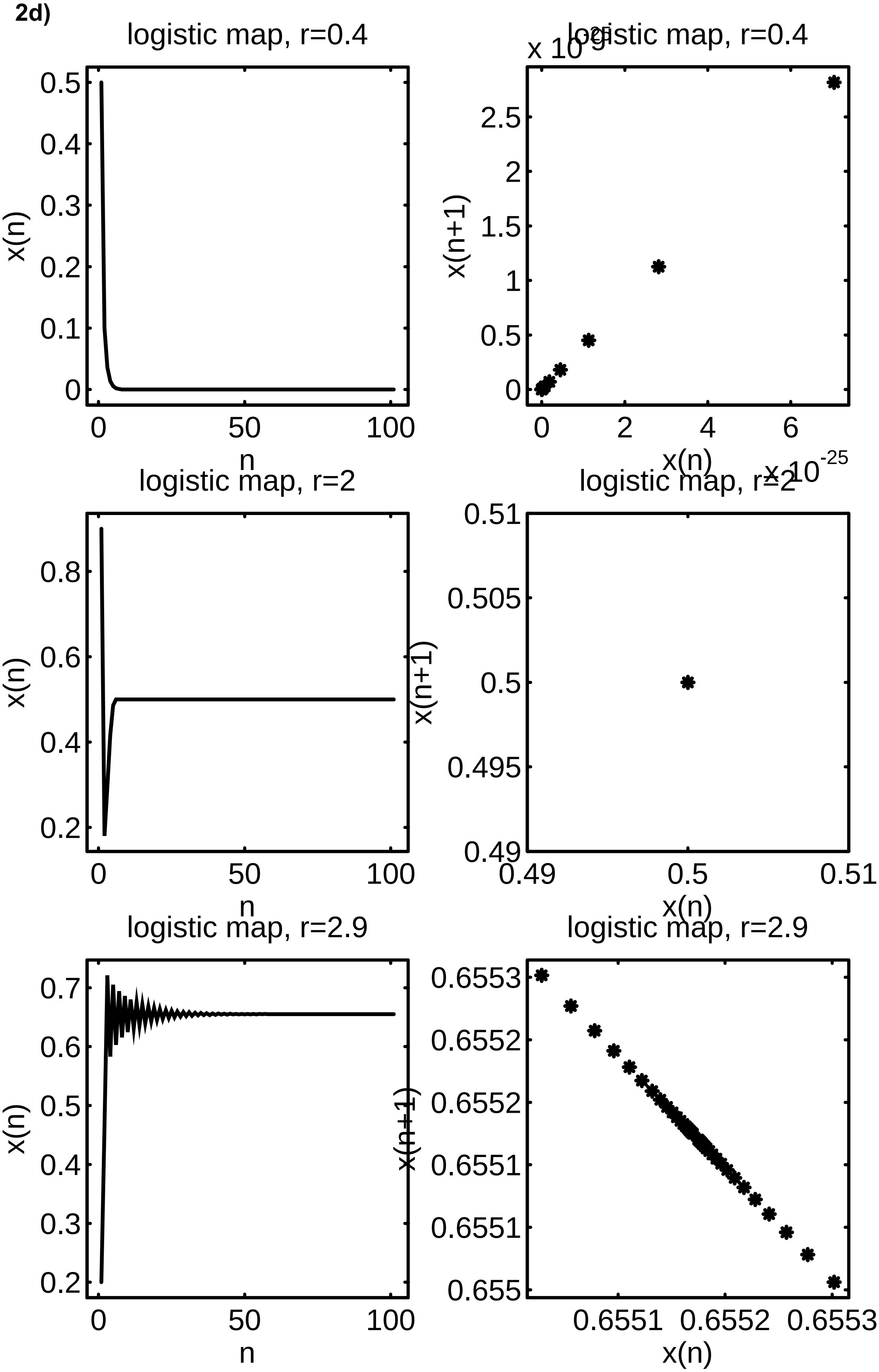
$3 < r \le 4$



$x^* = 0$ is unstable

$x^* = 1 - \frac{1}{r}$ is unstable

2d)



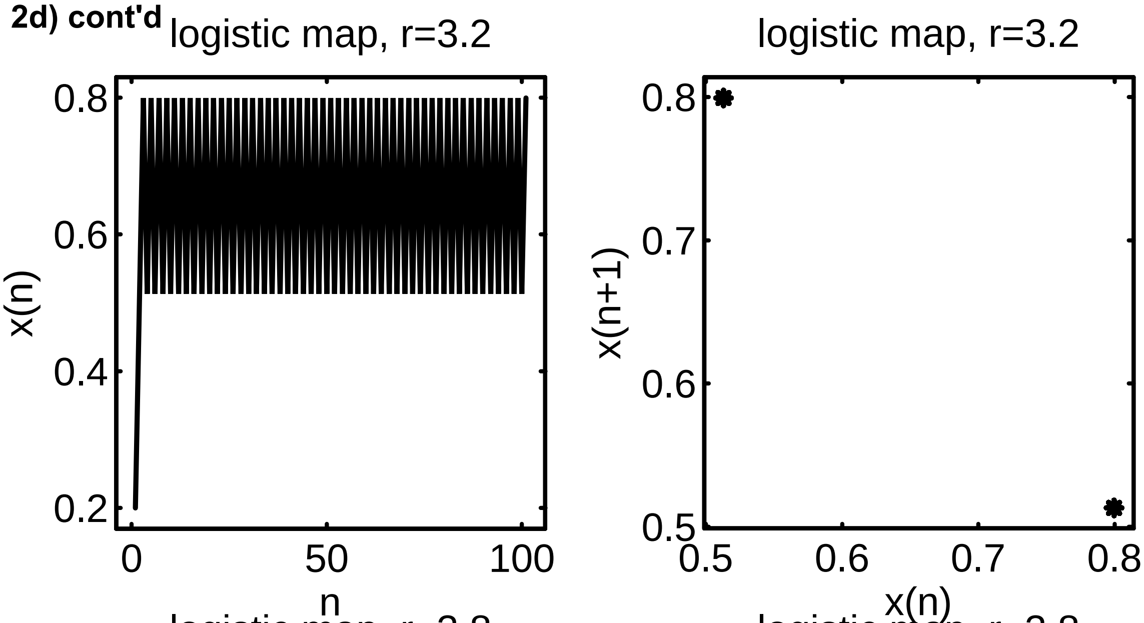
All initial values are mapped to zero: extinction

Stable fixed point at 1-1/r=1/2

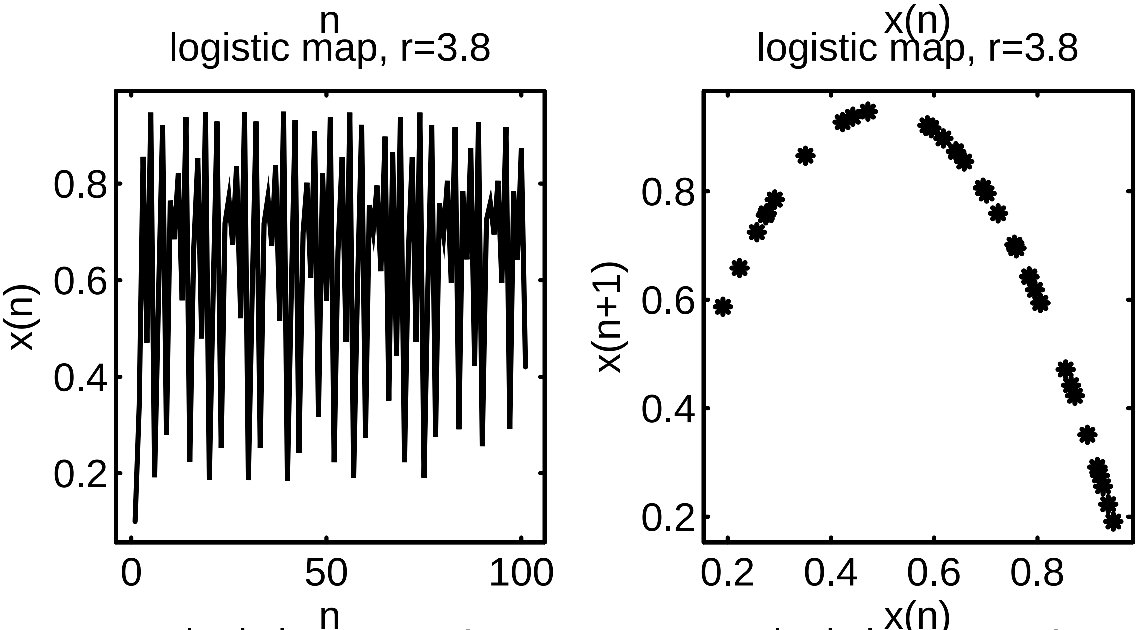
Stable oscillations around fixed point 1-1/r

Plots in right column only include 60<x<100

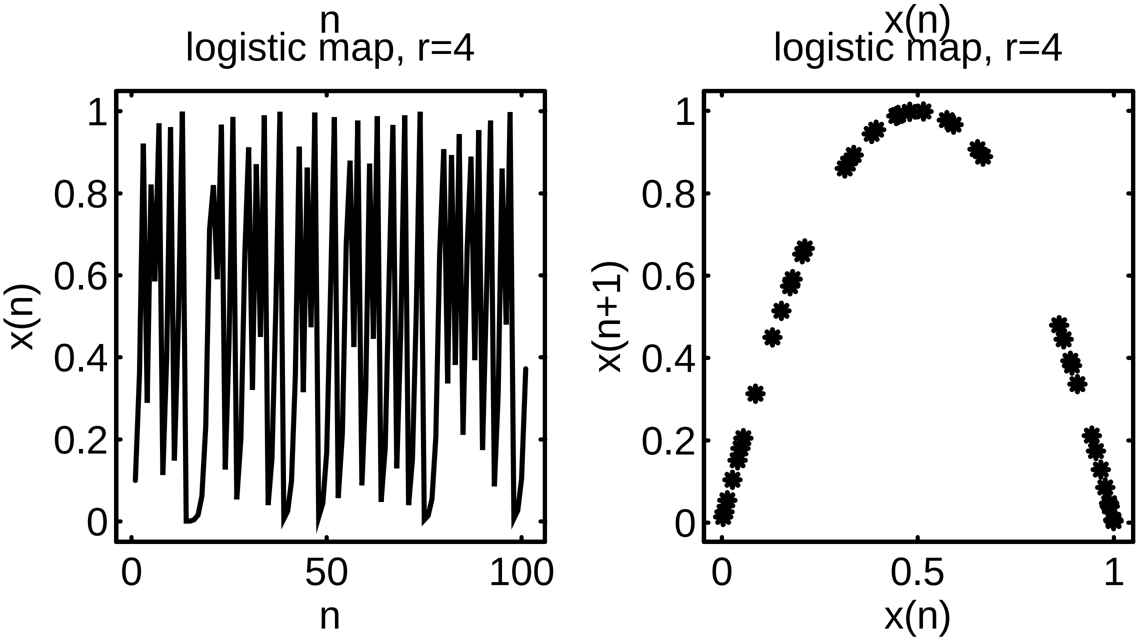
2d) cont'd



2-cycle fixed point, jumping between points on either side of (unstable) 1-1/r



A continuous range of values in [0,1] are being visited. No periodicity, no limiting value.



Now all values of x in [0,1] are visited; curve eventually becomes continuous x(n+1)=rx(n)(1-x(n)) for x in [0,1]. Varies erratically in range [0,1].

Plots in right column only include 60<x<100