Homework #8
Nonlinear dynamics and chaos

1. **Calculating δ from the linearized eigenproblem:**

   (a) Linearize the doubling transformation around $R \to R_\infty$ and obtain the expression for L_f.

   (b) Write the eigenvalue problem for the linearized doubling transformation: $L_g h(x) = \delta h(x)$, expand $h(x)$ in a Taylor expansion keeping only the first constant term $h(0)$; evaluate the eigenvalue equation at $x = 0$. Show that the resulting equation is $-\alpha\{g'[g(0)]+1\} = \delta$

   (c) To find $g'(1)$, differentiate the equation $g(x) = Tg(x)$ twice and evaluate it at $x = 0$. Use the fact that $g''(0) \neq 0$ for a quadratic map.

   (d) Use the above to show that $\delta \approx \alpha^2 - \alpha$ and calculate the numerical value from the known value of α. What is the error with respect to the actual value of δ?

2. **Intermittency type I numerics:** plot the iterates of the logistic map (x_n as function of n) at the point where it displays the intermittent chaotic behavior near the 3-period window for three different values of r. Connect the iterates by a line, and mark them with a symbol as well (thick dot or an "x"). Explain how the length of the periodic intervals varies with r in your results.

3. **Length of non-chaotic intervals for type III intermittency:** Find a map that appropriately describes a type III trapping region in the intermittency route to chaos. Explain why this map is the right one (can use Schuster or Ott...). Use $x_{n+2} - x_n \approx dx/dn$ (why is it appropriate to use this approximation in the trapping region? why $n+2$ rather than $n+1$?) and show that the length of non-chaotic intervals in this case behaves like ϵ^{-1}. Plot the iterates of the map in the trapping region.

4. **Quasi-periodicity route to chaos:** Use the Matlab program `pendulum.m` on the course home page for simulating a periodically forced and damped pendulum,

   (a) Choose some value for all model parameters, and increase the amplitude of the forcing from zero in order to see how the pendulum behavior changes from damped to chaotic. Explain what regimes are encountered on the way.

   (b) Plot the time series of $\theta(t)$ (either from actually running the Matlab program, or just schematically) from the solution of the forced and damped pendulum equation, for each of the regimes encountered from the damped to the chaotic regimes.

   (c) Plot a Poincare section based on sub-sampling the time series for $\theta(t)$ every period of the forcing. Do this for the different regimes that are encountered as the forcing amplitude increases from damped to chaotic regimes. Explain why the Poincare section looks the way it does for each of these regimes.

   (d) What would a quasi-periodic regime mean in this problem? Do you see it in your experiment as the forcing amplitude is increased? Why?

   (e) Could you anticipate the above results just by examining the equation for the pendulum? explain.

5. **Logistic map orbit diagram and some interesting patterns:**

   (a) Plot the orbit diagram for the logistic map in Matlab. Don't forget to iterate the map from the initial conditions until it converges to the attractor (fixed point, periodic orbit, or chaos) before starting to plot the iterations on the orbit diagram.

   (b) Plot another version of the orbit diagram in which you start from some arbitrary initial condition and plotting the iterates from this initial condition, rather than waiting for convergence to the attractor.

   (c) Strogatz 10.3.13 (Tantalizing patterns): (a) There are several smooth, dark tracks of points running through the chaotic part of the diagram. what are these curves? (hint: think about $f(x_m, r)$, where $x_m = 1/2$ is the maximum point of f). (b) Can you find the exact value of r at the corner of the "big wedge"? (hint: several of the dark tracks intersect at this corner.)