Homework #4

Nonlinear dynamics and chaos

1. Do problem 6.6.8 from Strogatz, page 191.

2. Find fixed points, draw vector field around them, and calculate their index for the system:
$$\dot{x} = xy; \qquad \dot{y} = x + y$$

3. Do the following systems have a limit cycle solution?

   (a) (Construct a Lyapunov function...):
   $$\dot{x} = y - x^3; \qquad \dot{y} = -x - y^3$$

   (b) does the following system have a limit cycle solution: (gradient system?)
   $$\dot{x} = y + 2xy; \qquad \dot{y} = x + x^2 - y^2$$

Do at least one of problems 4 and 5; optionally (challenge) both:

4. A glider: Let v = speed of glider and u = angle flight path makes with the horizontal. In the absence of drag (friction), the dimensionless equations of motion are:
$$dv/dt = -\sin u; \quad v du/dt = -\cos u + v^2$$

   (a) Using numerical integration, sketch the trajectories on a slice of the u-v phase plane between $-\pi < u < \pi, v > 0$.

   (b) Obtain an exact expression for the trajectories

   (c) Using your result in part b, obtain an exact expression for the separatrix in this system.

   (d) What does the flight path of the glider look like for motions inside the separatrix versus motions outside the separatrix? Sketch the glider's flight path in both cases.

   (e) If there is also a drag force, then
   $$dv/dt = -\sin u - Dv^2; \quad v du/dt = -\cos u + v^2.$$
   Describe what happens then.

5. Center manifold theorem: Analyze the bifurcation of the Lorenz system

$$\dot{x} = \sigma(y - x) \quad (1)$$

$$\dot{y} = \rho x - y - xz \quad (2)$$

$$\dot{z} = -\beta z + xy \quad (3)$$

at $\rho = 1$ and $(x, y, z) = (0, 0, 0)$ (verify that this is a fixed point):

(a) Find the eigenvalues and eigenvectors

(b) Use the eigenvectors to write the system in a Jordan form around the bifurcation point, let the new variables be (u, v, w).

(c) let the center manifold be in the u direction, and the z variable is transformed to $w = z$. Note that the w equation contains a u^2 term. Explain why this makes it non-invariant.

(d) transform the w equation to an invariant form using a nonlinear polynomial transformation $\tilde{w} = w - au^2$ and find the appropriate a.

(e) Verify that the transformed system is invariant to second order (to order of the square of the new transformed variables).

(f) Examining the form of the transformed equation on the center manifold, which bifurcation do you expect the Lorenz equations to have at this point.