AM 203 Homework #4 SOLUTIONS

Ian Eisenman

(1) Strogatz 6.6.8

$\dot{x} = \frac{\sqrt{2}}{4} x(x-1)\sin\varphi \qquad \dot{\varphi} = \frac{1}{2}\left(\beta - \frac{1}{\sqrt{2}}\left(1+\frac{x}{8}\right)\cos\varphi\right)$

$0 \le x \le 1, \quad -\pi \le \varphi \le \pi$

a) $\dot{x}(x,-\varphi) = -\dot{x}(x,\varphi)$ and $\dot{\varphi}(x,-\varphi) = \dot{\varphi}(x,\varphi)$

↳ reversible system (i.e., symmetric in $\varphi$ but arrows also reverse direction).

b) nullclines: $\dot{x} = 0$ at $x=0, x=1, \varphi=-\pi, \varphi=0, \varphi=\pi$

$\dot{\varphi} = 0$ at $x = 8\left(\frac{\sqrt{2}\beta}{\cos\varphi} - 1\right)$

fixed points: need $x = 8\left(\frac{\sqrt{2}\beta}{\cos\varphi} - 1\right)$ to intersect $\dot{x} = 0$.

The intersections in the $x$ and $\varphi$ range when $\frac{9}{8\sqrt{2}} > \beta > \frac{1}{\sqrt{2}}$ are

$\boxed{(x^*, \varphi^*) = (8(\sqrt{2}\beta - 1), 0)}$

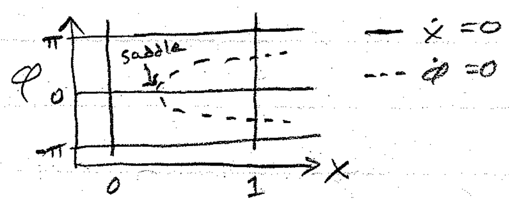
$\boxed{(x^*, \varphi^*) = \left(1, \pm\cos^{-1}\left(\frac{8\beta}{9\sqrt{2}}\right)\right)}$

Linearized Jacobian gives $\tau = 0$, $\Delta = \frac{1}{8}(9 - 17\sqrt{2}\beta + 16\beta^2) < 0$ for first $\Rightarrow$ saddle. For other 2, $\tau = \pm\frac{17}{288}\sqrt{162 - 256\beta^2}$

$\Delta = \frac{9}{64} - \frac{2}{9}\beta^2 > 0 \quad [\sqrt{162 - 256\beta^2} > 0] \Rightarrow$

$\varphi^* > 0$ f.p. unstable, $\varphi^* < 0$ f.p. stable.

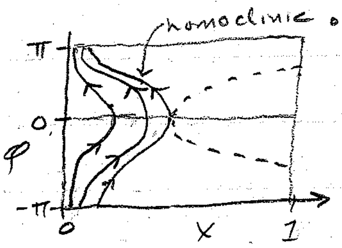
In this range of $\beta$, nullclines look are as follows:




Consider the saddle's unstable manifold in the $\varphi > 0$ half-plane. Since $\dot{x} < 0$ in $\varphi > 0$ and $\dot{\varphi} > 0$ outside the region enclosed by the $\dot{\varphi} = 0$ nullcline, and $\dot{x} = 0$ on the left edge, the trajectory must monotonically approach the $\varphi = \pi$ edge of the region. By reversability, then, it must return from $\varphi = -\pi$ back to the saddle point. Hence a homoclinic orbit.

① b) cont'd

The same argument applies to any trajectory beginning at $\varphi = 0$ and $x < x^*$ of the saddle $\Rightarrow$ band of closed orbits.

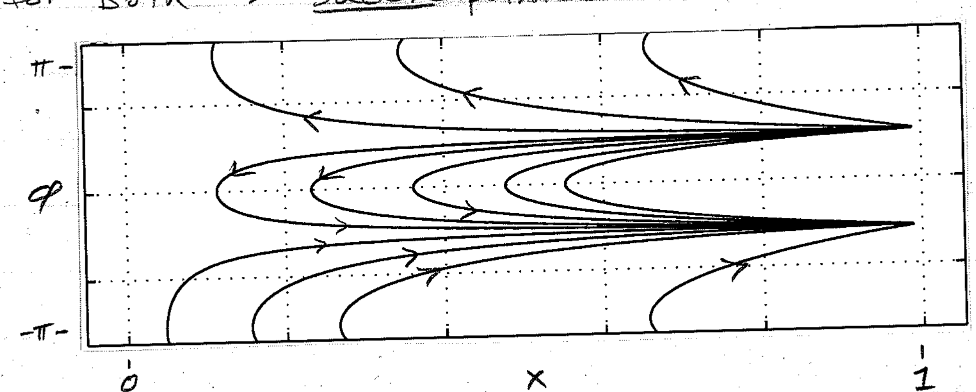



c) As $\beta \to \frac{1}{\sqrt{2}}$, $x^* = 8(\sqrt{2}\beta - 1) \to 0$ for the saddle. Since the homoclinic orbit has $x \leq x^*$ $\forall \varphi$, and the other closed orbits have smaller $x$, the homoclinic orbit chokes off the other closed orbits. When $\beta = \frac{1}{\sqrt{2}}$, $\dot{\varphi} \leq 0$ $\forall x$ at $\varphi = 0$ and $\dot{\varphi} \geq 0$ $\forall x$ at $\varphi$ sufficiently close to $\varphi = \pi$, $\varphi = -\pi$ $\Rightarrow$ no closed orbits winding around cylinder. Since all f.p.s are now on the region boundary, there can be no closed orbits around them. $\Rightarrow$ no closed orbits.

d) $\beta < \frac{1}{\sqrt{2}}$ brings 2 new f.p.s into region:

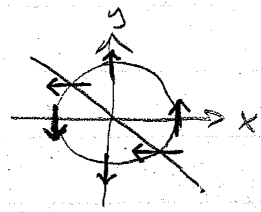
$(x^*, \varphi^*) = (0, \pm \cos^{-1} \sqrt{2}\, b)$

The linearized Jacobian gives $\tau = 0$, $\Delta = \frac{1}{8}(2\beta^2 - 1) < 0$ for both $\Rightarrow$ saddle points.



(2) $\dot{x} = xy$

$\dot{y} = x + y$




f.p.: $\boxed{(x^*, y^*) = (0,0)}$

nullclines: $\dot{x} = 0$ at $x=0$, $y=0$

$\dot{y} = 0$ at $y = -x$

Only need to consider directions at nullclines for index: arrow needs to cross a nullcline to rotate 45°.

$\boxed{\text{Index} = 0}$

(3) a) Try $V(x,y) = x^2 + ay^2$

$\dot{V} = 2x\dot{x} + 2ay\dot{y} = 2x(y - x^3) + 2ay(-x - y^3)$

$= 2(1-a)xy - 2(x^4 + ay^4)$

Let $a = 1 \Rightarrow V(x,y) = x^2 + ay^2$, $\dot{V} = -2(x^4 + y^4) < 0$

↳ no closed orbits

b) If $\dot{\vec{x}} = \begin{pmatrix} f(\vec{x}) \\ g(\vec{x}) \end{pmatrix}$ is a gradient system,

$\dot{x} = -\nabla V = -\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$

Hence, if $\dot{x} = -\nabla V$, $f_y - g_x = -V_{xy} + V_{yx} = 0$.

We can also show the converse ("only if"):

Consider a closed region in $(x,y)$, $C$, with area $A\hat{z}$.

$\oint_C \vec{f} \cdot d\vec{r} = \int_A (\nabla \times \vec{f}) \cdot \hat{z}\, da = \int_A (f_y - g_x)\, da$.

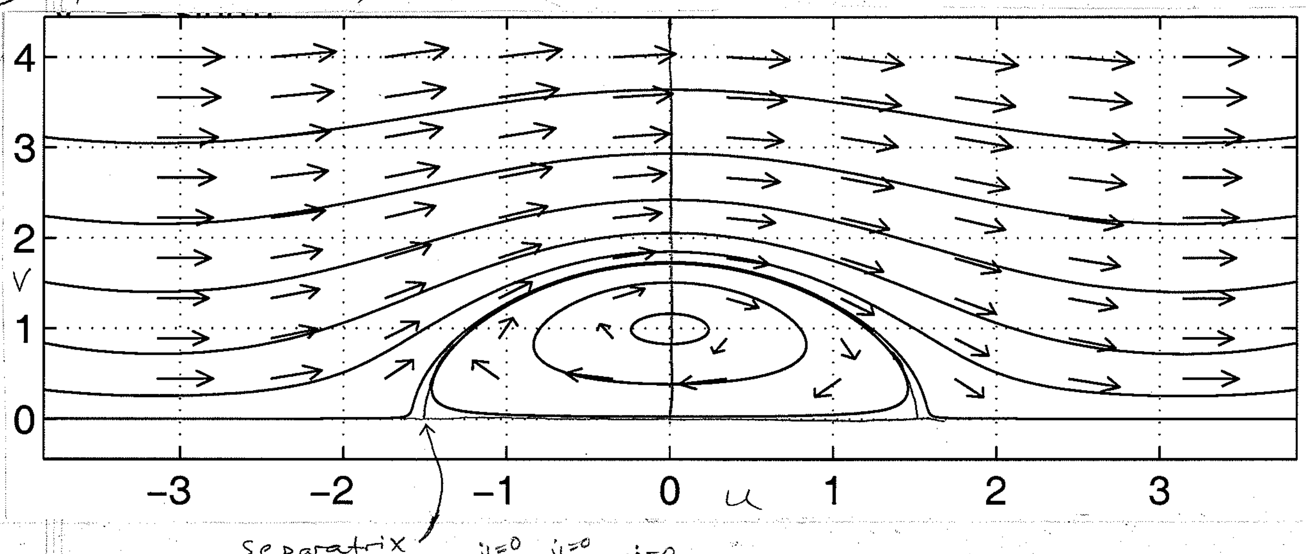
↳ if $f_y - g_x = 0$, then $\oint_C \vec{f} \cdot d\vec{r} = 0$ for any path $C$,

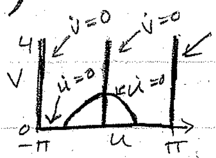
so $\int_A^B \vec{f} \cdot d\vec{r}$ is path independent and hence we can find a $V(\vec{x})$ s.t. $\int_A^B \vec{f} \cdot d\vec{r} = V(B) - V(A)$, in which case $\vec{f} = -\nabla V$.

$\Rightarrow$ Given $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$, if $f_y - g_x = 0$ then it's a gradient system and there are no closed orbits.

Here, $f_y - g_x = (1+2x) - (1+2x) = 0 \Rightarrow$ $\boxed{\text{no closed orbits}}$

(4) a) $\frac{dv}{dt} = -\sin u$ , $v\frac{du}{dt} = -\cos u + v^2$




Nullclines: 


Nullclines & reversibility suggest robust (nonlinear) center.

b) Try to integrate $\dot v, \dot u$:

$\frac{dv}{du} = \frac{\dot v}{\dot u} = \frac{v \sin u}{\cos u - v^2} \Rightarrow v \sin u \, du = (\cos u - v^2)\, dv$

Note $f\,dv + g\,du = 0$ implies $\frac{\partial g}{\partial v} = \frac{\partial f}{\partial u}$.

Let $(f, g) = \nabla \Psi$, and look for $\Psi$ s.t.

$0 = d\Psi = \frac{\partial \Psi}{\partial u} du + \frac{\partial \Psi}{\partial v} dv = (3v \sin u) du + (-3\cos u + 3v^2)\, dv$

$\hookrightarrow \Psi = v^3 - 3v\cos u + \text{const}$

Along trajectories $\boxed{v^3 - 3v\cos u = C}$, where $C$ is an expression of the (conserved) energy of the trajectory.

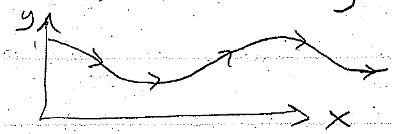
c) From the figure above, we see that in phase space the glider can make loops or wiggly right-moving trajectories (for higher speeds $v$). The separatrix is the trajectory separating the 2 regions. Based on figure, $v(u)$ is double-values inside and single-valued outside $\Rightarrow C = 0$. $\boxed{v = \sqrt{3\cos u}}$

($C < 0$ inside separatrix)

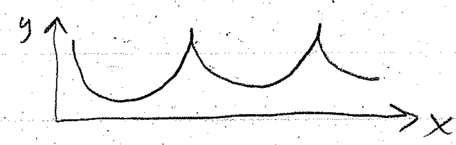
(4) d) $v = \left|\left(\frac{dx}{dt}, \frac{dy}{dt}\right)\right|$ and $u = \tan^{-1}\left(\frac{dy}{dx}\right)$. In physical $(x,y)$ space, trajectories inside the separatrix wiggle and trajectories outside the separatrix spiral (in a sense, opposite to $(u,v)$ behavior).

Inside separatrix, relatively slow wavy motion:

$c < 0$

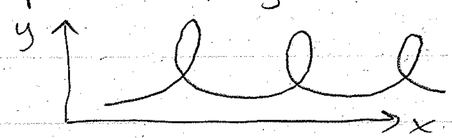



At separatrix, glider flips from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$ almost instantly: $c = 0$

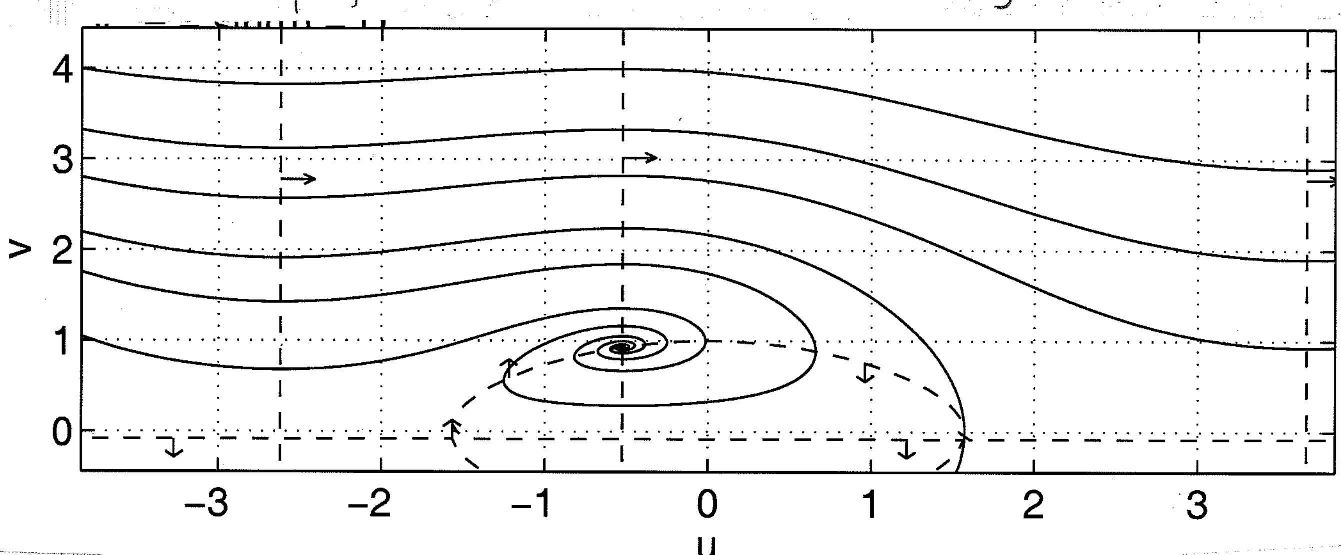



Outside separatrix, glider loops as it travels forward.

$c > 0$




e) $(u^*, v^*) = \left(-\tan^{-1}(D), (1+D^2)^{-1/4}\right)$ is attracting fixed point. All trajectories slow (fast ones loop in $(x,y)$ as they slow) until they reach the f.p., which describes a steady descent.



(5)

$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = \beta z + xy$$

$\vec{x}^* = 0$ is a f. p. Let $\rho = 1$

a) At $\vec{x}^* = 0$, Jacobian is $M = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$ $\quad (\dot{x} = Mx)$

The e'vectors / values are:

$\lambda_0 = -\beta, \quad \lambda_+ = 0, \quad \lambda_- = -(1+\sigma)$.

$V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad V_+ = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad V_- = \begin{pmatrix} \sigma \\ -1 \\ 0 \end{pmatrix}$

b) The e'vectors span the basis so no need for generalized e'vectors and we can rotate so Jacobian is diagonal. Use rotation matrix

$P = [\lambda_+, \lambda_-, \lambda_0] = \begin{pmatrix} 1 & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$J = P^{-1} M P = \begin{pmatrix} \lambda_+ & 0 & 0 \\ 0 & \lambda_- & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(1+\sigma) & 0 \\ 0 & 0 & -\beta \end{pmatrix} = \begin{pmatrix} J_c & 0 \; 0 \\ \begin{matrix} 0 \\ 0 \end{matrix} & (J_s) \end{pmatrix}$

$J_c = \lambda_+, \quad J_s = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_0 \end{pmatrix}$

Now we have $\vec{x} = P\vec{u}$. The linearized system is $\dot{\vec{u}} = J\vec{u}$. To include nonlinear terms, use

$\dot{\vec{x}} = \vec{f}(\vec{x}) \Rightarrow \dot{\vec{u}} = P^{-1}\dot{\vec{x}} = P^{-1}\vec{f}(P\vec{u})$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(1+\sigma) & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} -\frac{\sigma}{1+\sigma}(uw + vw\sigma) \\ \frac{1}{1+\sigma}(uw + vw\sigma) \\ u^2 - \sigma v^2 - uv + uv\sigma \end{pmatrix}$$

We can separate the equations for the trajectory along the center manifold and the stable manifold; they are linearly independent but have nonlinear coupling: $u_c \equiv u$, $\vec{u}_s \equiv \begin{pmatrix} v \\ w \end{pmatrix}$

$\dot{u}_c = J_c u_c + G_2(u_c, \vec{u}_s) + O(\vec{u}^3)$

$u_s = J_s u_s + H_2(u_c, \vec{u}_s) + O(\vec{u}^3)$

(5) b) cont'd

$G_2$ and $H_2$ are polynomials with only $O(\vec{u}^2)$ terms.

Recall that the point of all this is to write independent equations for motion in the center and stable manifolds so we can see what bifurcation happens in the center manifold. Here $u$ is tangent to the center manifold at the origin (bif'n point), but the $u_c^2$ term in $H_2$ (i.e., the $u^2$ term in $\dot{w}$) implies that the manifold turns away from the u-axis like $u^2$. Trajectories travel along the center manifold, i.e., it is invariant, and the $u^2$ term implies that trajectories leave the u-axis, i.e. it is not invariant, like $u^2$.

d) Let $\tilde{w} = w - au^2$. Find $a$ s.t. $\dot{\tilde{w}}$ has no $u^2$ term.

$$\dot{\tilde{w}} = \dot{w} - 2au\dot{u} = \left\{-\beta w + u^2 - \sigma v^2 - uv + uv\sigma\right\} - \left\{\frac{2au\sigma}{1+\sigma}(u+v\sigma)w\right\}$$

Insert $w = \tilde{w} + au^2$

$$\dot{\tilde{w}} = \left(-\beta\tilde{w} - \beta a u^2 + u^2\right) + \left\{-\sigma v^2 - uv + uv\sigma\right\} - \left\{\frac{2au\sigma}{1+\sigma}(u+v\sigma)(\tilde{w}+au^2)\right\}$$

Letting $a = \frac{1}{\beta}$ will make $(\,) = -\beta\tilde{w}$. $\{\}$ and $\{\}$ remain with no $u^2$ term. Hence, $\boxed{\tilde{w} = w - a^2}$ eliminates $u^2$.

(5) e) Examining the solution to part (b), we see that this transformation that made $\tilde{w}$ independent of $u$ to $O(u^2)$ did not introduce any $O(u^2)$ term in $v$ (just a $vu^2$ term). Since the equations for $v$ and $\tilde{w}$ have no terms of the form $u^2$, the $u$ axis in now invariant to second order.

f) Since $\tilde{w} = w + O(\vec{u}^2)$, we still have

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \lambda_+ & 0 & 0 \\ 0 & \lambda_- & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + O(\vec{u}^2)$$

Now we want to vary $\rho$ (no longer $\rho=1$) to find the bifurcation. We can re-compute the e'value $\lambda_+$, getting

$$\lambda_+ = -\frac{1}{2}\left(1+\sigma - \sqrt{1-2\sigma+4\rho\sigma+\sigma^2}\right) = \frac{\sigma}{1+\sigma}(\rho-1) + O(\rho-1)^2$$

The full equation for $\dot{u}$ is

$$\dot{u} = \lambda_+ u - \frac{\sigma}{1+\sigma}(u+v\sigma)\left(\tilde{w} + \frac{u^2}{\beta}\right)$$

So, along the $u$-axis (approximately the center manifold) it is

$$\dot{u} = \lambda_+ u - \frac{\sigma}{\beta(1+\sigma)} u^3$$

Inserting $\lambda_+$ to $O(\rho-1)$, it is

$$\boxed{\dot{u} = \frac{\sigma}{1+\sigma}\left((\rho-1)u - \frac{1}{\beta}u^3\right)}$$

We could easily transform this to normal form ($\mu = \rho-1$)

$\dot{U} = \mu U - U^3$ <u>supercritical</u> <u>pitchfork</u>

↳ Lorentz system has a pitchfork bifurcation at $\rho = 1$