Bifurcations:

1. Which bifurcation occurs in the following equations? (i) Plot vector field for values of the bifurcation parameters around the bifurcation point. (ii) Write the first terms in Taylor expansion near the bifurcation point and show that they resemble the normal form. Plot (qualitatively) the fixed point locations (x^*) as function of parameter value around the bifurcation point.

$$\begin{aligned} \dot{x} &= r + x - \ln(1+x) \\ \dot{x} &= r^2 - x^2 \\ \dot{x} &= x(r - e^x) \\ \dot{x} &= rx - \sinh x \end{aligned}$$

2. Analyze the case of imperfect transcritical bifurcation

$$\dot{x} = h + rx - x^2.$$

(i) Plot the bifurcation diagram for $h < 0, h = 0, h > 0$;

(ii) Sketch the different regions in the (r,h) plane that have qualitatively different vector fields. which bifurcations occur on the boundaries of these regions.

(ii) Analyze the case of an imperfect saddle node bifurcation. What happens if a small term h is added to the normal form. Why?

3. **Numerics: optional, extra credit problem** Show that the leapfrog scheme may be unstable, as follows. Consider the 1d advection equation

$$\partial F / \partial t + c \partial F / \partial x = 0; c > 0$$

Using leap frog in time, center difference in space, means that we are solving

$$(F_{m,n+1} - F_{m,n-1})/(2\Delta t) = -c(F_{m+1,n} - F_{m-1,n})/(2\Delta x) \equiv (\partial F / \partial t)_n \quad (1)$$

where $F_{m,n}$ is the value at location $m\Delta X$ and time $n\Delta t$, and equation 1 should be interpreted as an expression for the unknown $F_{m,n+1}$ in terms of previous, known, time steps.

To study the stability of this term, assume F to have some periodic structure in space, with a wave length of $2\pi/\mu$, and try a solution of the form

$$F_{m,n} = B^{n\Delta t} e^{i\mu m\Delta x}$$

where B may be complex. Derive a quadratic equation for $\sigma = (c\Delta t/\Delta x)\sin(\mu\Delta x)$ and show that the solution for $F_{m,n}$ grows exponentially for certain values of σ. What are these values? How does the exponential growth look like? Does the solution to the original continuous equation behave like this unstable finite difference solution?

(p.s. To eliminate this non physical numerical instability, one can use something like the Robert filter:

$$F_{n+1} = \bar{F}_{n-1} + 2\Delta t\,(\partial F/\partial t)_n \tag{2}$$

where

$$\bar{F}_n = F_n + \gamma(F_{n+1} - 2F_n + \bar{F}_{n-1}). \tag{3}$$

This means that there is a forward step, and then a filter operation which is equivalent to adding subtracting the second time derivative, which therefore has a smoothing effect.)