Homework #7
Nonlinear dynamics and chaos

1. Calculate a numerical approximation for

$$\lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

for the following two maps

$$x_{n+1} = r\sin(\pi x_n), \quad 0 \le r \le 1 \qquad \text{(the sine map)} \tag{1}$$

$$x_{n+1} = r - x_n^4 \tag{2}$$

by iterating the maps for different r-values and finding the r values at which period doubling(s) occur. Compare the results to δ for the logistic map and explain.

2. (Strogatz 10.7.3,4) some simple renormalization-related issues:

   (a) Show that if $g(x)$ is a fixed point of the doubling transformation, that is,

   $$g(x) = -\alpha g\left[g\left(\frac{x}{-\alpha}\right)\right] \equiv T[g],$$

   so is $\mu g(x/\mu)$.

   (b) show that $g(x)$ crosses the line $y = \pm x$ an infinite number of times by showing that if x^* is a fixed point of $g(x)$, so is $-\alpha x^*$.

   (c) Calculate an approximation to the universal α for the period doubling route to chaos. Start with the map $f(x,r) = r - x^2$, assume a two-term expansion for the universal function:

   $$g(x) = 1 + c_2 x^2.$$

   and calculate c_2 and α that approximately satisfy the functional equation for $g(x)$.

3. Show that

$$g_{i-1}(x) = (-\alpha) g_i\left[g_i\left(-\frac{x}{\alpha}\right)\right] \; (\equiv T[g_i(x)]).$$

Explain each stage in your derivation. (Schuster derives this, so you just need to explain what he does).

4. Do **only one** of the following two questions:

   (a) read, understand, and reproduce the approach of Strogatz “renormalization for pedestrians” pages 384-387 in order to analytically calculate an approximate to both δ and α for a quadratic maximum map. Skip example 10.7.2, but do example 10.7.3.

   (b) **Challenge question/ extra(!) credit:** First the easier part: Find α for quartic functions (such as $x_{n+1} = r - x_n^4$) using the approach of question 2c
   Next: a challenge in the best sense of the word (i.e. I have not tried this myself, and I don’t know that it is possible). Follow the approach of Strogatz “renormalization for pedestrians” on page 384-387 in order to analytically calculate both δ and α for a quartic maximum function (such as $x_{n+1} = r - x_n^4$). Compare your analytically derived results to the numerical approximation for δ from question 1.