APM203 Homework #2 SOLUTIONS

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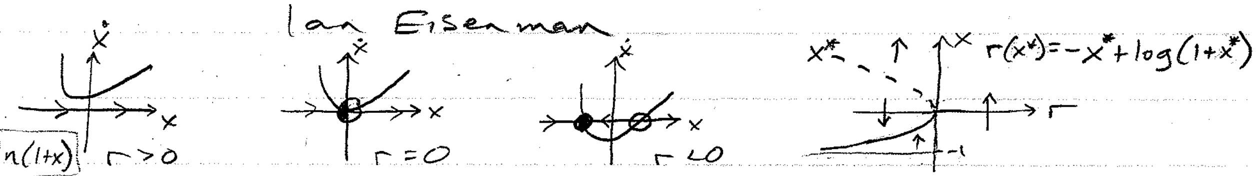
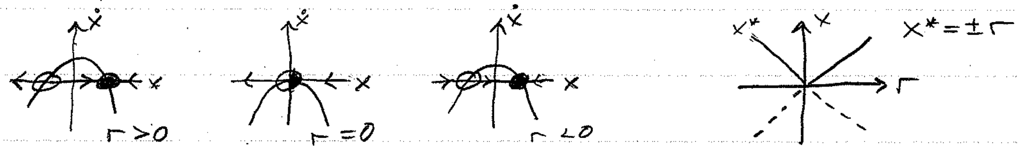
① a) $\boxed{\dot{x} = r + x - \ln(1+x)}$



<u>Saddle-node</u> (bif'n point $r=0$, $x^*=0$)

Near B.P., $\dot{x} = f(x) \overset{\text{Taylor}}{\simeq} r + x - \left(x - \frac{x^2}{2}\right) = \boxed{r + \frac{x^2}{2}}$

b) $\boxed{\dot{x} = r^2 - x^2}$



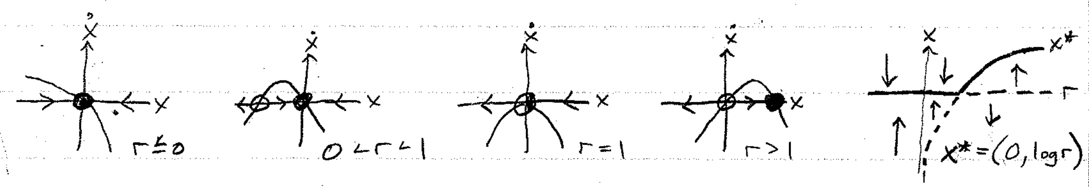
Here there are 2 f.p.s, then 1 at bif pt, then 2 again:

It's a <u>transcritical</u> (bif pt $r=0$, $x^*=0$).

With the tranformation $y \equiv -x + r$

$\dot{x} = \dot{y} = \boxed{2yr - y^2}$

c) $\boxed{\dot{x} = x(r - e^x)}$

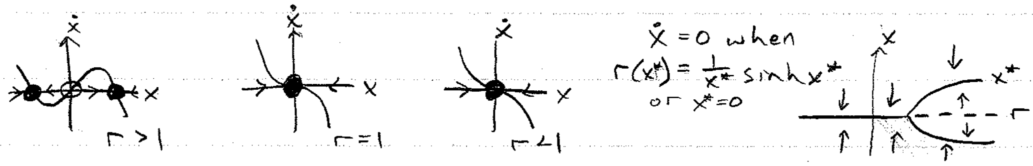


<u>transcritical</u> (BP: $r=1$, $x^*=0$)

Near BP, $\dot{x} = f(x) \simeq x(r - 1 - x) = \boxed{(r-1)x - x^2}$

With $R = r-1$, $\dot{x} = Rx - x^2$ (Normal form)

d) $\boxed{\dot{x} = rx - \sinh x}$



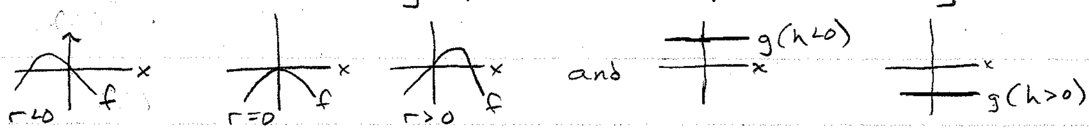
<u>Supercritical</u> <u>pitchfork</u> (BP: $r=1$, $x^*=0$)

Near BP, $\dot{x} = (r-1)x + x - \left(x + \frac{x^3}{6}\right) = \boxed{(r-1)x - \frac{x^3}{6}}$

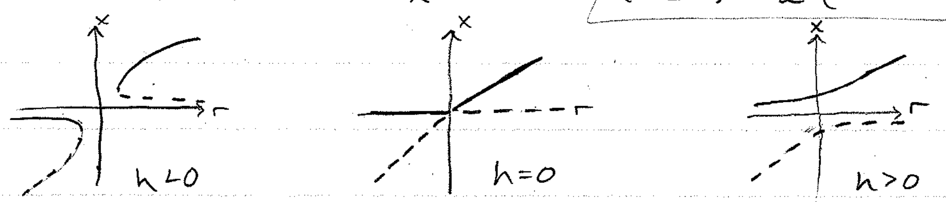
With $R = r-1$, $\dot{x} = Rx - \frac{x^3}{6}$

(2) $\dot{x} = h + rx - x^2$

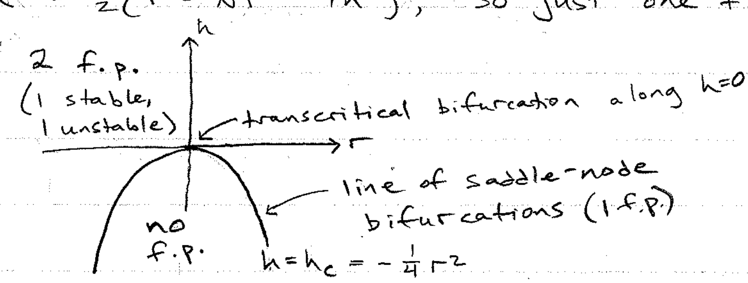
i. Let $f(r) = rx - x^2$, $g(h) = -h \Rightarrow$ f.p. where $f = g$



One could sketch the bif diagrams from this alone, but we'll use $\dot{x} = 0$ at $\boxed{x^*(r) = \frac{1}{2}\left(r \pm \sqrt{r^2 - 4h}\right)}$

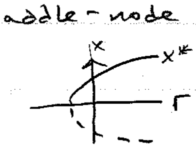


ii. $x^* = \frac{1}{2}\left(r \pm \sqrt{r^2 - 4h}\right)$, so just one f.p. (bif pt.) when $r^2 - 4h = 0$



Regions: below $h = h_c$, no f.p.; on $h = h_c$, 1 f.p.; below $h = h_c$, 2 f.p.

iii. With a perturbation/imperfection, the saddle-node bif'n diagram looks like this:



It's still a saddle-node, just shifted (made clear by the transformation $R = r + h$ to $\dot{x} = h + r - x^2$). The saddle-node, unlike the pitchforks and transcritical, is robust to the addition of an imperfection h (or even hx).

③ $\frac{\partial F}{\partial t} + c\frac{\partial F}{\partial x} = 0, \quad c>0$

This equation describes advection of F at a speed c. The solution can be written $F(x,t) = G(x+ct)$. In other words, the curve $F(x, t=0)$ is just shifted along the x-axis at velocity c as time evolves.

Substitute ansatz $F_{mn} = B^{n\Delta t} e^{i\mu m\Delta x}$ into discretized eq'n (1):

$(B^{\Delta t} - B^{-\Delta t})/2\Delta t = -c\,(e^{i\mu\Delta x} - e^{-i\mu\Delta x})/2\Delta x = -\frac{c}{\Delta x}\, i\sin(\Delta x)$

With $\sigma \equiv \frac{c\Delta t}{\Delta x}\sin(\mu\Delta x)$, this is

$(B^{\Delta t} - B^{-\Delta t}) = -2i\sigma \Rightarrow \boxed{B^{\Delta t} = -i\sigma \pm \sqrt{1-\sigma^2}}$

Since $F_{nm} \propto (B^{\Delta t})^n$, the scheme will be unstable if $|B^{\Delta t}| > 1$ for one of the two roots, which happens when $\boxed{|\sigma| > 1}$

So this harmless and totally stable advection equation can appear unstable when solves with the leapfrog scheme: in this scenario, a disturbance $F(x,t=0)$ will not only propegate at speed c, but wiggles will appear and their amplitude will grow exponentially as an artefact of the numerical scheme.

The conclusion is that if we wish to use the leapfrog scheme (without a Robert filter) to numerically solve a diff. eq. we must choose Δx and Δt carefully so that $|\sigma| \leq 1$ to avoid artificial instability.