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Introduction to Physical Oceanography
Homework 8 - Solutions

1. Geostrophy: The velocity of the flow is defined by

$$u = u_0 cos\left(\frac{\pi y}{2L}\right) e^{z/H} \quad (1)$$

   (a) Figure 1a shows the velocity as function of latitude at $z = 0$ and fig. 1b shows a contour plot of the velocity as function of latitude and depth (don't forget that z is negative since we take $z = 0$ to be the mean sea level - otherwise the velocity will blow up!).

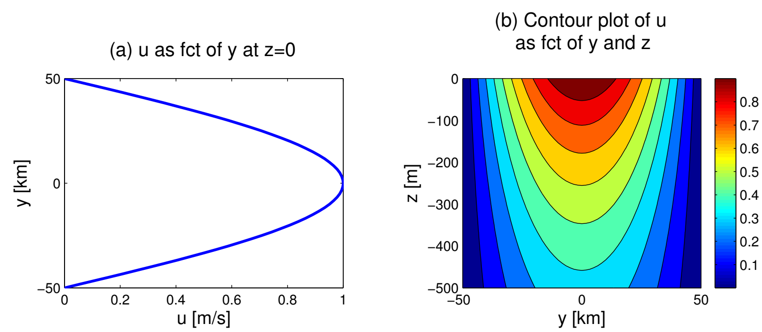

Figure 1: (a) Velocity as function of latitude; (b) contour plot of the velocity as function of latitude and depth.

   (b) The flow is assumed to be in geostrophic and hydrostatic balance and has only a $y-$component, therefore the governing equations are given by

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \quad (2)$$

$$\frac{\partial p}{\partial z} = -\rho g \quad (3)$$

(From the Boussinesq approximation: in the horizontal momentum equation, we can assume that ρ equal some reference density ρ_0 but we cannot assume that for the vertical momentum equation).

By integrating vertically the hydrostatic equation, we obtain

$$p(z) = \langle\rho\rangle g (\eta(y) - z) \quad (4)$$

where $\eta(y)$ is the sea surface elevation and

$$<\rho>= \frac{1}{\eta - z}\int_z^{\eta} g\rho dz$$

. In this question, we are interested in region near the surface, therefore we can set this average density $<\rho>$ to be equal to the reference density ρ_0 such that

$$p(z) = \rho_0 g\,(\eta(y) - z) \tag{5}$$

By plugging eq. 5 into eq. 2, we obtain that

$$fu = -g\frac{\partial\eta}{\partial y} \Rightarrow \frac{\partial\eta}{\partial y} = -\frac{fu}{g} \tag{6}$$

We can integrate this equation at the surface ($z = 0$) from $\eta(y = -L)$ to $\eta(y)$ such that

$$\int_{\eta(y=-L)}^{\eta(y)} \partial\eta = -\int_{y=-L}^{y} \frac{fu}{g}\partial y = -\frac{fu_0}{g}\int_{y=-L}^{y} cos\left(\frac{\pi y}{2L}\right)\partial y \tag{7}$$

leading to

$$\eta(y) - \eta(y = -L) = -\frac{2Lfu_0}{g\pi} sin\left(\frac{\pi y}{2L}\right)\Big|_{-L}^{y} \tag{8}$$

$$\eta(y) = -\frac{2Lfu_0}{g\pi}\left[sin\left(\frac{\pi y}{2L}\right) + 1\right] + \eta(y = -L) \tag{9}$$

$$\eta(y) = -\frac{2Lfu_0}{g\pi} sin\left(\frac{\pi y}{2L}\right) \tag{10}$$

I've used

$$\eta(y = L) - \eta(y = -L) = -\frac{4Lfu_0}{g\pi}$$

and assumed that $\eta(L) = -\eta(-L)$ for simplicity. Figure 2 shows the sea surface height as function of latitude. The gradient in sea surface height is equal to $-fu/g$ where for this problem u and g are positive and the Coriolis parameter $f = 2\Omega sin\,(45°S) = -\sqrt{2}\Omega$ is negative, so we expect the gradient in sea surface height to be positive and the sea surface height η must increase as y increases as seen in the plot.

(c) To find an expression for $\Delta\rho = \rho(y,z) - \rho(0,z)$, we need to use the thermal wind equation. For that we take the $z-$derivative of the geostrophic given by eq. 2 and make use of the hydrostatic equation (eq. 5) such that

$$\frac{\partial u}{\partial z} = -\frac{1}{f\rho_0}\frac{\partial}{\partial z}\frac{\partial p}{\partial y}$$

$$\frac{\partial u}{\partial z} = -\frac{1}{f\rho_0}\frac{\partial}{\partial y}(-\rho g)$$

leading to

$$\frac{\partial u}{\partial z} = \frac{g}{f\rho_0}\frac{\partial\rho}{\partial y} \tag{11}$$

(Note that ρ in the hydrostatic equation can not be approximated as a constant since its horizontal variation are setting the shear of the flow over the entire column of water). We have

$$\frac{\partial\rho}{\partial y} = \frac{fu_0\rho_0}{gH} cos\left(\frac{\pi y}{2L}\right) e^{z/H} \tag{12}$$

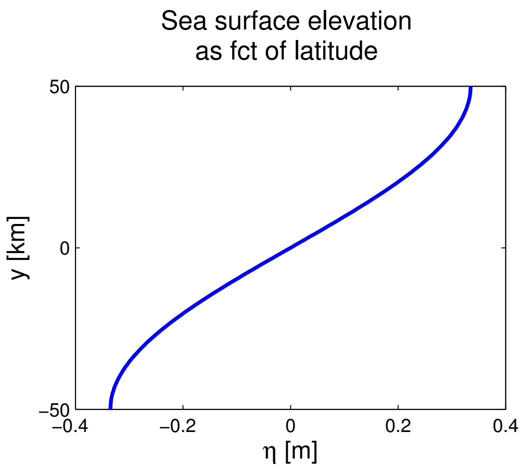

Figure 2: Sea surface height η as function of latitude.

We need to integrate this equation from $y = 0$ to y such that

$$\begin{aligned}\int_{\rho(0,z)}^{\rho(y,z)} \partial\rho &= \rho(y,z) - \rho(0,z) = \int_{y=0}^{y} \frac{f u_0 \rho_0}{gH} cos\left(\frac{\pi y}{2L}\right) e^{z/H} \partial y \\ \Delta\rho &= \frac{2Lf u_0 \rho_0}{gH\pi} sin\left(\frac{\pi y}{2L}\right) e^{z/H}\end{aligned}$$

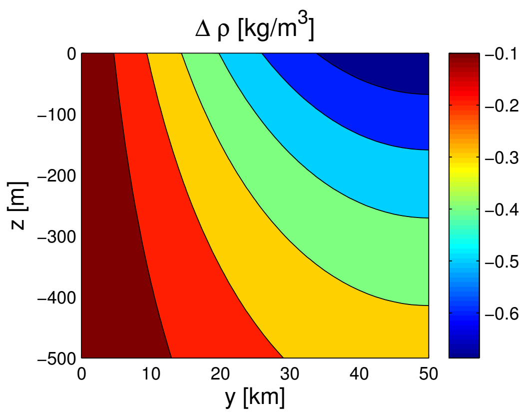

Figure 3: Contour plot of $\Delta\rho = \rho(y,z) - \rho(0,z)$ as function of depth and latitude for $0 < y < L$.

(d) From the thermal wind equation, we have

$$\frac{\partial u}{\partial z} = \frac{g}{f\rho_0}\frac{\partial \rho}{\partial y} \tag{13}$$

Assuming that the density is given by the linear equation of state

$$\rho = \rho_0 \left(1 - \alpha(T - T_0)\right)$$

we obtain

$$\frac{\partial u}{\partial z} = -\frac{g\alpha}{f}\frac{\partial T}{\partial y} \tag{14}$$

such that

$$\frac{\partial T}{\partial y} = -\frac{fu_0}{g\alpha H} cos\left(\frac{\pi y}{2L}\right) e^{z/H} \tag{15}$$

and integrating from $y = -L$ to $y = L$ we obtain

$$\begin{aligned}
\int_{T(y=-L)}^{T(y=L)} \partial T &= -\frac{fu_0}{g\alpha H}\int_{y=-L}^{y=L} cos\left(\frac{\pi y}{2L}\right) e^{z/H}\partial y \\
T(L,z) - T(-L,z) &= -\frac{2Lfu_0}{g\alpha H\pi} e^{z/H} sin\left(\frac{\pi y}{2L}\right)\Big|_{y=-L}^{y=L} \\
\Delta T(z) &= -\frac{4Lfu_0}{g\alpha H\pi} e^{z/H}
\end{aligned}$$

For the reference density given in this problem $\rho_0 = 1025\,kg/m^3$, we find in the table posted on the course homepage that the thermal expansion coefficient α is equal to $2489 * 10^{-7}K^{-1}$. At $z = -300\,m$, we have

$$\Delta T(z = -300m) = -\frac{4\,(50km)\,(-\sqrt{2}\Omega s^{-1})\,(1m \cdot s^{-1})}{(9.81m \cdot s^{-2})\,(2489 \cdot 10^{-7}K^{-1})\,(500m)\,\pi} e^{-300m/500m} \approx 3\,K = 3\,°C$$

Therefore there is a difference of roughly $3°C$ over the domain at a depth of $300m$.

2. Acceleration: The Eulerian flow is given by

$$\begin{aligned}
u &= a(x+y) \\
v &= a(x+y)
\end{aligned}$$

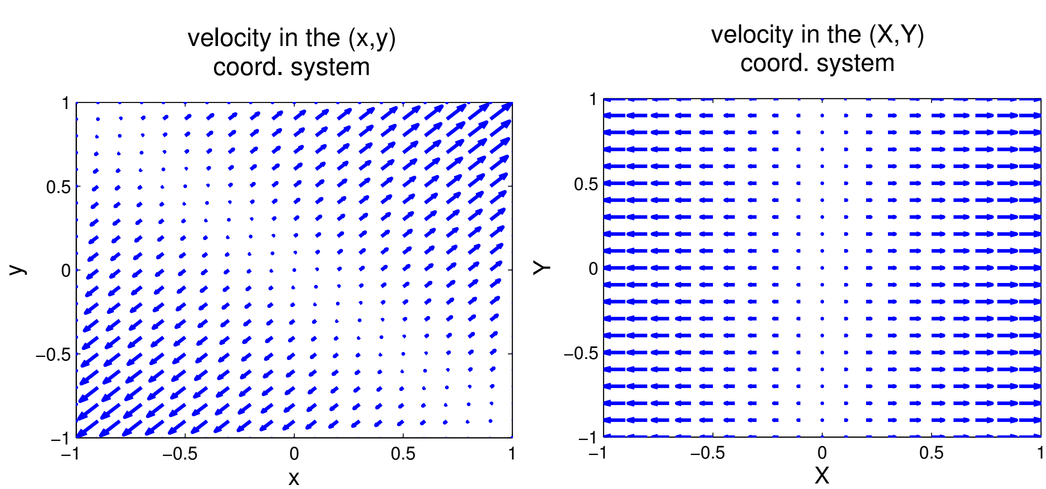

Figure 4: Quiver plot of the velocity - Left: in the frame of reference (x,y); Right: in the new reference frame (X,Y).

From the left panel in figure 4, we see that the velocity has always an angle of 45° with the $x-$axis, therefore it seems like a good idea to use a transformation of coordinates. The new coordinate system (X,Y) is obtained by a counterclockwise rotation of 45° of the original coordinate system (x,y) such that

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} cos 45^\circ & sin 45^\circ \\ -sin 45^\circ & cos 45^\circ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} x+y \\ -x+y \end{pmatrix}$$

It is now very easy to transform our u and v velocities from the (x,y) coordinate system into U and V in the new (X,Y) coordinate system:

$$\begin{aligned} U &= \frac{\sqrt{2}}{2}(u+v) = 2aX \\ V &= \frac{\sqrt{2}}{2}(-u+v) = 0 \end{aligned}$$

We need to consider only the X component of the velocity, since the Y component is 0 (which was the goal of our transformation!). The velocity field in the (X,Y) coordinate system is shown on the right panel of figure 4. Let consider only $X > 0$, and we see that the value of the velocity U increases as X increases. Physically, this problem can be thought of a channel in the X direction getting narrower as X increases: as the width decreases the velocity needs to increase to conserve mass.

What is the width of the channel $W(X)$ in the (X,Y) coordinate system? we assume that the mass flux across the channel is conserved such that

$$U(X) \cdot W(X) = C \Rightarrow W(X) = \frac{C}{2aX} \tag{16}$$

where C is a constant. The width is the x,y coordinate system is therefore

$$W(x,y) = \frac{C}{a\sqrt{2}(x+y)} \tag{17}$$

Acceleration of the fluid particles: I will first find the acceleration A in the (X,Y) coordinate system and then transform it back into the (x,y) coordinate system.

Acceleration expressed in terms of Eulerian quantities:

$$A = \frac{DU}{Dt} = U\frac{\partial U}{\partial X} = 4a^2 X \Rightarrow a(x,y) = 2\sqrt{2}a^2(x+y) \tag{18}$$

where D/Dt is the material derivative and U is a function of X only.

Acceleration expressed in terms of Lagrangian quantities: first we use the velocity to find the particle trajectory such that

$$U = \frac{DX}{Dt} = 2aX \Rightarrow X(t;X_0) = X_0 e^{2at} \tag{19}$$

where the particle at $t = 0$ was at $X(t = 0) = X_0$. The velocity and acceleration in terms of Lagrangian quantity are respectively given by

$$\begin{aligned} U &= 2aX_0e^{2at} \\ A &= 4a^2X_0e^{2at} \end{aligned}$$

3. Continuity equation: I will show 2 different ways to derive this equation

   1. Very mathematical: Consider the continuity equation for an incompressible fluid such that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{20}$$

We have shown previously that the horizontal velocities u and v in the interior can be assumed to be depth independent. Using this assumption, we can integrate vertically the continuity equation

$$\begin{aligned} \int \partial w &= -\int \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \partial z \\ w(z) &= -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) z + f(x,y) \end{aligned}$$

We assume that the bottom is flat and that we cannot have vertical velocity at the bottom, this boundary condition can be written as

$$\begin{aligned} w(-H) &= 0 \Rightarrow w(-H) = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) H + f(x,y) = 0 \\ f(x,y) &= -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) H \end{aligned}$$

where $z = -H$ is the bottom of the ocean. The vertical velocity is therefore

$$w(z) = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(H + z) \tag{21}$$

We can now apply the fact that at the surface $z = \eta$, the velocity is given by $w(\eta) = \frac{D\eta}{Dt}$ such that

$$w(\eta) = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \tag{22}$$

leading to

$$-\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(H + \eta) = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \tag{23}$$

or

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} + (H + \eta)\frac{\partial u}{\partial x} + (H + \eta)\frac{\partial v}{\partial y} = 0 \tag{24}$$

Since we assumed that the bottom of the ocean is flat, H is constant and doesn't depend on x and y (or t), the above equation can be written as

$$\frac{\partial}{\partial t}(H + \eta) + \frac{\partial}{\partial x}u(H + \eta) + \frac{\partial}{\partial y}v(H + \eta) = 0 \Rightarrow h_t + (uh)_x + (vh)_y = 0 \tag{25}$$

where $h(x,y,t) = H + \eta(x,y,t)$ is the total thickness of the water column. This equation states that if the local horizontal divergence of volume, $\nabla \cdot (\vec{u}_h h) = (uh)_x + (vh)_y$, is positive, it must be balanced by a local decrease of the layer thickness due to a drop in the free sea surface.

2. More intuitive: We assume again that the motion is not depth dependent and the ocean has a uniform density. We consider a volume of water bounded between x and $x+dx$ and, y and $y+dy$,

The mass of this volume is equal to

$$dm = \rho h(x,y,t) dx dy$$

where as above h is total thickness of the water column such that $h(x,y) = H + \eta$. To keep in mind: we cannot create or destroy mass! therefore to look at its rate of change, we need to consider the mass fluxes across the interfaces.

In the x direction, the mass flux entering the volume at (x,y) is equal to

$$\rho u(x,y) h(x,y) dy$$

while the one leaving the volume at $(x+dx,y)$ is

$$\rho u(x+dx,y) h(x+dx,y) dy$$

In the y direction, the mass flux entering the volume at (x,y) is equal to

$$\rho v(x,y) h(x,y) dx$$

while the one leaving the volume at $(x,y+dy)$ is

$$\rho v(x,y+dy) h(x,y+dy) dx$$

. To summarize this, we have

$$\begin{aligned} \frac{\partial m}{\partial t} &= \rho u(x,y)h(x,y)dy - \rho u(x+dx,y)h(x+dx,y)dy \\ &+ \rho v(x,y)h(x,y)dx - \rho v(x,y+dy)h(x+dx,y+dy)dx \\ \frac{\partial}{\partial t}(\rho h(x,y,t)dxdy) &= -\rho \frac{\partial (uh)}{\partial x} dxdy - \rho \frac{\partial (vh)}{\partial y} dxdy \end{aligned}$$

such that

$$h_t + (uh)_x + (vh)_y = 0 \tag{26}$$

4. Width of the western boundary current: the N-S velocity in the western boundary current is given by

$$v = v_0 e^{-\beta(x-x_0)/J}$$

   (a) The decay scale in the E-W direction is therefore given by J/β.

(b) If the width of the Gulf Stream is equal to the decay scale, we should have

$$W = J/\beta$$

and the friction coefficient is equal to $J = \beta W$. At a latitude $30°N$, the value of β is given by

$$\beta = \frac{2\Omega}{R} cos(\theta_0) = \frac{2(7.3 \cdot 10^{-5})}{6371 \cdot 10^3} cos(30\pi/180) \approx 1.9846 \cdot 10^{-11} (ms)^{-1}$$

The friction coefficient is therefore

$$J = 1.9846 \cdot 10^{-11} \cdot 50 \cdot 10^3 \approx 9.9231 \cdot 10^{-7} s^{-1}$$

(c) If u and v are of the same order of magnitude U, we can write

$$\begin{aligned} O(fv) &= [fU] = 2\Omega sin(30°N)U \approx \Omega U \\ O(Ju) &= [JU] \end{aligned}$$

The ratio of the 2 terms is

$$\frac{fU}{JU} = \frac{\Omega}{J} \approx 74$$

We can conclude that the friction term in the interior is about 74 times smaller than the Coriolis term.

5. Planetary vs. relative vorticity: Open University p88

(a) question 4.3: (a) At latitude ϕ, the angular velocity due to the rotation of the Earth around the vertical axis is given by $\Omega sin\phi$. Since the vorticity similar to a solid body rotation is given by 2 times the angular velocity, the planetary vorticity will be equal to $2\Omega sin\phi$. We recognize this value to be the Coriolis parameter f. (b) At the North Pole, the planetary vorticity will be equal to 2Ω, so an observer standing at the North pole will experience a counterclockwise rotation. At the South Pole the vorticity will be -2Ω, and the observer will experience a clockwise rotation. At the equator the planetary vorticity is 0.

(b) question 4.4: (a) A body of water is carried southwards from the Equator. (i) At the equator the planetary vorticity of the water is zero, as it moves south, its planetary vorticity f is moving into regions of increasingly negative planetary vorticity. (ii) The absolute vorticity $\zeta + f$ needs to remain constant, therefore the decrease in planetary vorticity f needs to be balance by an increase in relative vorticity ζ. (b) (i) If winds are blowing in a clockwise direction, the contribution made to the water will be a negative relative vorticity; (ii) if cyclonic winds are blowing in the Southern Hemisphere, meaning that the wind are blowing in a clockwise direction (cyclonic=clockwise in the SH and counterclockwise in the NH), the contribution made to the water will be again a negative relative vorticity.

6. Ekman pumping:

(a) Experiment: for a cup with a width of $10cm$, I had to wait about $20sec$ before the spinning of the tea stopped.

(b) $x-$momentum equation that includes the balance between local acceleration and scale selective friction:

$$\frac{\partial u}{\partial t} = A_\nu \frac{\partial^2 u}{\partial x} \tag{27}$$

We assume that the solution to this equation is of the form

$$u = sin\left(\frac{x}{L}\right) e^{-t/\tau} \tag{28}$$

By plugging this solution into our equation, we obtain a decay time scale given by

$$\tau = \frac{L^2}{A_\nu}$$

We recognize the simple diffusion timescale by viscosity. The viscosity of water is about $1cP = 0.001 N s m^{-2}$ for a density of roughly $\rho = 1000 kg m^{-3}$ such that $A_v = 10^{-6} m^2 s^{-1}$

$$\tau = 10000 sec \approx 2.7 hours$$

This time scale is way longer than the one I've observed! why?

(c) Short explanation for those who are interested: at the bottom of the cup, a very thin Ekman layer appears. We could solve for this layer in the same way that we solved for the surface Ekman layer in HW-06 but it is not necessary since we already know that the Ekman depth is given by $\delta_E = \sqrt{2A_\nu/f}$, where $f = 2\Omega sin\theta \approx 2\Omega$ for this case.

Assuming that the cup has a cylindrical form, the total mass in the cup is given by $M = \rho\pi(L/2)^2 L = \rho\pi L^3/4$ where the height of the cup is taken to be L. The mass flux in the boundary layer defined by the Ekman layer is given by $\rho\delta_E\pi L v$, where v is the angular velocity $v = \Omega L/2$. The decay time scale taking into account the Ekman boundary layer is given by

$$\tau_E = \frac{M}{\rho\delta_E 2\pi L v} = \frac{\rho\pi L^3}{4\rho\delta_E\pi L^2\Omega}$$

$$\tau_E = \frac{L}{4\delta_E\Omega} = \frac{L}{4\sqrt{A_v\Omega}} = \frac{L\delta_E}{4A_v}$$

therefore to explain the time scale that we've observed, we need an Ekman layer of $\delta_E \approx 1mm$.

In addition, if you place leaves in your tea cup and stir the water, you will notice that the leaves tend to pile up in the center of the cup due to presence of the Ekman layer leading to convergence of the water in the center of the cup and Ekman pumping.

7. Challenge problem:

(a) Knauss: the explanation provided by Knauss was the explanation provided by Henry Stommel in his original paper on western intensification. He considered an anticyclonic gyre assuming no fast boundary current, but just a northward return flow that occupies half of the basin (see Fig 6.14 in Knauss). The purpose was to shown that we need a fast western boundary current to close the vorticity balance.

In this anticyclonic symmetric gyre, the curl of the wind stress provides the ocean with a negative vorticity everywhere (clockwise rotation shown by the arrow in eq. 6.29 in Knauss). On the western side of the gyre, the flow is northward and therefore the planetary vorticity βv is positive (counterclockwise arrow); while on the eastern side of the gyre due to a negative velocity the planetary vorticity is negative (clockwise arrow). The vorticity dissipation by friction is given by $J(v_x - u_y)$ and assuming that the flow is geostrophic, we can relate the velocity gradients to the sea surface height gradients such that the vorticity dissipation is now $-J\nabla^2\eta$. In this gyre the contribution by $\nabla^2\eta$ is positive on both side of the ocean (counterclockwise arrow). Assuming that the wind stress and the advection of planetary vorticity are order 1 terms (Sverdrup balance) and that the friction is one order of magnitude smaller, we have

Ocean	*wind stress*	*friction*	*planetary vorticity*	*total*
Western side	-1	0.1	-1	$\Rightarrow -1.9$
Eastern side	-1	0.1	1	$\Rightarrow 0.1$

The vorticity is almost in balance in the eastern part of the ocean, but not in the western part. Since the vorticity is not in balance in the ocean, we might want to assume that the return flow should be confined into a narrow boundary rather than half of the domain. We need to choose an asymmetrical domain with a boundary layer corresponding to a strong narrow northward flow, due to friction the velocity is increasing away from the coast. :

1. Western boundary current:

Ocean	*wind stress*	*friction*	*planetary vorticity*	*total*
Western b.c.	-1	10	-9	$\Rightarrow 0$
East interior	-1	0.1	0.9	$\Rightarrow 0$

2. Eastern boundary current:

Ocean	*wind stress*	*friction*	*planetary vorticity*	*total*
West interior	-1	0.1	0.9	$\Rightarrow 0$
Eastern b.c.	-1	-10	-9	$\Rightarrow -20$

We can clearly see that the only way to achieve a vorticity balance in the ocean will be to have a western boundary current.

(b) Open University: in this book the argument is the same than the one described in Knauss with one major difference , the direction of the arrows is different! why? because they look at the contribution of each term to the relative vorticity and not the total vorticity. Also they don't explain why the boundary needs to be at in the western side of the gyre and not in the eastern side (from my point of view, their explanation is a bit weak).