Laure Zanna
11/07/05

Introduction to Physical Oceanography
Homework 6 - Solutions

1. Buoyancy oscillations with friction: the expression equivalent to

   linearized vertical acceleration = buoyancy + vertical (non-scale selective) friction

   is given by

$$\frac{\partial w}{\partial t} = -\frac{\rho_0 - \rho}{\rho_0} g - Jw \tag{1}$$

   where ρ_0 is the density of the environment and ρ is the density of a fluid parcel.

   For a small vertical displacement δz of a parcel of fluid, the density ρ using a Taylor expansion around its rest position at $z = 0$ is equal to

$$\rho(z) = \rho_0 + \frac{d\rho}{dz}\delta z = \rho_0\left(1 + \frac{1}{\rho_0}\frac{d\rho}{dz}\delta z\right) \tag{2}$$

   By definition, the velocity is given by

$$w = \frac{\delta z}{\delta t} \tag{3}$$

   leading to

$$\frac{\delta^2 z}{\delta t^2} = \frac{g}{\rho_0}\frac{d\rho}{dz}\delta z - J\frac{\delta z}{\delta t} \tag{4}$$

   or

$$\frac{\delta^2 z}{\delta t^2} + J\frac{\delta z}{\delta t} - \frac{g}{\rho_0}\frac{d\rho}{dz}\delta z = 0 \Rightarrow \frac{\delta^2 z}{\delta t^2} + J\frac{\delta z}{\delta t} + N^2\delta z = 0 \tag{5}$$

   such that $N^2 = -(g/\rho_0)d\rho/dz$ where N is the Brunt-Vaisala frequency (note: we assumed that we are in a stably stratified ocean therefore $d\rho/dz < 0$ and $N^2 > 0$). We expect the solution to this 2nd ODE to be of the form $\delta z = Ae^{\lambda t}$ such that

$$\lambda^2 + J\lambda + N^2 = 0 \Rightarrow \lambda = -\frac{J}{2} \pm \sqrt{\frac{J^2}{4} - N^2} \tag{6}$$

   The solution is then given by

$$\delta z = Ae^{0.5\left(-J+\sqrt{J^2-4N^2}\right)t} + Be^{0.5\left(-J-\sqrt{J^2-4N^2}\right)t} \tag{7}$$

   The coefficients A and B can be determined from the initial conditions. Assuming that $\delta z(t = 0) = \delta z_0$ and $\delta z/\delta t|_{t=0} = 0$, we find that

$$\begin{aligned} A &= \frac{\delta z_0}{2}\left(1 + \frac{J}{\sqrt{J^2 - 4N^2}}\right) \\ B &= \frac{\delta z_0}{2}\left(1 - \frac{J}{\sqrt{J^2 - 4N^2}}\right) \end{aligned}$$

The solution is

$$\delta z = \frac{\delta z_0}{2} e^{-0.5Jt} \left[\left(1 + \frac{J}{\sqrt{J^2 - 4N^2}}\right) e^{0.5\sqrt{J^2-4N^2}t} + \left(1 - \frac{J}{\sqrt{J^2 - 4N^2}}\right) e^{-0.5\sqrt{J^2-4N^2}t} \right] \quad (8)$$

Two cases arise from this solution

- If $J^2 > 4N^2$ (the argument under the square root is positive), then there are no oscillations. The friction coefficient is larger than N (see Fig. 1 case 1). In this case, the friction is so large that the fluid parcel returns to its equilibrium position without oscillating.
- If $J^2 < 4N^2$ (the argument under the square root is negative leading to an imaginary number). We can write

  $$\frac{J^2}{4} - N^2 = i^2 \left(N^2 - \frac{J^2}{4}\right)$$

  and the solution is now

  $$\delta z = \delta z_0 e^{-0.5Jt} \left[cos\left(\sqrt{N^2 - \frac{J^2}{4}}t\right) + \frac{J}{\sqrt{4N^2 - J^2}} sin\left(\sqrt{N^2 - \frac{J^2}{4}}t\right)\right] \quad (9)$$

  In this case, the parcel oscillates around its equilibrium position and the amplitude of the oscillations decays in time (see Fig. 1 case 2). The frequency of the oscillations is now equal to the Brunt-Vaisala frequency but to $\sqrt{N^2 - \frac{J^2}{4}}$.

How do we know if the parcel of fluid will be oscillating before decaying? Let's compare the values of N and $J/2$ in the ocean.

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz} = -\frac{9.81}{1026}\frac{-3}{5000} \approx 5.7 \cdot 10^{-6} s^{-2} \Rightarrow N \approx 0.0024 s^{-1}$$

Friction in the ocean is roughly taken to be of the order $J = O(10^{-3}, 10^{-4})$. Therefore, both cases described above are theoretically possible.

Note: this problem is equivalent to a damped harmonic oscillator: for example, an object attached to a spring assuming that there is friction between the table and the object.

2. Surface Ekman Spiral: Consider the balance of Coriolis force and vertical friction:

$$-fv = A_v u_{zz} \quad (10)$$

$$fu = A_v v_{zz} \quad (11)$$

(a) We wish to find a single equation for u. First we take $\partial^2/\partial z^2$ of eq. 10

$$-fv_{zz} = A_v u_{zzzz} \Rightarrow v_{zz} = -\frac{A_v}{f} u_{zzzz} \quad (12)$$

Substitute v_{zz} obtained above into eq. 11

$$fu = -A_v \frac{A_v}{f} u_{zzzz} \Rightarrow u_{zzzz} + \frac{f^2}{A_v^2} u = 0. \quad (13)$$

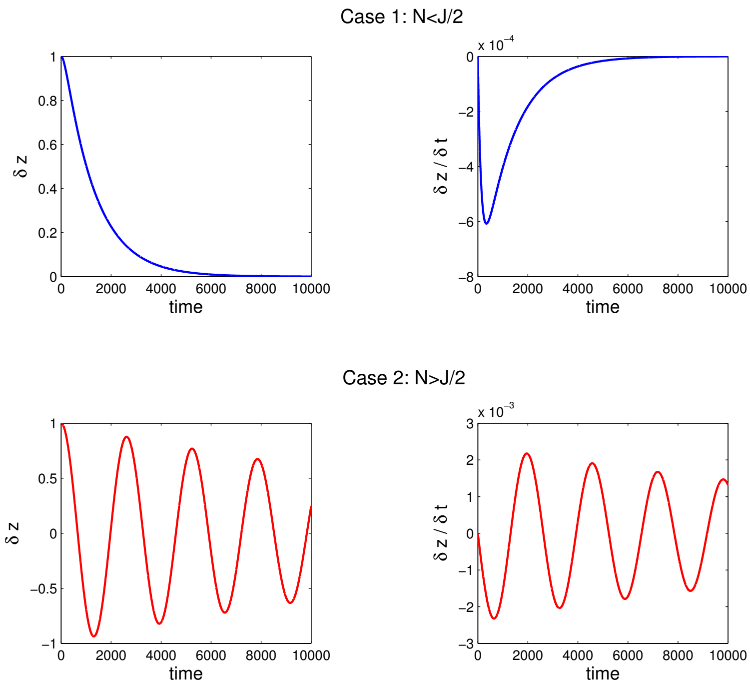

Figure 1: Displacement δz and vertical velocity $\delta z/\delta t$ for 2 different cases: 1. $J/2 = 4 * 10^{-3} > N = 0.0024$, and 2. $J/2 = 10^{-4} > N = 0.0024$.

(b) We can assume that the solution is of the form $u = e^{az}$ such that

$$a^4 + \frac{f^2}{A_v^2} = 0 \Rightarrow a = \pm\sqrt{\pm\frac{if}{A_v}} \tag{14}$$

Using $i = e^{i\pi/2}$, $i^{\pm 1/2} = e^{\pm i\pi/4} = 1/\sqrt{2} \pm i/\sqrt{2}$, we get

$$a_1 = (1+i)\sqrt{\frac{f}{2A_v}}\,;\, a_2 = (1-i)\sqrt{\frac{f}{2A_v}}\,;\, a_3 = -(1+i)\sqrt{\frac{f}{2A_v}}\,;\, a_4 = -(1-i)\sqrt{\frac{f}{2A_v}}$$

(c) The solution for $u(z)$ is given by a linear combination of the four exponents found above such that

$$u(z) = b_1 e^{\gamma(1+i)z} + b_2 e^{\gamma(1-i)z} + b_3 e^{-\gamma(1+i)z} + b_4 e^{-\gamma(1-i)z} \tag{15}$$

where $\gamma = \sqrt{f/2A_v}$ and the coefficients $b_{1,2,3,4}$ are defined from the boundary conditions. Away from the surface, we expected the solution to decay exponentially: since the forcing is applied at $z = 0$ and will decay with depth, as $z \to -\infty$ the solution should be bounded. Therefore we require $b_3 = b_4 = 0$, and we are left with

$$u(z) = b_1 e^{\gamma(1+i)z} + b_2 e^{\gamma(1-i)z} \tag{16}$$

(d) We can now find $v(z)$ such that

$$-fv = A_v u_{zz} \Rightarrow v = -i\left(a_1 e^{\gamma(1+i)z} - a_2 e^{\gamma(1-i)z}\right) \tag{17}$$

Using the boundary conditions $u(z=0)=1$ and $v(z=0)=0$, we obtain

$$b_1+b_2=1\,;b_1=b_2\Rightarrow b_1=b_2=\frac{1}{2} \tag{18}$$

such that

$$\begin{aligned} u(z) &= e^{\gamma z}cos(\gamma z) \\ v(z) &= e^{\gamma z}sin(\gamma z) \end{aligned}$$

where $\gamma=\sqrt{f/2A_v}$.

(e) From fig. 2, we see that our solution describes a spiral, in 3D we can see the decay and rotation of the velocity vector.

(f) The decay scale of the velocity (called also the Ekman depth) is given by

$$\delta_E=\frac{1}{\gamma}=\sqrt{\frac{2A_v}{f}} \tag{19}$$

Note that the Ekman depth depends on the Coriolis parameter and on the friction coefficient. The Ekman depth increases with increasing friction (the stronger friction will penetrate deeper into the Ekman layer) and with decreasing latitude (for the same friction, the Ekman layer thickness will be larger near the equator than at higher latitudes). Using $f=2\Omega sin(45°N)\approx 1.0324\cdot 10^{-4}sec^{-1}$ and $A_v=100cm^2/sec$, the Ekman depth is estimated to be $\delta_E\approx 14m$.

3. Review of equations

(a) 3D momentum equation
Horizontal momentum equations:

$$\underset{hm_1}{\frac{\partial \vec{u}_H}{\partial t}} \quad \underset{hm_2}{+(\vec{u}_H\cdot\nabla_H)\vec{u}_H} \quad \underset{hm_3}{+w\frac{\partial \vec{u}_H}{\partial z}} \quad \underset{hm_4}{+f\hat{k}\times\vec{u}_H} = \quad \underset{hm_5}{-\frac{1}{\rho}\nabla_H p} \quad \underset{hm_6}{+A_h\nabla_H^2\vec{u}_H} \quad \underset{hm_7}{+A_z\frac{\partial^2\vec{u}_H}{\partial z^2}} \quad \underset{hm_8}{-J\vec{u}_H}$$

where horizontal velocity vector is given by $\vec{u}_H=(u,v)$, the 2D operator $\nabla_H=(\frac{\partial}{\partial x},\frac{\partial}{\partial y})$.
Vertical momentum equation:

$$\underset{vm_1}{\frac{\partial w}{\partial t}} \quad \underset{vm_2}{+(\vec{u}_H\cdot\nabla_H)w} \quad \underset{vm_3}{+w\frac{\partial w}{\partial z}} = \quad \underset{vm_4}{-g\frac{\rho}{\rho_0}} \quad \underset{vm_5}{-\frac{1}{\rho_0}\frac{\partial p}{\partial z}} \quad \underset{vm_6}{+A_h\nabla_H^2 w} \quad \underset{vm_7}{+A_z\frac{\partial^2 w}{\partial z^2}} \quad \underset{vm_8}{-Jw}$$

(b) Continuity equation for an incompressible fluid

$$\underset{c_1}{\nabla_H\cdot\vec{u}_H} \quad \underset{c_2}{+\frac{\partial w}{\partial z}}=0$$

Temperature equation

$$\underset{t_1}{\frac{\partial T}{\partial t}} \quad \underset{t_2}{+(\vec{u}_H\cdot\nabla_H)T} \quad \underset{t_3}{+w\frac{\partial T}{\partial z}} = \quad \underset{t_4}{\kappa_h\nabla_H^2 T+} \quad \underset{t_5}{\kappa_v\frac{\partial^2 T}{\partial z^2}+} \quad \underset{t_6}{H^{air-sea}}$$

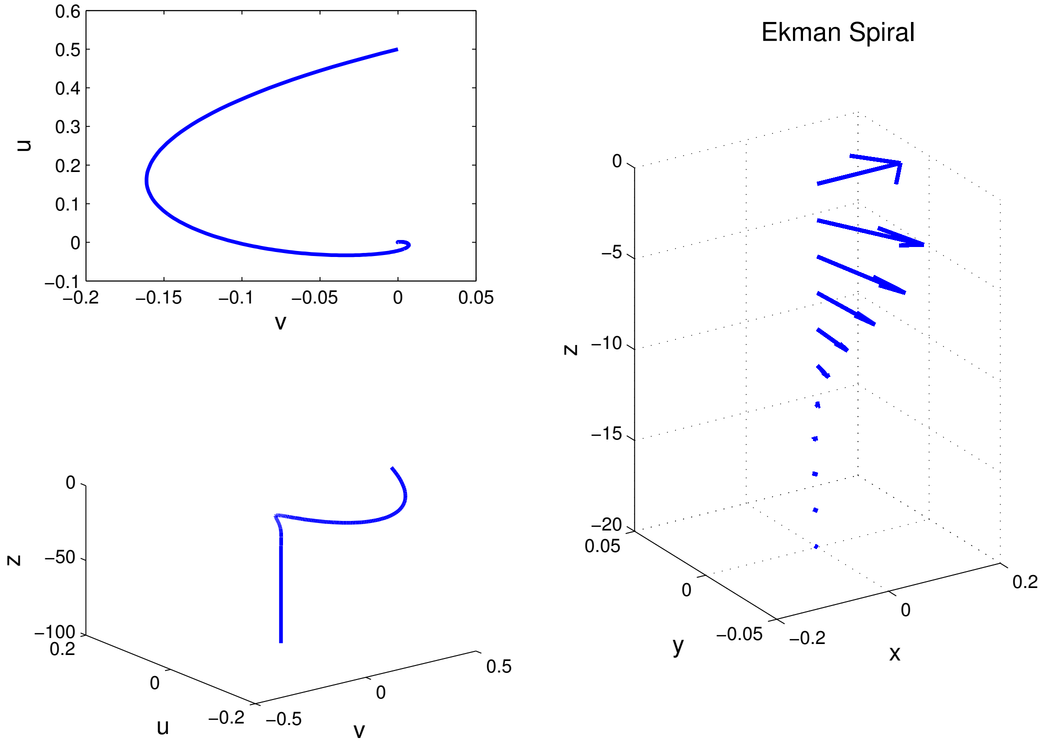

Figure 2: Ekman spiral

(c) Physical Meaning:
Horizontal momentum equation: hm_1: local rate of change of horizontal velocity; $hm_{2,3}$: horizontal and vertical advection of horizontal velocity; hm_4: Coriolis force ; hm_5: horizontal pressure gradient; $hm_{6,7}$: scale selective friction; hm_8: bottom friction
Vertical momentum equation: vm_1: local rate of change of vertical velocity; $vm_{2,3}$: horizontal and vertical advection of vertical velocity; vm_4: gravity; vm_5: vertical pressure gradient; $vm_{6,7}$: scale selective friction; vm_8: bottom friction
Continuity equation: c_1: horizontal divergence; c_2: vertical divergence
Temperature equation: t_1: local rate of change of temperature; $t_{2,3}$: horizontal and vertical advection of temperature; $t_{4,5}$: horizontal and vertical diffusion of temperature; t_6: air-sea heat flux

Some of the important phenomena described in class
geostrophy: $hm_4 \approx hm_5$
hydrostatic balance: $vm_4 \approx vm_5$
inertial oscillations: $hm_1 \approx hm_4$
buoyancy oscillations: $vm_1 \approx vm_4$
Ekman layer: $hm_4 \approx hm_7$
abyssal recipes: $t_3 \approx t_5$

(d) We wrote 4 equations: horizontal momentum equation, vertical momentum equation, continuity equation and temperature equation. Since the horizontal momentum equation is in a vector form for $\vec{u}_H = (u, v)$, this equation is equivalent to 2 scalar equations.

Therefore, we have 5 scalar equations.
We have 6 unknowns which are: u, v, w, p, ρ, T, and only 5 equations. The missing equation will be the equation of state given by

$$\rho = \rho(T, S, p) \tag{20}$$

and S is given by the salinity equation (similar to the heat equation).